Math 4310 Homework 9 Solutions

Problem 1. We want to produce an orthogonal basis $\{f_1, f_2, f_3\}$ via Gram-Schmidt. We can take $f_1 = x$, the first vector in our initial list. Then we have to take

$$f_2 = e^x - \frac{\langle x, e^x \rangle}{\langle x, x \rangle} x = e^x - \frac{1}{1/3} x = e^x - 3x,$$

and finally

$$f_3 = \cos(\pi x) - \frac{\langle x, \cos(\pi x) \rangle}{\langle x, x \rangle} x - \frac{\langle e^x - 3x, \cos(\pi x) \rangle}{\langle e^x - 3x, e^x - 3x \rangle} (e^x - 3x).$$

Computing these integrals gives that this is

$$f_3 = \cos(\pi x) - \frac{-2/\pi^2}{1/3} x - \frac{6/\pi^2 - (1+e)/(1+\pi^2)}{(e^2 - 7)/2} (e^x - 3x).$$

This ends up being a bit more of a mess than I was intending, so at this point you can try to normalize things but the length of f_3 ends up being something horrible.

Problem 2. (a) We start by using the definition and expanding out the LHS:

$$\|v - w\|^2 = \langle v - w, v - w \rangle = \langle v, v \rangle - \langle w, v \rangle - \langle v, w \rangle + \langle w, w \rangle.$$

The first and last terms are $\|v\|^2$ and $\|w\|^2$, respectively. The middle two terms become $-2\langle v, w \rangle$ by symmetry, and finally we use that the angle θ is defined by $\cos(\theta) = \langle v, w \rangle / (\|v\| \|w\|)$.

(b) Again, we just expand out the two terms on the LHS as in the previous part:

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \qquad \|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2.$$

Adding these equations together gives what we want.

Problem 3. (a) The key here is we want to define $T : P_k(\mathbb{R}) \to P_k(\mathbb{R})$ so that $\int_0^4 T(p)(x) T(q)(x) dx$ is equal to $\int_{-1}^1 p(u) q(u) du$ by integration-by-substitution. There's a lot of flexibility for how to define $T(p)$, but it's enough to look for parameters $A, B, C \in \mathbb{R}$ such that $T(p) = Ap(Bx + C)$ works. (Note that for any choice of A, B, C this $T(p)$ is a polynomial, and is pretty straightforwardly linear in p; also as long as A and B are nonzero it's fairly easy to see we can do another linear transformation to invert it).

So consider

$$\int_0^4 \big(Ap(Bx + C)\big)\big(Ap(Bx + C)\big) dx = A^2 \int_0^4 p(Bx + C) q(Bx + C) dx.$$

We want to do the change-of-variables $u = Bx + C$, and we want this to satisfy $u(0) = -1$ and $u(4) = 1$. This forces us to take the parameters $C = -1$ and $B = 1/2$; then $du = 1/2dx$ and

$$\int_0^4 \big(Ap(12x - 1)\big)\big(Ap(12x - 1)\big) dx = A^2 \int_{-1}^1 p(u) q(u) 2du.$$

Since we want this to equal $\int_{-1}^q p(u) q(u) du$ we need $A^2 2 = 1$, or $A = 1/\sqrt{2}$.

We conclude that if we define a linear transformation $T : P_k(\mathbb{R}) \to P_k(\mathbb{R})$ by

$$T(p) = \frac{1}{\sqrt{2}} p(\tfrac{1}{2}x - 1)$$

then $\langle\langle T(p), T(q) \rangle\rangle = \langle p, q \rangle$. So this T is an isometry from V to V', and thus is an isometric isomorphism because V and V' have the same dimension (since their underlying spaces are the same).

(b) Since isometries preserve orthonormality, $T(L_0), \ldots, T(L_k)$ is an orthonormal basis of V', i.e.

$$\frac{1}{\sqrt{2}} L_0(\tfrac{1}{2}x - 1), \ldots, \frac{1}{\sqrt{2}} L_k(\tfrac{1}{2}x - 1).$$

Problem 4. Let $\{w_1, \dots, w_{n-1}\}$ be an orthonormal basis of W. Then $\{w_1, \dots, w_{n-1}, v\}$ is also orthonormal (since v is perpendicular to each w_i, and is normalized by our assumption) and thus an orthonormal basis for V. So we know we can write any vector u in terms of this orthonormal basis as

$$u = \langle u, w_1 \rangle w_1 + \dots + \langle u, w_{n-1} \rangle w_{n-1} + \langle u, v \rangle v.$$

Then, since we know T acts on the basis by $T(w_i) = w_i$ and $T(v) = -v$ we get

$$T(u) = \langle u, w_1 \rangle w_1 + \dots + \langle u, w_{n-1} \rangle w_{n-1} - \langle u, v \rangle v.$$

But this is just $u - 2\langle u, v \rangle v$.

Problem 5. (a) Linearity in each coordinate is an easy consequence of matrix multiplication (and the fact that taking transposes is a linear operator):

$$B(ax + bx', y) = (ax + bx')^\top Ay = (ax^\top + b(x')^\top)Ay = ax^\top Ay + b(x^\top)Ay = aB(x, y) + bB(x', y),$$

$$B(x, ay + by') = xA(ay + by') = axAy + bxAy' = aB(x, y) + bB(x, y').$$

Moreover note that $x^\top Ay$ is a 1×1 matrix so is trivially equal to its own transpose, i.e.

$$B(x, y) = x^\top Ay = (x^\top Ay)^\top = y^\top A^\top x;$$

if A is symmetric, i.e. $A = A^\top$, then this is $B(y, x)$ by definition so B is symmetric.

(b) Note that if A is any matrix and $x = \sum x_i e_i$ and $y = \sum y_j e_j$ we have

$$x^\top Ay = x^\top \begin{bmatrix} \sum_j a_{1j} y_j \\ \vdots \\ \sum_j a_{nj} y_j \end{bmatrix} = \sum_{i,j} x_i a_{ij} y_j.$$

On the other hand, if B is any bilinear form we have

$$B(x, y) = B\Big(\sum x_i e_i, \sum y_j e_j\Big) = \sum_{i,j} x_i y_j B(e_i, e_j).$$

So if we start with B and define A by $a_{ij} = B(e_i, e_j)$ we conclude $x^\top Ay = B(x, y)$. Moreover, if B is symmetric then $a_{ij} = B(e_i, e_j) = B(e_j, e_i) = a_{ji}$ so A is symmetric too.

(c) If V is an n-dimensional vector space and $\mathcal{B}$ is a basis, we know $x \mapsto [x]_\mathcal{B}$ gives an isomorphism $V \cong F^n$. Using this isomorphism we get a bijective correspondence between bilinear forms B' on F^n and B on V by $B(x, y) = B'([x]_\mathcal{B}, [y]_\mathcal{B})$. Then applying (a) and (b) we know bilinear forms B' on F^n are in bijective correspondence with matrices A by $B'(v, w) = v^\top Aw$, so combining these bijections gives what we want.

Problem 6. (a) Linearity in the first coordinate is straightforward, as is additivity in the second coordinate. For the rest of antilinearity we see that

$$z \cdot aw = \sum_{i=1}^n z_i \overline{aw_i} = \overline{a} \sum_{i=1}^n z_i \overline{w}_i = \overline{a} z \cdot w.$$

Similarly, for Hermitian-ness we have

$$\overline{z \cdot w} = \overline{\sum_{i=1}^n z_i \overline{w_i}} = \sum_{i=1}^n \overline{z}_i w_i = \sum_{i=1}^n w_i \overline{z}_i = w \cdot z.$$

Finally, positive-definiteness is because

$$z \cdot z = \sum_{i=1}^{n} z_i \overline{z}_i = \sum_{i=1}^{n} |z_i|^2$$

is always a nonnegative real number, and is positive if some z_i is nonzero.

(b) First of all, since $\langle x, y \rangle_{\mathbb{C}}$ is sesquilinear, it's in particular $\mathbb{R}$-bilinear since if $a \in \mathbb{R}$ then $\overline{a} = a$; and composing an $\mathbb{R}$-bilinear map with the $\mathbb{R}$-linear map $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ gives that $\langle\ ,\ \rangle_{\mathbb{R}}$ is bilinear. Also since $\langle x, y \rangle_{\mathbb{C}}$ is Hermitian, $\langle x, y \rangle_{\mathbb{C}}$ and $\langle y, x \rangle_{\mathbb{C}}$ are complex conjugates and thus their real parts are the same, giving symmetry of $\langle\ ,\ \rangle_{\mathbb{R}}$. Finally, since $\langle x, x \rangle_{\mathbb{C}}$ is always a positive real number if $x \neq 0$, we get $\langle x, x \rangle_{\mathbb{R}} = \langle x, x \rangle_{\mathbb{C}}$ is always a positive real number if $x \neq 0$.

(c) Since $\langle x, x \rangle_{\mathbb{C}}$ is real, its real part $\langle x, x \rangle_{\mathbb{R}}$ is equal to it, and thus $\sqrt{\langle x, x \rangle_{\mathbb{C}}} = \sqrt{\langle x, x \rangle_{\mathbb{R}}}$. However, note that $\langle x, y \rangle_{\mathbb{R}}$ is zero whenever $\langle x, y \rangle_{\mathbb{C}}$ is purely imaginary, so any two vectors with $\langle x, y \rangle_{\mathbb{C}} = i$, say, will be not orthogonal with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$ but will be orthogonal with respect to $\langle\ ,\ \rangle_{\mathbb{R}}$. A simple example is $x = ie_1$ and $y = e_1$ for $e_1 \in \mathbb{C}^n$ the first standard basis vector.