Math 4310 Homework 7 - Due April 4

Problem 1. Compute the determinants of the following matrices. (You should be able to do this by without any brute-force computations that are too annoying).

(a) The matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \in M_5(\mathbb{R}).$$

(b) The matrix

$$B = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} \in M_3(F),$$

where x, y, z are arbitrary elements of the field F. (Your answer should end up as a product of linear factors).

Problem 2. Compute the signs of the following permutations on the set $\{1, \ldots, n\}$. Your answers will depend on the value of n, but in each case you should be able to express it in terms of residues of n modulo some particular number.

(a) The permutation σ taking the list $(1, \ldots, n)$ to $(2, \ldots, n, 1)$ (i.e. given by $\sigma(i) = i + 1$ for $i < n$ and $\sigma(n) = 1$).

(b) The permutation τ taking $(1, 2, \ldots, n)$ to $(n, n-1, \ldots, 1)$ (i.e. given by $\tau(i) = n + 1 - i$).

Problem 3. (a) What are the eigenvalues and associated eigenvectors of the following matrix over $\mathbb{Q}$ (i.e. treated as an element of $M_4(\mathbb{Q})$ and defining a $\mathbb{Q}$-linear transformation $\mathbb{Q}^4 \to \mathbb{Q}^4$)? What about if we view it as a matrix over $\mathbb{R}$? Over $\mathbb{C}$?

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(b) What are the eigenvalues and associated eigenvectors of the following matrix over $\mathbb{F}_7$?

$$B = \begin{bmatrix} \overline{5} & \overline{3} \\ \overline{4} & \overline{2} \end{bmatrix}.$$

Problem 4. Consider the linear transformation $M_n(F) \to M_n(F)$ given by $A \mapsto A^\top$, i.e. taking a matrix to its transpose. Describe what the eigenspaces of this transformation are. Is it diagonalizable?

Problem 5. Suppose $A \in M_n(\mathbb{Q})$ is an invertible matrix such that A and A^{-1} both only have *integer* entries. What are the possible values for $\det(A)$? (Show that the values you claim are possible, and show that no others can arise).

Problem 6. Suppose that A, B, C, D are $n \times n$ matrices, and consider the $2n \times 2n$ block matrix

$$M = (m_{ij}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

(a) If C is the zero matrix, prove $\det(M) = \det(A)\det(D)$ by working with the formula

$$\det(M) = \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) m_{\sigma(1),1} m_{\sigma(2),2} \cdots m_{\sigma(2n),2n}.$$

(Hint: Think about which terms in the sum are automatically zero by our hypothesis on C, and what you can say about the σ which are possibly not zero).

(b) Give an example of a matrix M for which $\det(M) \neq \det(A)\det(D) - \det(B)\det(C)$. (Hint: 4×4 is the smallest place you can get an example - try making a matrix M which has $\det(M) = 0$ but the other quantity nonzero).

Extended Glossary. Give a definition of an **symmetric multilinear form**. Give an example and a non-example of a symmetric multilinear form. Then state and prove a theorem about symmetric multilinear forms.