# Math 4310 Homework 4 Solutions

**Problem 1.** (a) By adding and subtracting the equations, you can see that any element $(x, y, z)$ satisfying both equations satisfies $z = 2x$ and $y = -3x$, and conversely any vector $(x, -3x, 2x)$ satisfies both. The one-element set $\{(1, -3, 2)\}$ is a basis for the space of such vectors.

(b) An arbitrary polynomial in $\mathcal{P}_4(\mathbb{R})$ is of the form $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$; computing $x^4 f(1/x)$ for this and setting it equal to $f(x)$ we find the requirement for being in the subspace is

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4,$$

i.e. $a_4 = a_0$ and $a_3 = a_1$. Thus we can take a basis to be $\{x^4 + 1, x^3 + x, x^2\}$.

(c) We can take a basis to be the one-element set $\{(1, 0)+W\}$. This coset is a nonzero vector in $V/W$ because $(1, 0) \notin W$, and any set with one nonzero element is linearly independent. To see it spans, note that an arbitrary element $(x, y) + W$ of $V/W$ can be written as $(x - y, 0) + W$ (by picking a different representative) and thus is a scalar multiple $(x - y) \cdot [(1, 0) + W)]$.

**Problem 2.** Let $\{w_1, \ldots, w_m\}$ be a basis of $W$; as a linearly independent set in $V$ the basis extension theorem lets us extend it to a basis

$$\{w_1, \ldots, w_m, w'_1, \ldots, w'_k\}$$

of $V$. Take $W' = L(w'_1, \ldots, w'_k)$. Then $W + W' = V$ because any element of $V$ can be written as a linear combination

$$v = (a_1w_1 + \cdots + a_mw_m) + (b_1w'_1 + \cdots + b_kw'_k) \in W + W',$$

and $W \cap W' = 0$ because any element in both can be simultaneously written as a linear combination in $\{w_1, \ldots, w_m\}$ and $\{w'_1, \ldots, w'_k\}$; if the linear combinations were anything other than zero then subtracting one from the other would give a nontrivial dependence relation.

**Problem 3.** (a) By definition, $\Phi_i(a_j) = 0$ for $j \neq i$ and $\Phi_i(a_i) = 1$. So, if we have any dependence relation

$$\alpha_0\Phi_0 + \cdots + \alpha_n\Phi_n = 0,$$

this is an equality of polynomials and we can evaluate both sides at $x = a_i$ to get

$$\alpha_0\Phi_0(a_i) + \cdots + \alpha_n\Phi_n(a_i) = 0,$$

and the LHS here is $\alpha_i \cdot 1$. So we conclude $\alpha_i = 0$ and thus the set is linearly independent. Since it consists of $n + 1$ linearly independent vectors in a dimension-$n + 1$ space, it is a basis.

(b) The polynomials you get are

$$\Phi_0(x) = \frac{-1}{6}(x^3 - 6x^2 + 11x - 6) \qquad \Phi_1(x) = \frac{1}{2}(x^3 - 5x^2 + 6x)$$

$$\Phi_2(x) = \frac{-1}{2}(x^3 - 4x^2 + 3x) \qquad \Phi_3(x) = \frac{1}{6}(x^3 - 3x^2 + 2x)$$

(c) To check that the polynomial $f(x) = x^p - x \in \mathbb{F}_p[x]$ always takes the value zero, we need to go back to how we defined $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$: it consists of all cosets $[a] = a + p\mathbb{Z}$ for $a \in \mathbb{Z}$. Then, showing $f([a]) = 0$ is asking that $[a]^p - [a] = [a^p - a] = 0$ in $\mathbb{Z}/p\mathbb{Z}$, i.e. that $a^p - a \equiv 0 \pmod{p}$ for any $a$. But if $p|a$ this is clear, and if $p \nmid a$ then "Fermat's little theorem" tells us $a^{p-1} \equiv 1 \pmod{p}$ and we can multiply by $a$ and rearrange to get $a^p - a \equiv 0 \pmod{p}$.

**Problem 4.** (a) In this case, take $\{s_1, \ldots, s_m\}$ to be a basis of $S$ and $\{t_1, \ldots, t_n\}$ to be a basis of $T$; we claim that $\{s_1, \ldots, s_m, t_1, \ldots, t_n\}$ is a basis of $S+T$ (which proves the claim by looking at the sizes of these sets). It certainly spans, so the only thing we need to check is linear independence. But if we have a dependence relation

$$\sum a_i s_i + \sum b_j t_j = 0,$$

rearranging we get $(\sum a_i s_i) = -(\sum b_j t_j)$; since $\sum a_i s_i \in S$ and $\sum b_j t_j \in T$, if they're equal their common value is in $S \cap T$ and thus is zero. So $\sum a_i s_i = \sum b_j t_j = 0$, and linear independence of both sets implies all of the coefficients are zero.

(b) This time, let $u_1, \ldots, u_r$ be a basis for $S \cap T$, and then use the basis extension theorem to extend it to bases $\{u_1, \ldots, u_r, s_1, \ldots, s_m\}$ of $S$ and $\{u_1, \ldots, u_r, t_1, \ldots, t_n\}$ of $T$. We claim

$$\{u_1, \ldots, u_r, s_1, \ldots, s_m, t_1, \ldots, t_n\}$$

is a basis of $S+T$; looking at the sizes of these sets we conclude

$$\dim(S+T) = r+m+n = (m+r)+(n+r)-r = \dim(S)+\dim(T)-\dim(S\cap T).$$

Again, it's easy to see this set spans $S+T$: it contains sets that span both $S$ and $T$. For linear independence, suppose that we have a dependence relation

$$\sum a_i s_i + \sum b_j t_j + \sum c_k u_k = 0.$$

Rearranging gives

$$\sum a_i s_i + \sum c_k u_k = -\sum b_j t_j,$$

with the LHS in $S$ and the RHS in $T$ and thus both sides in $S \cap T$. Then the RHS $-\sum b_j t_j$ is a linear combination of $\{t_1, \ldots, t_n\}$ that lies in the span of $\{u_1, \ldots, u_r\}$; but $L(t_1, \ldots, t_n) \cap L(u_1, \ldots, u_r) = 0$ because these sets are disjoint and combine to form a basis. So $-\sum b_j t_j = 0$ and thus each $b_j$ individually is zero. Looking at the LHS we get $\sum a_i s_i + \sum c_k u_k = 0$, and thus each $a_i$ and $c_k$ is zero because the set of all $s_j$ and $u_k$ is linearly independent.

(c) As suggested, take $V = \mathbb{R}^2$, and take $S, T, U$ to be three distinct lines through the origin. (You're almost forced to choose this if you want anything interesting to happen - if you want to get into a "bad" situation that doesn't reduce to the previous part, you need $S, T, U$ to all be different and also you don't want to choose the "trivial" subspaces 0 or $\mathbb{R}^2$). Any two distinct lines span $\mathbb{R}^2$, so certainly three distinct lines do, so $S+T+U = \mathbb{R}^2$ and $\dim(S+T+U) = 2$. On the other hand, since they're distinct lines all of the intersections are trivial: $S \cap T = S \cap U = T \cap U = S \cap T \cap U = 0$. Then we compute

$$\begin{aligned}\dim(S)+\dim(T)+\dim(U)-\dim(S\cap T)-\dim(S\cap U)-\dim(T\cap U)+\dim(S\cap T\cap U)&\\ = 1+1+1-0-0-0+0 = 3,&\end{aligned}$$

which is different from $\dim(S+T+U)$.