Math 4310 Homework 3 Solutions

Problem 1. (a) This is a subspace. We check:

- It's nonempty because it contains the zero vector $(0, \dots, 0)$.
- If $(x_1, \dots, x_n)$ is in the set then $x_1 + \cdots + x_n = 0$, and multiplying by any scalar a we get $ax_1 + \cdots + ax_n = 0$ so $a(x_1, \dots, x_n) = (ax_1, \dots, ax_n)$ is in the set.
- If $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are in the set then adding the equations $x_1 + \cdots + x_n = 0$ and $y_1 + \cdots + y_n = 0$ and rearranging tells us $(x_1 + y_1, \dots, x_n + y_n)$ is in the set.

(b) This is a subspace. It certainly contains the zero polynomial, and is closed under scalar multiplication and addition because those are done coefficientwise and if we start with the x coefficient equal to zero we'll end up with that as well.

(c) This is not a subspace. It's not closed under scalar multiplication or addition: we have x^2 is in the set but $2 \cdot x^2 = x^2 + x^2$ is not.

Problem 2. (a) We can prove this by induction on n. The base case of $n = 0$ is trivial; $\mathcal{P}_0(F)$ is the space of constant polynomials, so the set $\{1\}$ is linearly independent (because 1 is nonzero) and spans (because any constant polynomial is a multiple of 1).

For the inductive step, assume that $n > 1$ and that we know the result for $\mathcal{P}_{n-1}(F)$. To see the set spans, consider an arbitrary polynomial $f(x) \in \mathcal{P}_n(F)$. If a_n is the coefficient of x^n in $f(x)$, then $f(x) - a_n(x+1)^n$ has 0 as the coefficient of x^n and thus is a polynomial of degree at most $n - 1$; by inductive hypothesis it can be written as a linear combination of $1, (x+1), \dots, (x+1)^{n-1}$. Rearranging we have $f(x)$ written as a linear combination of $1, (x+1), \dots, (x+1)^n$, proving that it spans.

Similarly, for linear independence assume we have a dependence relation

$$0 = a_0 \cdot 1 + a_1(x+1) + \cdots + a_{n-1}(x+1)^{n-1} + a_n(x+1)^n.$$

Comparing coefficients of x^n for both sides we get that $0 = a_n$, and thus we actually have a dependence relation of the form

$$0 = a_0 \cdot 1 + a_1(x+1) + \cdots + a_{n-1}(x+1)^{n-1}$$

in $\mathcal{P}_{n-1}(F)$; by induction all of the coefficients must be zero, so the dependence relation is trivial.

(b) This is essentially immediate from the definition. A polynomial is a finite sum of the form $a_0 + a_1 x + \cdots + a_n x^n$, i.e. a linear combination from the set of monomials, so this set spans. Moreover, the set is linearly independent because if we have an equality of polynomials $0 = a_0 + a_1 x + \cdots + a_n x^n$ then by definition this means the coefficients a_i are all zero.

Problem 3. To prove $E \cup W$ spans V, we need to show that for any $v \in V$ we can write v as a linear combination of elements in E. By assumption that E' spans V/W, if we take the coset $v + W$ we can write $v + W$ as a linear combination of elements of E', i.e.

$$v + W = a_1(e_1 + W) + \cdots + a_n(e_n + W) = (a_1 e_1 + \cdots + a_n e_n) + W$$

for $a_i \in F$ and $e_i + W \in E'$ (so $e_i \in E$). Then by definition of equality of cosets, this means we can write

$$v = (a_1 e_1 + \cdots + a_n e_n) + w = a_1 e_1 + \cdots + a_n e_n + 1 \cdot w$$

for some $w \in W$; so we've written v as a linear combination in $E \cup W$ as desired.

Problem 4. As stated, this is a formal but somewhat tedious check; for the following we let (v, w), (v_1, w_1), (v_2, w_2), and (v_3, w_3) denote an arbitrary element $V \times W$ and a, b denote arbitrary elements of F.

1. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$.
2. $[(v_1, w_1) + (v_2, w_2)] + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = \big((v_1 + v_2) + v_3, (w_1 + w_2) + w_3\big)$ $= \big(v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)\big) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1, w_1) + [(v_2, w_2) + (v_3, w_3)]$.
3. The additive identity is $(0, 0)$ because $(v, w) + (0, 0) = (v + 0, w + 0) = (v, w)$.
4. The additive inverse of (v, w) is $(-v, -w)$ because $(v + w) + (-v, -w) = (v - v, w - w) = (0, 0)$.
5. $a\big((v_1, w_1) + (v_2, w_2)\big) = a(v_1 + v_2, w_1 + w_2) = (a(v_1 + v_2), a(w_1 + w_2)) = (av_1 + av_2, aw_1 + aw_2)$ $= (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$.
6. $(a + b)(v, w) = ((a + b)v, (a + b)w) = (av + bv, aw + bw) = (av, aw) + (bv, bw) = a(v, w) + b(v, w)$.
7. $a\big(b(v, w)\big) = a(bv, bw) = (a(bv), a(bw)) = ((ab)v, (ab)w) = (ab) \cdot (v, w)$.
8. $1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w)$.

Problem 5. (a) The set $\ell^\infty(\mathbb{R})$ is nonempty because the zero sequence is certainly uniformly bounded (say, by $M = 1$). It's closed under scalar multiplication because if (a_i) is a sequence uniformly bounded by some constant M, and b is a scalar in $\mathbb{R}$, then $b \cdot (a_i) = (b \cdot a_i)$ is uniformly bounded by $|b|M + 1$ (if $|a_i| < M$ for all i, then $|ba_i| < |b|M$ for all M, except in the trivial case $b = 0$). It's closed under addition because if (a_i) is uniformly bounded by M and (b_i) is uniformly bounded by N, then $(a_i) + (b_i) = (a_i + b_i)$ is uniformly bounded by $M + N$ (we have $|a_i + b_i| \le |a_i| + |b_i| < M + N$).

(b) Again, the zero sequence is in $\ell^1(\mathbb{R})$ since $\sum_{n=0}^\infty 0$ converges. This set is closed under scalar multiplication because we know if $\sum |a_n|$ converges then so does $\sum b|a_n|$ for any scalar b, and closed under addition because if $\sum |a_n|$ and $\sum |b_n|$ converge then so does $\sum(|a_n| + |b_n|)$ and thus so does $\sum |a_n + b_n|$ by the comparison test.

Problem 6. (a) First of all, $\bigoplus_{i=1}^\infty V_i$ is nonempty because it contains the vector of all zeros (since *all* of its coordinates are equal to zero, so certainly all but finitely many are). It's closed under scalar multiplication because of all but finitely many coordinates of (v_i) are zero, then the same entries of $a \cdot (v_i) = (av_i)$ are zero. It's closed under addition because if all but finitely many coordinates of both (v_i) and (w_i) are zero, then so are all but finitely many coordinates of $(v_i) + (w_i) = (v_i + w_i)$ (certainly the only coordinates that can be nonzero are the finitely many where at least one of $v_i \ne 0$ or $w_i \ne 0$).

(b) A bijective correspondence between $\prod_{n=1}^\infty F$ and $F[\![x]\!]$ can be given by having a sequence (a_i) correspond to the formal power series $\sum_{n=0}^\infty a_{n+1}x^n$. (We need to use this shift because I wrote my infinite product starting at $n = 1$, but the power series starts at $n = 0$). This is certainly a bijection because the power series is exactly determined by its sequence of coefficients!

Moreover, under this bijection, the subspace $\bigoplus_{n=1}^\infty F$ corresponds to the subspace of polynomials $F[x]$. This is because a polynomial can be thought of as a formal power series with only finitely many terms - i.e. such that all but finitely many of the coefficients are zero! So a sequence (a_n) is in the direct sum iff all but finitely many of the coordinates are zero, iff all but finitely many of the associated power series $\sum a_n x^n$ are zero, iff that power series is actually a polynomial.