Math 4310 Homework 3 - Due February 17

Problem 1. For each of the following subsets of a vector space over a field F determine (with proof) whether or not they are a subspace.

(a) The subset of F^n consisting of vectors $(x_1, \dots, x_n)$ with $x_1 + \cdots + x_n = 0$.

(b) The subset of $F[x]$ consisting of polynomials with no linear term (e.g. polynomials $a_0 + a_1 x + \cdots + a_n x^n$ with $a_1 = 0$).

(c) The subset of $F[x]$ consisting of polynomials with quadratic term equal to $1 \cdot x^2$ (e.g. polynomials $a_0 + a_1 x + \cdots + a_n x^n$ with $a_2 = 1$).

Problem 2. For each of the following subsets of polynomial vector spaces over a field F, show that they span the vector space in question and that they're linearly independent.

(a) The set $1, x+1, (x+1)^2, \dots, (x+1)^n$ of the space $\mathcal{P}_n(F)$ of polynomials of degrees $\leq n$.

(b) The set of all monomials $\{1, x, x^2, x^3, \dots\}$ in the space of all polynomials $F[x]$.

Problem 3. Let V be a vector space, W a subspace, and V/W the associated quotient space. Suppose that $E \subseteq V$ is a set such that $E' = \{e + W : e \in E\} \subseteq V/W$ spans V/W. Prove that $E \cup W$ spans V.

Problem 4. If V, W are two vector spaces over a field F, we define its *direct product* $V \times W$ (also sometimes called the *direct sum* $V \oplus W$) as the set of pairs (v, w) with $v \in V$ and $w \in W$. We define addition and scalar multiplication on $V \times W$ by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \qquad \text{and} \qquad a \cdot (v, w) = (a \cdot v, a \cdot w),$$

respectively. Write out a verification that $V \times W$ satisfies the axioms of being a vector space. (This will be a bit tedious, but it's the only time I'll ask you to actually check all of the axioms like this.)

Problem 5. The function space $\mathcal{F}(\mathbb{N}, \mathbb{R})$ can be thought of as the space of all infinite sequences in $\mathbb{R}$ (since a sequence $(a_0, a_1, a_2, \dots)$ can be considered to be a function $\mathbb{N} \to \mathbb{R}$). Prove that the following two subsets of $\mathcal{F}(\mathbb{N}, \mathbb{R})$ are actually subspaces:

(a) The space $\ell^\infty(\mathbb{R})$ of all sequences (a_n) that are uniformly bounded (i.e. such that there exists a single real number M with $|a_n| < M$).

(b) The space $\ell^1(\mathbb{R})$ of all sequences (a_n) such that the infinite series $\sum_{n=0}^\infty |a_n|$ converges.

In fact, for any p with $1 < p < \infty$ we can also define $\ell^p(\mathbb{R})$ as the set of sequences (a_n) satisfying $\sum |a_n|^p < \infty$; these can be proven to be a subspace, too, but doing so is a fair bit more involved.

Problem 6. In Problem 4, we defined the "direct product" of two vector spaces, and said it was also denoted the "direct sum". The reason we have two different names for the same thing is that these concepts actually become different if you define them for infinite sets. Namely, if $V_1, V_2, V_3, \dots$ is an infinite sequence of vector spaces over a field F, we define the *direct product* $\prod_{i=1}^\infty V_i$ as the set of all infinite sequences $(v_1, v_2, v_3, \dots)$ with $v_i \in V_i$; this is a vector space when we define addition and scalar multiplication coordinatewise. We then define the *direct sum* $\bigoplus_{i=1}^\infty V_i$ as the subset of $\prod_{i=1}^\infty V_i$ consisting of sequences $(v_1, v_2, v_3, \dots)$ in which *all but finitely many* of the v_i's are equal to zero.

(a) Prove that $\bigoplus_{i=1}^\infty V_i$ is a subspace of $\prod_{i=1}^\infty V_i$, so it's actually a vector space itself.

(b) Recall that in class we defined the space of *formal power series* over F, denoted $F[\![x]\!]$, as the set of all sums $\sum_{i=0}^\infty a_n x^n$ with $a_n \in F$ (where we interpret this sum as a purely formal object, without caring about convergence or its "value"). Give a bijective correspondence between $F[\![x]\!]$ and $\prod_{n=1}^{\infty} F$, such that if we treat the space of polynomials $F[x]$ as a subspace of $F[\![x]\!]$, it corresponds to the subspace $\bigoplus_{n=1}^{\infty} F$ of $\prod_{n=1}^{\infty} F$.

Extended Glossary. Give a definition of a **field extension**. Give an example and a non-example of a congruence. Then state and prove a theorem about field extensions.