Math 4310 Homework 8 Solutions

Problem 1. (a) Start off by subtracting the first column from each of the others; via these elementary column operations that don't change the determinant we get

$$\det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = \det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then, add each each of the other columns to the first one; again this doesn't change the determinant and we get

$$\det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \det\begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

This matrix is upper-triangular so the determinant is the product of the diagonal entries, i.e. $\det(A) = 4 \cdot (-1)^4 = 4$.

(b) Subtracting the first row from the second and third gives us

$$\det\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} = \det\begin{bmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x \end{bmatrix}.$$

Doing a cofactor expansion on the first row we get

$$\det(B) = \det\begin{bmatrix} y-x & y^2-x^2 \\ z-x & z^2-x \end{bmatrix}.$$

Since det is multilinear in the rows, we can factor out $(y-x)$ from the first row and $(z-x)$ from the second to get

$$\det(B) = (y-x)(z-x)\det\begin{bmatrix} 1 & y+x \\ 1 & z+x \end{bmatrix}.$$

This remaining determinant we compute directly as $(z+x)-(y+x) = (z-y)$ so

$$\det(B) = (y-x)(z-x)(z-y).$$

Problem 2. (a) This can be realized by using $n-1$ swaps: for example first swapping 1 and 2, then 1 and 3, and so on up to swapping 1 and n. So the sign is $(-1)^{n-1}$, i.e. $\text{sgn}(\sigma)$ is 1 if n is odd and -1 if n is even.

(a) This permutation is just pairwise swapping each of the first $\lfloor n/2 \rfloor$ entries with something in the last $\lfloor n/2 \rfloor$ entries, so there are $\lfloor n/2 \rfloor$ swaps. In particular this means if $n \equiv 0, 1 \pmod 4$ then $\lfloor n/2 \rfloor$ is even so $\text{sgn}(\sigma) = 1$, and if $n \equiv 2, 3 \pmod 4$ then $\lfloor n/2 \rfloor$ is odd so $\text{sgn}(\sigma) = -1$.

Problem 3. (a) We compute the characteristic polynomial to be $(x^2+1)(x^2-2)$. So over $\mathbb{Q}$ it has no linear factors and thus no eigenvalues and no eigenvectors. Over $\mathbb{R}$ it factors as $(x^2+1)(x-\sqrt{2})(x+\sqrt{2})$ and accordingly has eigenvectors v_3 and v_4 of eigenvalues $\pm\sqrt{2}$, which we can find to be. Over $\mathbb{C}$ it factors completely as $(x-i)(x+i)(x-\sqrt(2))(x+\sqrt(2))$ and thus has eigenvectors v_1, v_2 of eigenvalues $\pm i$ in addition to v_3, v_4. We can compute the v_i's explicitly as:

$$v_1 = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\sqrt{2}-1 \end{bmatrix}.$$

(b) The characteristic polynomial here is

$$(\overline{5}-x)(\overline{2}-x) - \overline{3}\cdot\overline{4} = x^2 - \overline{7}x + \overline{10} - \overline{12}.$$

Since we're working mod 7, this reduces to $x^2 - \overline{2}$ and we can factor it as $(x-\overline{4})(x-\overline{3})$; thus the eigenvalues are $\overline{3}$ and $\overline{4}$. We can then find the associated eigenvectors to be

$$v_1 = \begin{bmatrix} \overline{1} \\ \overline{4} \end{bmatrix} \qquad v_2 = \begin{bmatrix} \overline{1} \\ \overline{2} \end{bmatrix}.$$

(Your eigenvectors may look different, but you should be able to check they're multiples of these!)

Problem 4. The answer turns out to be "yes" if the characteristic of F is not 2, and "no" if the characteristic of F is 2. First of all, we need to see what the eigenvalues actually are. You can do this by computing the characteristic polynomial to be $(x+1)^{n(n+1)/2}(x-1)^{n(n-1)/2}$ (it's not too bad if you use the standard basis - you get a $n^2 \times n^2$ matrix but that's built from 1×1 or 2×2 blocks along the diagonal). Alternatively, we can argue that if A is a nonzero matrix and $A^\top = cA$ then $(A^\top)^\top = (cA)^\top = c^2 A$, so looking at a nonzero entry we conclude $c^2 = 1$, and thus $c = \pm 1$. So the only eigenvalues are 1 and -1 (and remember that these coincide of $\text{char}(F) = 2$).

Then, it's straightforward to see that the $n(n+1)/2$-dimensional space of symmetric matrices is the eigenspace for 1 (i.e. the set of A such that $A^\top = A$). Also, if $-1 \neq 1$ (i.e. the characteristic is not 2) we can see that the $n(n-1)/2$-dimensional space of antisymmetric matrices is the eigenspace for -1 (i.e. the set of A such that $A^\top = -A$). So, if $\text{char}(F) \neq 2$ we've exhibited two distinct eigenspaces with dimensions that sum to $n^2 = \dim(F)$; thus we can use them to create a basis of eigenvectors and see the linear transformation is diagonalizable. If $\text{char}(F) = 2$, though, we just have the one $n(n+1)/2$-dimensional eigenspace and no others, so the transformation is not diagonalizable.

Problem 5. We can only have $\det(A) = \pm 1$. These both clearly arise, for instance from a diagonal matrix with one entry of -1 and the other diagonal entries 1 (which is its own inverse). On the other hand, by looking at the formula

$$\det(A) = \sum_{\sigma\in S_n} \text{sgn}(\sigma) m_{\sigma(1),1} m_{\sigma(2),2} \cdots m_{\sigma(n),n}.$$

we see that if A is a matrix of integers then $\det(A)$ must also be an integer. Similarly if A^{-1} only has integer entries then $\det(A^{-1})$ is also an integer. But $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$, so $\det(A)$ and $\det(A^{-1})$ have to be multiplicative inverses in $\mathbb{Z}$, and the only integers with multiplicative inverses are ± 1.

Problem 6. (a) Start with the formula

$$\det(M) = \sum_{\sigma\in S_{2n}} \text{sgn}(\sigma) m_{\sigma(1),1} m_{\sigma(2),2} \cdots m_{\sigma(2n),2n}.$$

Now, the lower-left block C is assumed to be zero, and this corresponds to saying $m_{ij} = 0$ for $i > n$ and $j \leq n$. Thus for any σ, if any $j \in \{1, \ldots, n\}$ has $\sigma(j) \notin \{1, \ldots, n\}$ the corresponding term of the sum equals zero, so we can reduce to σ with the property that they take $\{1, \ldots, n\}$ into itself. But this actually means σ

is composed of one permutation σ_1 of $\{1,\ldots,n\}$ and one permutation σ_2 of $\{n+1,\ldots,2n\}$. So this reduces to a sum over pairs of permutations (σ_1,σ_2):

$$\det(M) = \sum_{\sigma_1,\sigma_2} \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2) m_{\sigma_1(1),1}\cdots m_{\sigma_1(n),n} m_{\sigma_2(n+1),n+1}\cdots m_{\sigma_2(2n),2n}.$$

Since σ_1 and σ_2 are independent of each other we can actually factor this as a product of two sums:

$$\det(M) = \left(\sum_{\sigma_1} \operatorname{sgn}(\sigma_1) m_{\sigma_1(1),1}\cdots m_{\sigma_1(n),n}\right)\left(\sum_{\sigma_2} \operatorname{sgn}(\sigma_2) m_{\sigma_2(n+1),n+1}\cdots m_{\sigma_2(2n),2n}\right).$$

But the first term is $\det(A)$ and the second is $\det(D)$, since it's the usual definition of the determinant (with the indices shifted because of where D is in the matrix).

(b) As suggested we want to try a 4×4 matrix which we'll force the determinant of to be zero. One way to do this is to make two columns exactly the same, and then do whatever we want in the other two columns. A simple instance of that idea is taking the matrix

$$M = \begin{bmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 1 \\ c & 0 & d & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Since the second and fourth columns are the same, no matter what we plug in for a, b, c, d we'll always have $\det(M) = 0$. On the other hand, if we take determinants of the individual blocks we get

$$\det(A)\det(D) - \det(B)\det(C) = ad - bc.$$

So we just need to pick a, d, b, c so that $ad - bc \neq 0$. For example, we can take

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$