Math 4310 Homework 6 - Due March 15

Problem 1. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials of degree ≤ 2, and consider the two bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, (x+1), (x+1)^2\}$ of it. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be a linear transformation defined on the basis $\mathcal{B}$ by $T(1) = -1$, $T(x) = 3x + 4$, and $T(x^2) = 2x^2 - x - 1$.

(a) Compute the coordinate matrix $[T]_{\mathcal{B},\mathcal{B}}$ with respect to $\mathcal{B}$.

(b) Compute the coordinate matrix $[T]_{\mathcal{C},\mathcal{B}}$ where the domain is with respect to $\mathcal{B}$ and the codomain with respect to $\mathcal{C}$.

(c) Compute the coordinate matrix $[T]_{\mathcal{C},\mathcal{C}}$ with respect to $\mathcal{C}$.

(d) Which of the matrices you computed are similar?

Problem 2. Recall a square matrix $A \in M_n(F)$ is *nilpotent* if there's some exponent n such that $A^n = 0$. For each of the following parts, either prove that the claimed statement is true or provide an explicit counterexample to show it isn't. (For the ones that are false it should be enough to look at 2×2 matrices for counterexamples).

(a) If $A, B \in M_n(F)$ are two nilpotent matrices, is $A + B$ necessarily nilpotent?

(b) If $A, B \in M_n(F)$ are two nilpotent matrices that commute (i.e. satisfy $AB = BA$), is AB necessarily nilpotent?

(c) If $A, B \in M_n(F)$ are two nilpotent matrices (that are *not* necessarily assumed to commute), is AB necessarily nilpotent?

Problem 3. Recall that in class we've defined the *rank* of a linear transformation $T : V \to W$ (with V, W finite-dimensional) as the dimension of the image: $\operatorname{rank}(T) = \dim(\operatorname{img} T)$.

(a) You might also remember from an earlier linear algebra class that if A is an $m \times n$ matrix, then $\operatorname{rank}(A)$ was defined as the dimension of the column space (i.e. the subspace of F^m spanned by the columns of A). Prove that if $T : F^m \to F^n$ is the linear transformation associated to A using the standard bases, then $\operatorname{rank}(T)$ (as defined above) equals $\operatorname{rank}(A)$ (as defined here).

(b) Show in fact that if $T : V \to W$ is any linear transformation and $A = [T]_{\mathcal{C},\mathcal{B}}$ is the matrix of T with respect to arbitrary bases $\mathcal{B}, \mathcal{C}$, then $\operatorname{rank}(T) = \operatorname{rank}(A)$.

(c) Conclude that if A, B are equivalent matrices then $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Problem 4. (a) Let A be an $m \times n$ matrix and suppose that there exists a $n \times m$ matrix B such that $AB = I_m$ is the $m \times m$ identity matrix. Prove that $m \leq n$.

(b) Conclude that if A is an $m \times n$ matrix and there exists a $n \times m$ matrix B with $AB = I_m$ and $BA = I_n$, then actually $m = n$ and A is square. This is why we usually only even look at square matrices when defining invertibility!

(c) Show that if $m < n$, then you can find a $m \times n$ matrix A and a $n \times m$ matrix B such that $AB = I_m$. Show actually that given your A, you can find many other matrices B' with $AB' = I_m$ too.

Problem 5. Recall that two matrices $A, B \in M_{m\times n}(F)$ are called *equivalent* if there existed invertible matrices $P \in M_m(F)$ and $Q \in M_n(F)$ such that $PAQ = B$.

(a) Prove that this definition of "equivalent" gives an equivalence relation on $M_{m\times n}(F)$.

(b) Show that if $\operatorname{rank}(A) = r$ then A is equivalent to the matrix J_r which has 1's on the first r diagonal entries and 0's everywhere else. (Hint: let $T : F^n \to F^m$ be defined by A, and choose bases $\mathcal{B}, \mathcal{C}$ that give

$[T]_{\mathcal{C},\mathcal{B}} = J_r$). So every $m \times n$ matrix is equivalent to one of the matrices $J_0, J_1, \dots, J_d$ for $d = \min\{m, n\}$.

(c) In Problem 3, you showed that if A and B are equivalent then their rank was the same; use this to conclude that the different J_r's are not actually equivalent to each other. Conclude that there are exactly $d+1$ equivalence classes of $m \times n$ matrices under "equivalence of matrices", and they are characterized entirely by rank: A and B are equivalent if and only if $\text{rank}(A) = \text{rank}(B)$.

Extended Glossary. Give a definition of an **algebra over a field** for an arbitrary field F. (Depending on what source you look at, you may find slightly different lists of axioms for an algebra - you can use whichever variant you want). Give an example and a non-example of an algebra. Then state and prove a theorem about algebras.