The Arc Algebra of a Surface

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Carleton College

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Outline

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1. A crash course in knot theory

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2. An introduction to the Kauffman bracket

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3. Generalizing the Kauffman bracket to an algebraic structure

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2. An introduction to the Kauffman bracket
3. Generalizing the Kauffman bracket to an algebraic structure
4. Our project

Section I

What is a knot?

Knots

Let's play with some knots...

Knot Theory (A Crash Course)

Definition

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- embedding means the knot cannot intersect itself
- an isotopy is a stretching and moving that does not break the knot or cause it to intersect itself at any point

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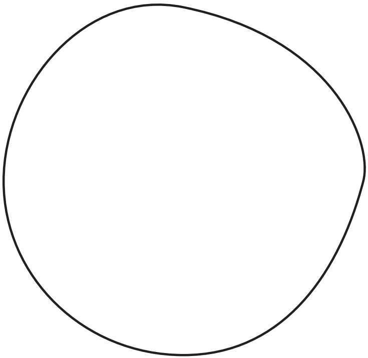
- each individual knot in a link is called a connected component of the link

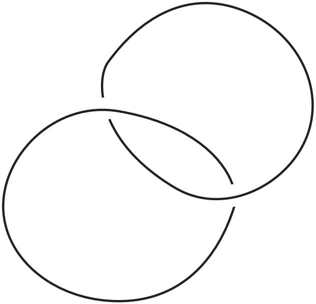
Some Examples

Unknot



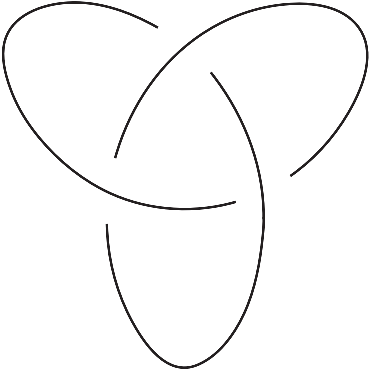
Some Examples

Hopf Link



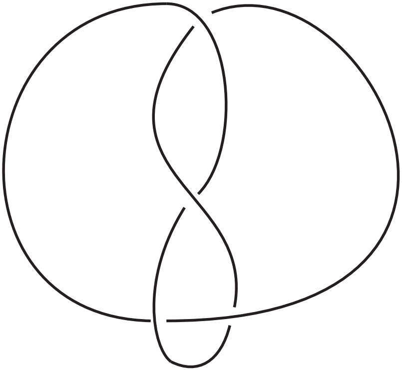
Some Examples

Trefoil



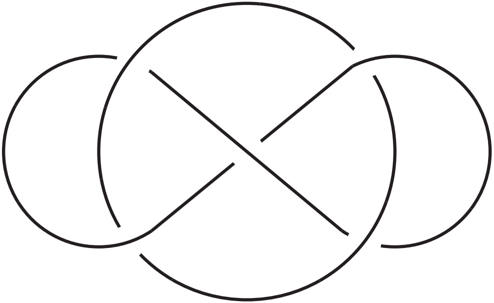
Some Examples

Figure-Eight Knot



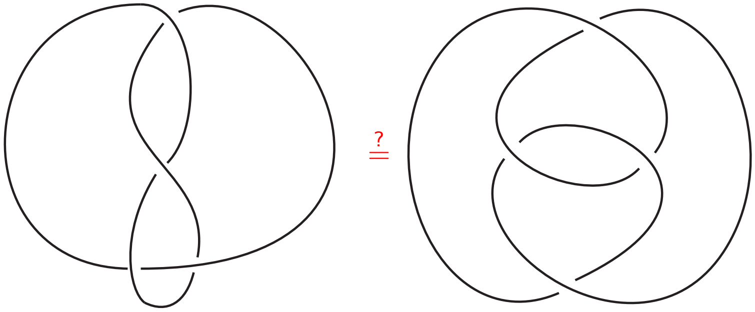
Some Examples

Whitehead Link



The Fundamental Problem of Knot Theory

The main problem in knot theory is determining when two knots are equivalent. This is nontrivial! Example:



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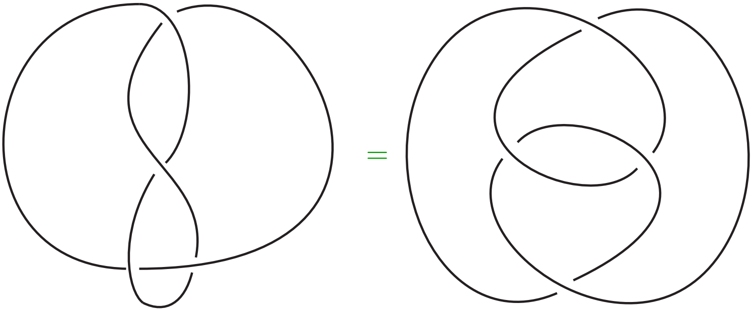

These are different diagrams of the same knot.

Invariants

So how do you tell things apart?

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- other common people invariants are hair color, eye color, and DNA sequence

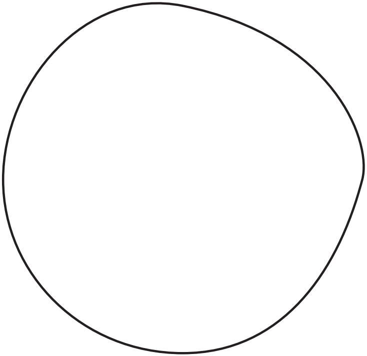
Knot Invariants

We need knot invariants:

- should help us tell knots apart
- will probably involve characteristics of knots
- should not depend on a particular representation

Connected Components

One simple invariant of links is the number of connected components.



$\rightarrow \quad 1$

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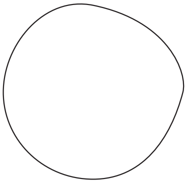

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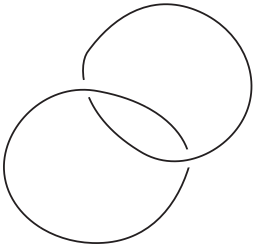
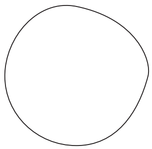


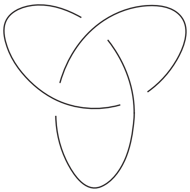
$\rightarrow \quad 2$

These two links have a different value for this invariant, so they are different links.

Connected Components

Two different links may have the same number of connected components, e.g.:

 $\rightarrow \quad 1$

 $\rightarrow \quad 1$

The trefoil and the unknot both have 1 connected component.

Completeness and Computability of Invariants

Invariants have varying degrees of completeness and computability.

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  Height is very computable but laughably incomplete.
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  For knots:
  Connected components is very computable but terribly incomplete.
  The knot complement and the Kontsevich integral are very complete but incredibly difficult to compute.

Kontsevich Integral is Scary

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{min}<t_1<\cdots<t_m<t_{max} \\ t_j \text{ noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

The Great Tragedy of Knot Theory

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We can, however, split the difference.

Section II

The Kauffman Bracket Polynomial

Let's Make a Polynomial

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3. After breaking all crossings of a complicated knot we are left with unknots, and then we recursively build up.

What in the world are you talking about?

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Stay with us here.

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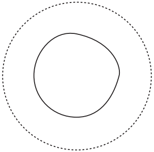
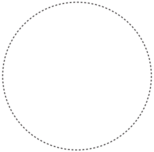
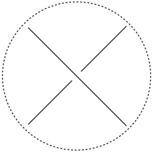
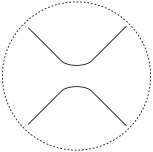
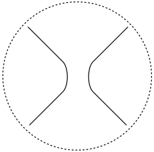
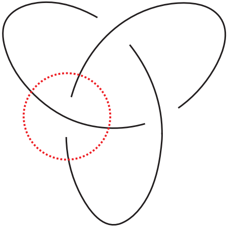
Define:

 $= (-A^2 - A^{-2})$ 

 $= A$  $+ A^{-1}$ 

Computation of Kauffman Bracket for Trefoil



$$=$$

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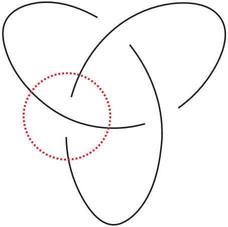

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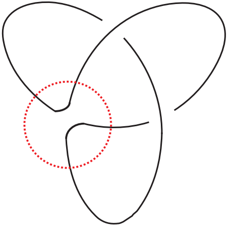
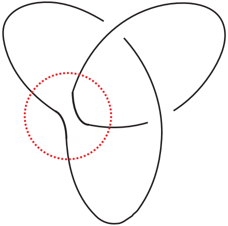


$+A^{-1}$



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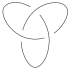

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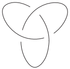 $=$ A  $+ A^{-1}$ 

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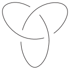
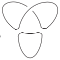
 $=$ A  $+ A^{-1}$ 

$=$ $A \Big(A$  $+ A^{-1}$  $\Big) + A^{-1} \Big(A$  $+ A^{-1}$  $\Big)$

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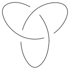
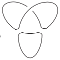
 $=$ A  $+ A^{-1}$ 

$=$ $A \Big(A$  $+ A^{-1}$  $\Big) + A^{-1} \Big(A$  $+ A^{-1}$  $\Big)$

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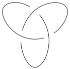
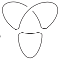
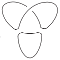
 $=$ A  $+ A^{-1}$ 

$=$ $A\Big(A$  $+ A^{-1}$  $\Big) + A^{-1}\Big(A$  $+ A^{-1}$  $\Big)$

$=$ AA  $+ AA^{-1}$  $+ A^{-1}A$  $+ A^{-1}A^{-1}$ 

$=$ $\cdots\cdots$

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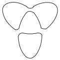
 $=$ AAA  $+ AAA^{-1}$ 

$+ AA^{-1}A$  $+ AA^{-1}A^{-1}$ 

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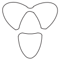
Computation of Kauffman Bracket for Trefoil

 $=$ AAA  $+\, AAA^{-1}$ 

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\end{aligned}$$

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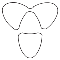
 $=$ AAA  $+\, AAA^{-1}$ 

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$$
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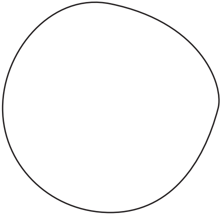
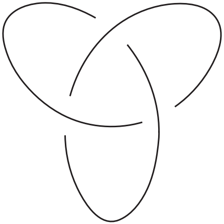
& \quad\quad +A^{-1}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 \\
= \quad & -A^9 + A + A^{-3} + A^{-7}
\end{aligned}
$$

The Trefoil is not an Unknot!



$$\rightarrow \quad -A^2 - A^{-2}$$



$$\rightarrow \quad -A^9 + A + A^{-3} + A^{-7}$$

As the Kauffman bracket polynomial is different, these are actually different knots!

Do you believe us?

Do you believe us? (Hint: you shouldn't)

Do you believe us? (Hint: you shouldn't)

- we have not shown that the Kauffman bracket is an invariant of knots

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- if we tried, we would find that it almost is

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- we would like to understand three-dimensional spaces

Section III

Generalizing the Kauffman Bracket

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(things are going to get weird)

Three-dimensional spaces?

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- as topologists, we want to understand 3-dimensional spaces

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- this is hard

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- as topologists, we want to understand 3-dimensional spaces
- this is hard
- it hurts our brains

Three-dimensional spaces?

- as topologists, we want to understand 3-dimensional spaces
- this is hard
- it hurts our brains
- so we start simple

Simple 3-Dimensional Spaces

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- $\mathbb{R}^3$ is a very simple 3-dimensional space

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- $\mathbb{R}^3$ is a very simple 3-dimensional space
- we are going to instead consider thickened surfaces

Surfaces

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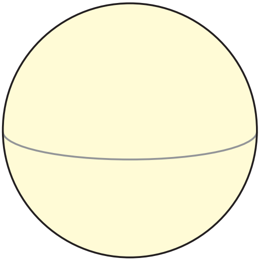

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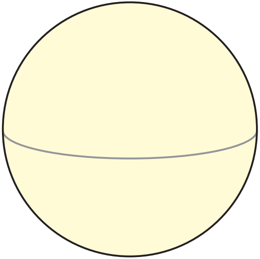

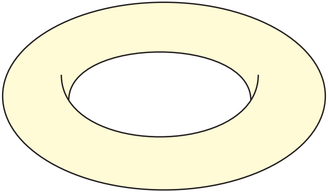

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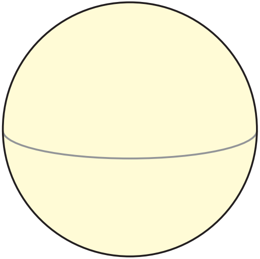

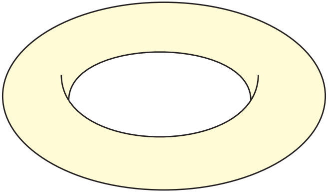

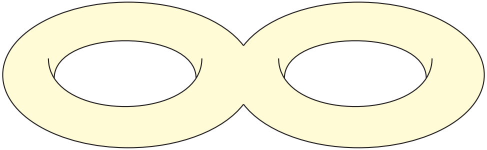

Thickened Surfaces

To form a thickened surface we start with a surface and give it thickness.

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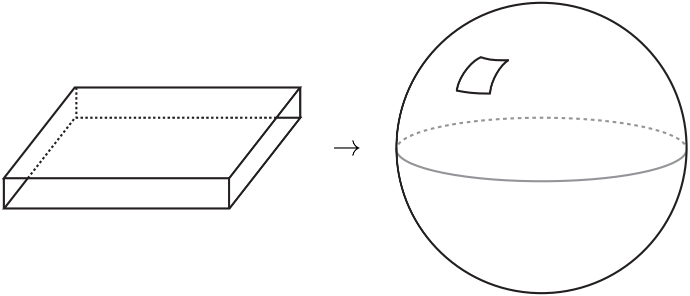

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Studying the knots that live in thickened surfaces can give us information about the structure of these spaces.

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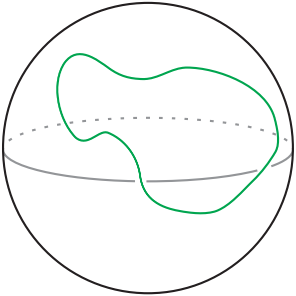

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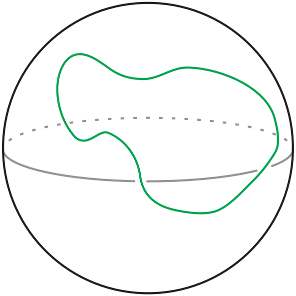
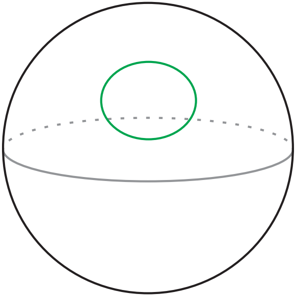


$\rightarrow$



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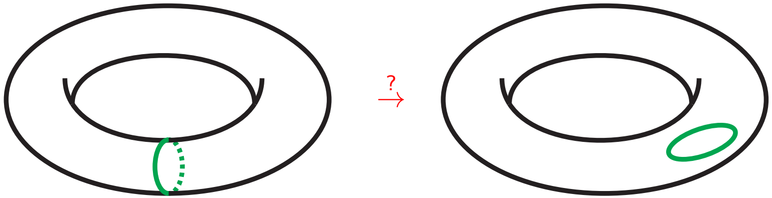

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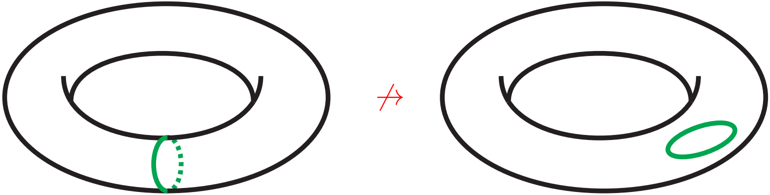

Knots on a Thickened Surface

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- we would like to study knots that live in thickened surfaces
- but there are a lot of them, and it's hard to tell them apart

Knots on a Thickened Surface

- we would like to study knots that live in thickened surfaces
- but there are a lot of them, and it's hard to tell them apart
- have we seen this problem before?

The Kauffman Bracket Returns

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Recall that in $\mathbb{R}^3$ the Kauffman bracket produced a polynomial when given a knot.

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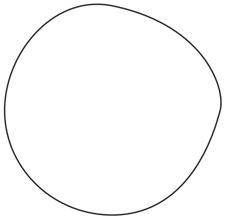
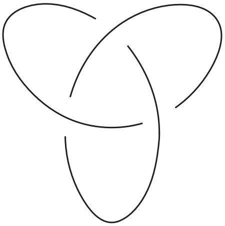
 $\rightarrow \quad -A^2 - A^{-2}$

 $\rightarrow \quad -A^9 + A + A^{-3} + A^{-7}$

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What does the Kauffman bracket do to knots on thickened surfaces?

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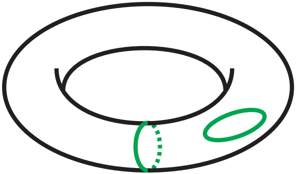

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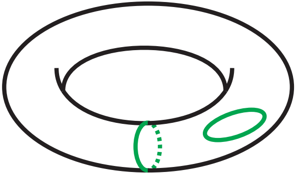
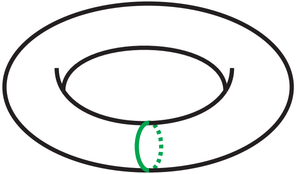
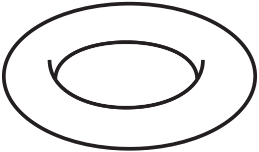
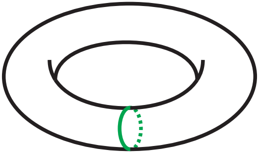


$$= (-A^2 - A^{-2})$$



Structure of the Kauffman Bracket

$(A+A^{-1})$  $+(-A^3-A^{-1})$ 

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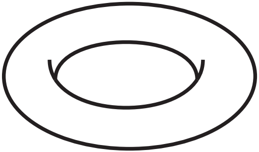
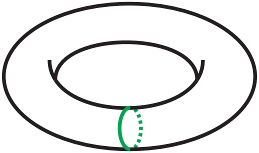
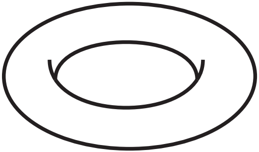
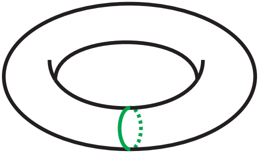
$(A+A^{-1})$  $+(-A^3-A^{-1})$ 

- We have objects

Structure of the Kauffman Bracket

$(A+A^{-1})$  $+(-A^3-A^{-1})$ 

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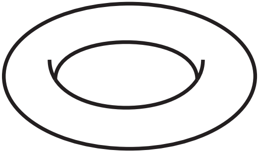
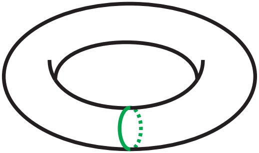
$(A+A^{-1})$  $+(-A^3-A^{-1})$ 

- We have objects
- with polynomial coefficients
- that we can add together.

What is this structure?

IT'S A VECTOR SPACE!

What is this structure?

IT'S A ~~VECTOR SPACE~~!

What is this structure?

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(Actually a module, as polynomials form a ring, not a field.)

But there's one more thing...

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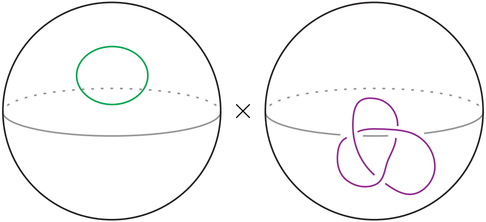
Multiplication



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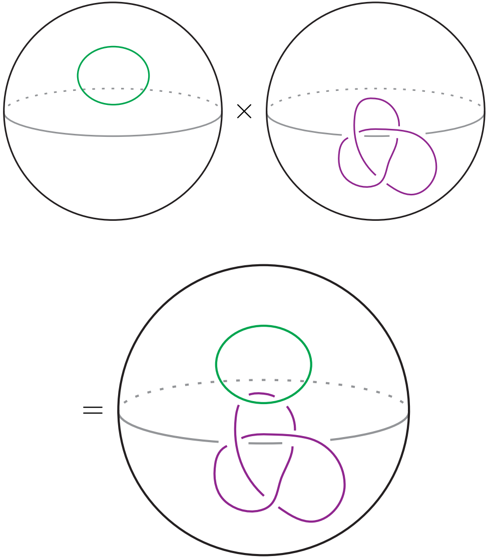

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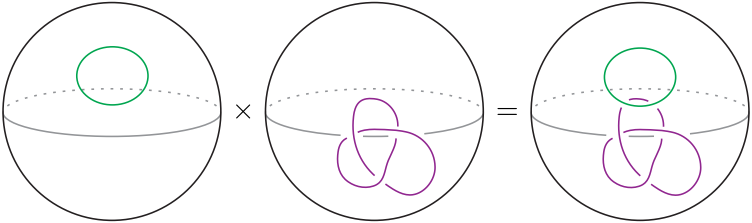

"This is not your grandfather's multiplication sign."
–Joe Silverman, Ph.D.

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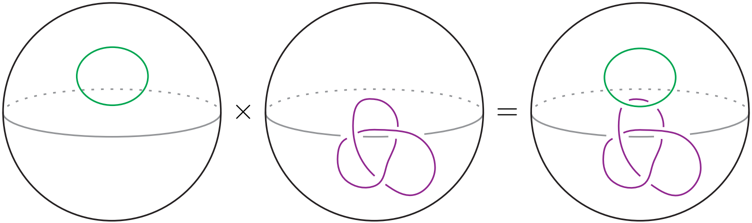

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Summary

In order to better understand thickened surfaces we have created a module with multiplication, which is an algebra.

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This algebra stores geometric information about the thickened surface in a more computationally approachable structure.

Section IV

Generalizing the Generalization of the Kauffman Bracket

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(things are going to get weirder)

Our Project

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The algebra defined in Section III has been known and studied since around 1990.

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Our project involved a generalization of this algebra to punctured surfaces developed by Julien Roger and Tian Yang in 2011.

Punctured Thickened Surfaces

Take a thickened surface and punch a hole in it

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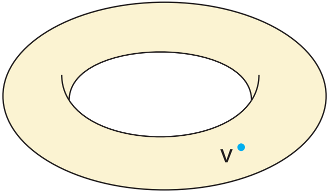
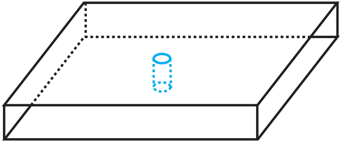


$\rightarrow$



Arcs

In addition to knots on punctured surfaces, we have arcs.

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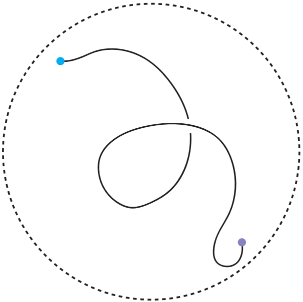

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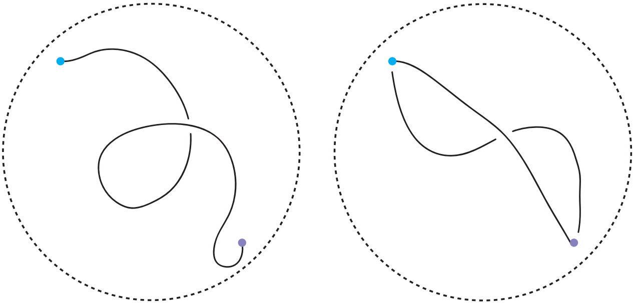

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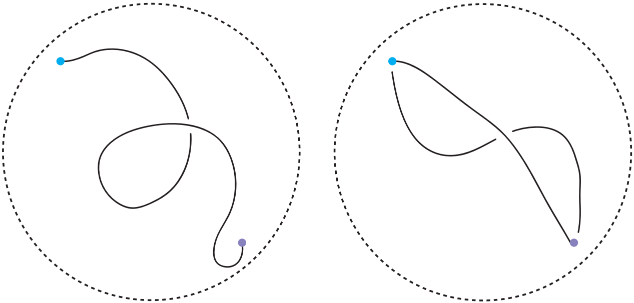

knots with two endpoints (thumbtacked at punctures)

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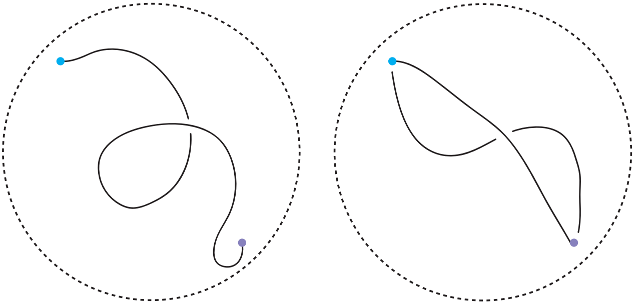

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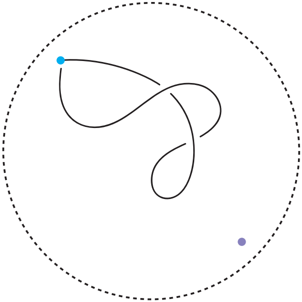

Puncture Relations

We need two more relations for punctured surfaces:

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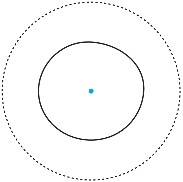
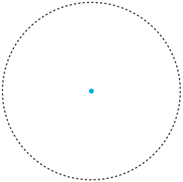
 $= (A + A^{-1})$ 

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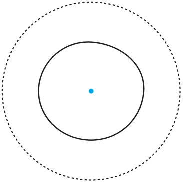
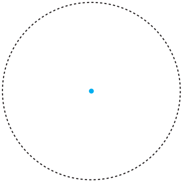
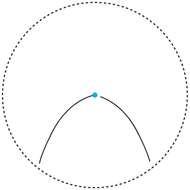
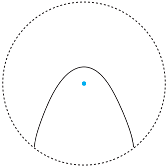
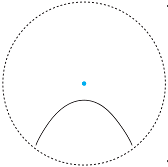
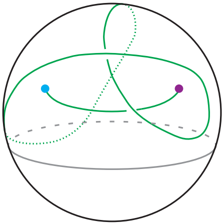
 $= (A + A^{-1})$ 

 $= v^{-1} \Big[A^{\frac{1}{2}}$  $+ A^{-\frac{1}{2}}$  $\Big]$

Let's see what happens

Let's use our rules on this...



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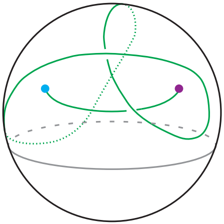
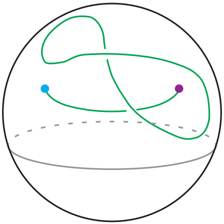


$=$



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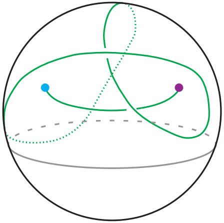

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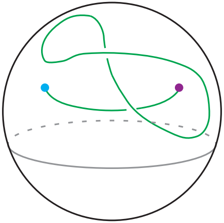
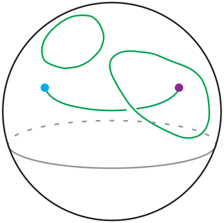
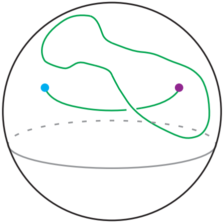


$= A$



$+ A^{-1}$



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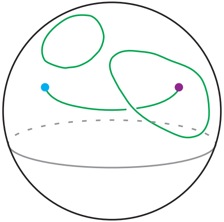
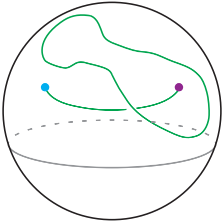
A  $+ A^{-1}$ 

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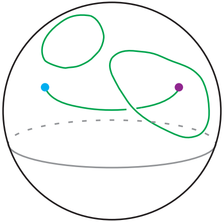
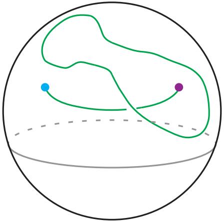
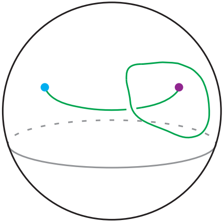
A  $+ A^{-1}$ 

$= [A(-A^2 - A^{-2}) + A^{-1}]$ 

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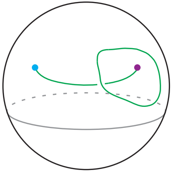

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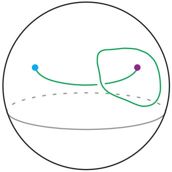
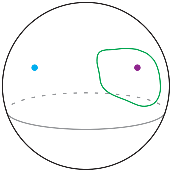
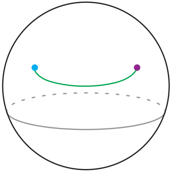
$[A(-A^2 - A^{-2}) + A^{-1}]$ 

$= [A(-A^2 - A^{-2}) + A^{-1}]$  $\times$ 

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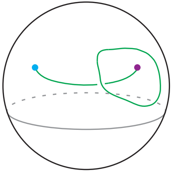
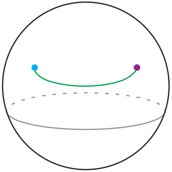
$$[A(-A^2 - A^{-2}) + A^{-1}]$$ 

$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$ 

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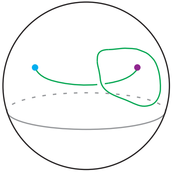
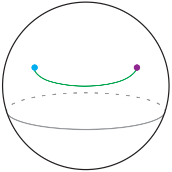
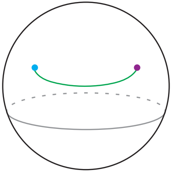
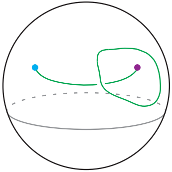
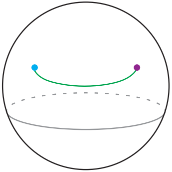
$$[A(-A^2 - A^{-2}) + A^{-1}]$$ 

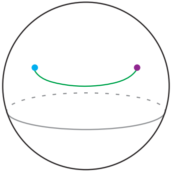
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$ 

$$= (-A^4 - A^2)$$ 

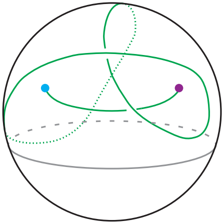
Let's see what happens

$[A(-A^2 - A^{-2}) + A^{-1}]$ 

$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$ 

$= (-A^4 - A^2)$  There is nothing left to do.

Reviewing Calculations



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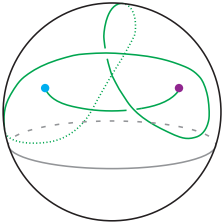

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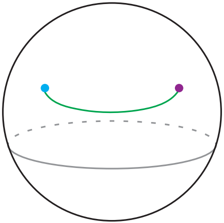
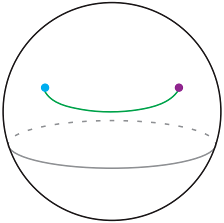


Generating Arc Algebras

This arc:



is special.

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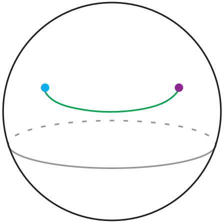
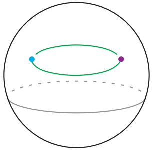


is special.
We'll come back to that...

Another Example



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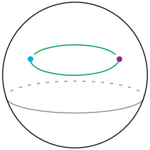
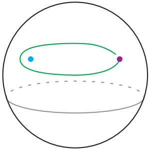
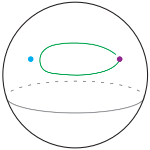


$= v^{-1} \Big[A^{1/2}$



$+ A^{-1/2}$



$\Big]$

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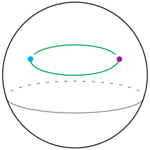
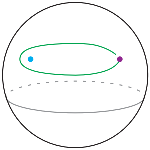
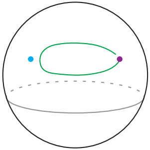
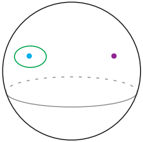
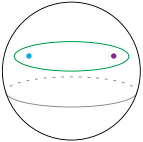
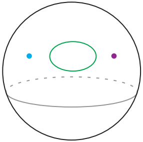
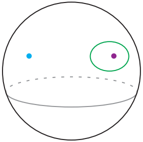
 $= v^{-1} \Bigg[A^{1/2}$  $+ A^{-1/2}$  $\Bigg] =$

$(vv)^{-1} \Bigg[A$  $+$  $+$  $+ A^{-1}$  $\Bigg]$

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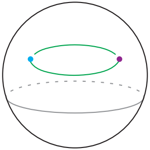
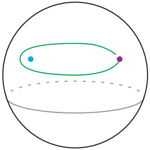
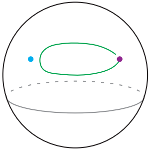
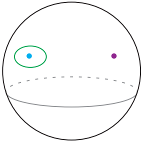
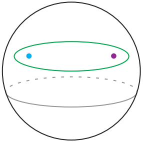
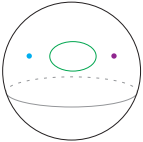
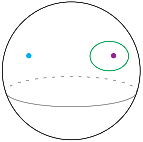
 $= v^{-1} \Big[A^{1/2}$  $+ A^{-1/2}$  $\Big] =$

$(vv)^{-1} \Big[A$  $+$  $+$  $+ A^{-1}$  $\Big]$

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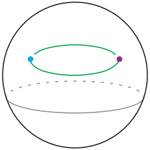
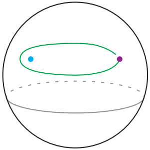
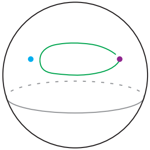
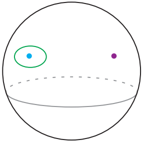
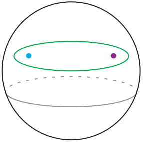
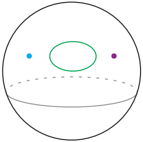
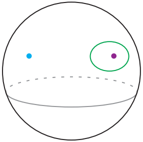
 $= v^{-1}\Big[A^{1/2}$  $+ A^{-1/2}$  $\Big] =$

$(vv)^{-1}\Big[A$  $+$  $+$  $+ A^{-1}$  $\Big]$

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5. What is left?

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1. There's no crossings (we removed them).

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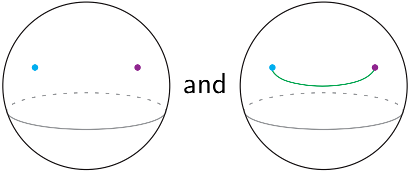
So we are left with:



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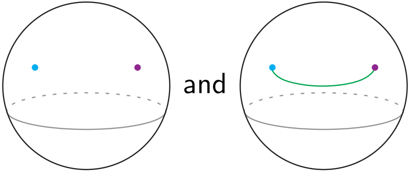

With polynomial coefficients in $A^{\frac{1}{2}}, v, v$.

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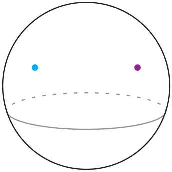
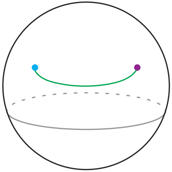
$k = p_0(A^{\frac{1}{2}}, v, v)$  $+ p_1(A^{\frac{1}{2}}, v, v)$ 

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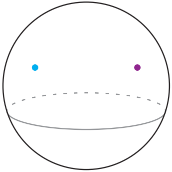
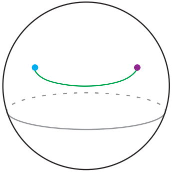
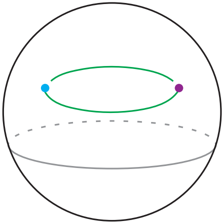
$k = p_0(A^{\frac{1}{2}}, v, v)$  $+ p_1(A^{\frac{1}{2}}, v, v)$ 

We say that the two diagrams in this sum generate this arc algebra.

Relation on the Generator



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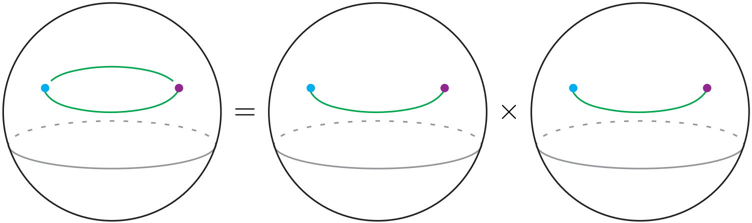

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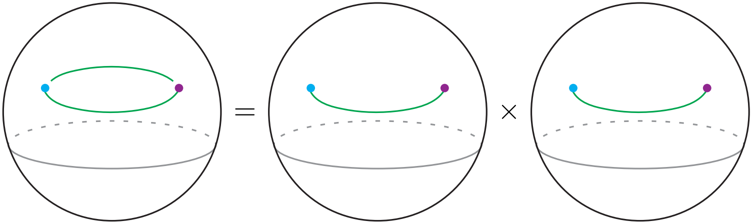

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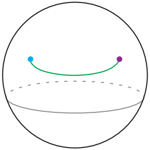
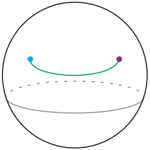
Relation on the Generator

 $\times$  $= (v v)^{-1}(-A^2 + 2 - A^{-2})$

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and we hope to achieve a general description.

References

[1] Doug Bullock. A finite set of generators for the Kauffman bracket skein algebra. *Math. Z.*, 231(1):91-101, 1999

[2] Doug Bullock and Józef H. Przytycki. Multiplicative structure of the Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.*, 128(3):923-931, 2000.

[3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space, arXiv:1110.2748v2, 2012.

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Thanks to you, as well, for coming to our talk.

Questions

We are prepared to answer any and all of your questions to the best of our ability.