Math 4310 Homework 1 Solutions

Problem 1. *Base case*: The base case here is $n = 1$, and we can just check that both sides of the equation give us a value of 1: the left-hand side is just $1^2 = 1$, and the right-hand side is $1 \cdot 2 \cdot 3/6 = 1$.

Inductive step: Assume that the formula holds for n; we want to prove it for $n + 1$. Starting with the left-hand side and using the inductive hypothesis we have

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

On the other hand, the formula we want for $n + 1$ is

$$\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

We can then just check that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

by expanding out both sides, e.g. by seeing they both equal

$$(n+1) \cdot \frac{(2n^2+8n+6)}{6}.$$

By induction, this proves the formula for all n.

Problem 2. (a) By definition, $b|a$ means there exists $x \in \mathbb{Z}$ with $a = bx$. Multiplying by this we get $ac = (bx)c = b(xc)$ and thus $b|ac$ by definition.

(b) Again, $b|a$ means there is x with $a = bx$, and similarly $c|b$ means there is y with $c = by$. Combining these equations we get $c = axy$ and thus $c|a$.

(c) Once more, there is x with $a = bx$, and here the second divisibility tells us $a + c = by$ for some y. Subtracting these two identities gives us $c = b(y - x)$ and thus $b|c$.

Problem 3. We have:

- $2^2 - 1 = 3$ is prime
- $2^3 - 1 = 7$ is prime
- $2^4 - 1 = 15 = 3 \cdot 5$
- $2^5 - 1 = 31$ is prime
- $2^6 - 1 = 63 = 3^2 \cdot 7$
- $2^7 - 1 = 127$ is prime
- $2^8 - 1 = 255 = 3 \cdot 5 \cdot 17$

(b) If n is composite, we can write $n = ab$ with $1 < a, b < n$. Then we have that $2^a - 1$ divides $2^n - 1 = (2^a)^b - 1$, in particular with

$$2^n - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + \cdots + 2^{(b-2)a} + 2^{(b-1)a}).$$

Since $1 < a < n$, we have $1 < 2^a - 1 < 2^n - 1$, so $2^a - 1$ is a proper factor and $2^n - 1$ can't be prime.

Extended Glossary. Since extended glossaries are open-ended, I won't be providing "solutions" to them.