Math 4310 Homework 7 Solutions

Problem 1. (a) This matrix determines a cyclic linear transformation. In particular we can choose the following b and obtain the following linearly independent vectors:

$$b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad T(b) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad T^2(b) = \begin{bmatrix} 10 \\ 3 \\ 1 \end{bmatrix}.$$

(b) Again this is cyclic, with an example basis given by

$$b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad T(b) = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \qquad T^2(b) = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}.$$

Problem 2. (a) If T is nilpotent then we know its nilpotence index is at most n. On the other hand if it has nilpotence index less than n then $T^{n-1}(b) = 0$ for all b, so $\{b, T(b), \ldots, T^{n-1}(b)\}$ can never be a basis. So if T is both nilpotent and cyclic it must have nilpotence index equal to n.

(b) By definition, a projection P has $P^2(v) = P(v)$ for every v; by induction we get $P^N(v) = P(v)$ for all N. So if P is nilpotent of index N, then $P^N(v) = 0$ for every v means $P(v) = 0$ for every v, i.e. $P = 0$. (And $P = 0$ is trivially both nilpotent and a projection).

(c) If $n \geq 3$ then a transformation T can never be both cyclic and a projection, since the vectors $b, T(b), T^2(b)$ can never be linearly independent because $T(b) = T^2(b)$. So this leaves us with the cases $n = 0, 1, 2$.

- If $n = 0$ so V is just the zero vector space, then the map $T(0) = 0$ is vacuously both cyclic and a projection.
- If $n = 1$, so V is one-dimensional and spanned by a single vector v, then any linear transformation T is of the form $T(v) = av$ for some $a \in F$. Then T is a projection iff $a^2 = a$, iff $a = 0$ or 1; but $T(v) = 0$ is not cyclic. So only the identity map $T(v) = v$ is a cyclic projection.
- If $n = 2$, we look at the possible projections of each rank we have. A rank-0 projection is the zero map, which is not cyclic. A rank-2 projection is the identity map, which is also not cyclic. I claim that all of the rank-1 projections are cyclic; these are the maps for which there's a basis $\{v, w\}$ such that $T(v) = v$ and $T(w) = 0$. Given such a T, we can take $b = v + w$ and then $\{b, T(b)\} = \{v + w, v\}$ is a basis.

Problem 3. We can row-reduce the given matrix to get

$$\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

thus we find y and w are the free variables, and the solutions to the system of equations are all vectors of the form

$$\mathbf{x} = \begin{bmatrix} 2y + w \\ y \\ -w \\ w \end{bmatrix}.$$

(b) We can check that $\mathbf{b}$ is not in the column space (which is two-dimensional, spanned by the first and third columns), so there are no solutions. On the other hand $\mathbf{c}$ is in the span, with one solution being 4 times the first column plus 2 times the second; thus the general solution set can be obtained as this particular solution plus the homogeneous solutions in part (a), i.e.

$$\mathbf{x} = \begin{bmatrix} 4+2y+w \\ y \\ 1-w \\ w \end{bmatrix}.$$

Problem 4. Row-reducing the augmented matrix gives

$$\left[\begin{array}{ccc|c} \overline{1} & \overline{0} & \overline{3} & \overline{4} \\ \overline{0} & \overline{1} & \overline{2} & \overline{1} \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{array}\right].$$

This lets us read off that z is a free variable and the solution is all vectors of the form

$$\mathbf{x} = \begin{bmatrix} \overline{4}+\overline{2}z \\ \overline{1}+\overline{3}z \\ z \end{bmatrix}.$$

Problem 5. We prove this by induction on k. The base case of $k=1$ is trivial since an eigenvector is nonzero so $\{v_1\}$ is automatically linearly independent. For the inductive step assume $v_1, \ldots, v_{k-1}$ are known to be linearly independent. Then if we have a dependence relation

$$a_1v_1 + \cdots + a_{k-1}v_{k-1} + a_kv_k = 0,$$

applying T to it gives another dependence relation

$$a_1\lambda_1v_1 + \cdots + a_{k-1}\lambda_{k-1}v_{k-1} + a_k\lambda_kv_k = 0.$$

Now, subtract λ_k times the first dependence relation from the second one. We get a dependence relation

$$a_1(\lambda_1 - \lambda_k) + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$$

on $\{v_1, \ldots, v_{k-1}\}$. By induction this dependence relation is trivial, i.e. $a_i(\lambda_i - \lambda_k) = 0$ for $1 \leq i \leq k-1$. Moreover, since all of the λ_i are assumed distinct, each $\lambda_i - \lambda_k$ is nonzero and we can divide by it to conclude $a_i = 0$ for $1 \leq i \leq k-1$. Then our original dependence relation reduces to $a_kv_k = 0$, which forces $a_k = 0$ too. So the dependence relation must be trivial, and $\{v_1, \ldots, v_k\}$ must be linearly independent.

Problem 6. (a) We prove this by induction on $k \geq 1$. The base case of $k=1$ is our assumption; for the inductive step we have

$$T^{k+1}(v) = T^k(T(v)) = T^k(\lambda v) = \lambda T^k(v) = \lambda \cdot \lambda^k v = \lambda^{k+1}v.$$

(b) In this case we have

$$\lambda T^{-1}(v) = T^{-1}(\lambda v) = T^{-1}(T(v)) = v.$$

Since $v \neq 0$ by assumption we must have $\lambda \neq 0$, and rearranging then gives $T^{-1}(v) = \lambda^{-1}v$ as desired.

(c) I originally meant to write "is every eigenvector of T^2 necessarily an eigenvector of T", not assuming T invertible - that problem has a straightforward solution of "no", that a nilpotent transformation like $T : F^2 \to F^2$ defined by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

does not have the vector $(0, 1)$ as an eigenvector, while $T^2 = 0$ does.

But the problem I actually wrote requires T^{-1} to be invertible, so you can't use nilpotent transformations. Here the answer is still "no", which you can see for instance by taking $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be defined by a matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which has *no* real eigenvectors but such that $T^2 = -I$ is a scalar matrix so has every (nonzero) vector as an eigenvector. (Note that this matrix still works if you interpret it as a linear transformation $\mathbb{C}^2 \to \mathbb{C}^2$, as well. Over $\mathbb{C}$ it has two eigenvalues i and $-i$, and corresponding eigenvectors v_1 and v_2. But then $v_1 + v_2$ is an eigenvector of $T^2 = -I$ but not of T).