

The Arc Algebra of a Surface

Martin Bobb Dylan Peifer

Carleton College

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Outline

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1. A crash course in knot theory

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2. An introduction to the Kauffman bracket

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3. Generalizing the Kauffman bracket to an algebraic structure

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2. An introduction to the Kauffman bracket
3. Generalizing the Kauffman bracket to an algebraic structure
4. Our project

Section I

What is a knot?

Knots

Let's play with some knots...

Knot Theory (A Crash Course)

Definition

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- ▶ **embedding** means the knot cannot intersect itself
- ▶ an **isotopy** is a stretching and moving that does not break the knot or cause it to intersect itself at any point

Links

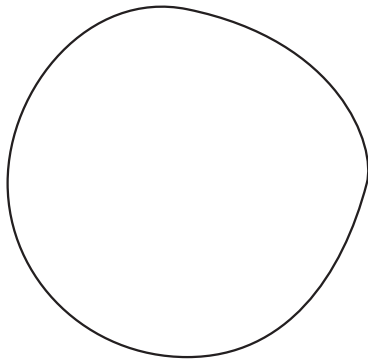
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- ▶ each individual knot in a link is called a connected component of the link

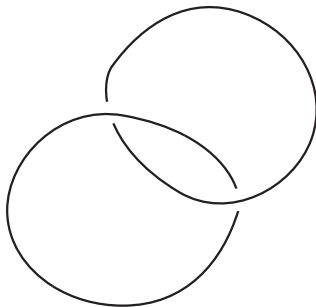
Some Examples

Unknot



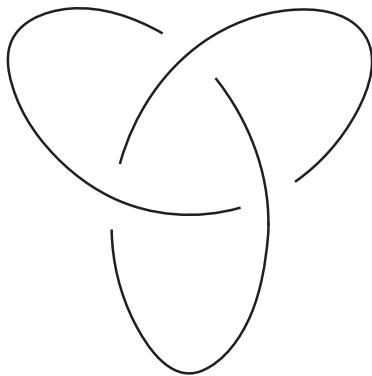
Some Examples

Hopf Link



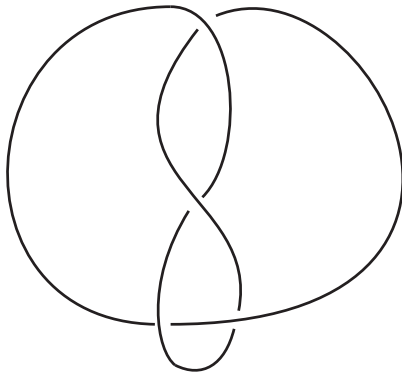
Some Examples

Trefoil



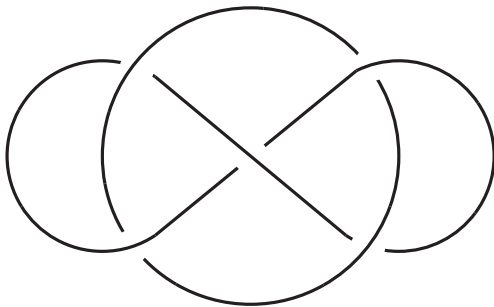
Some Examples

Figure-Eight Knot



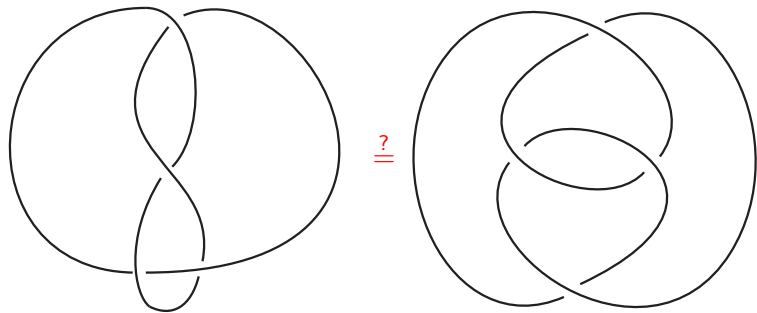
Some Examples

Whitehead Link



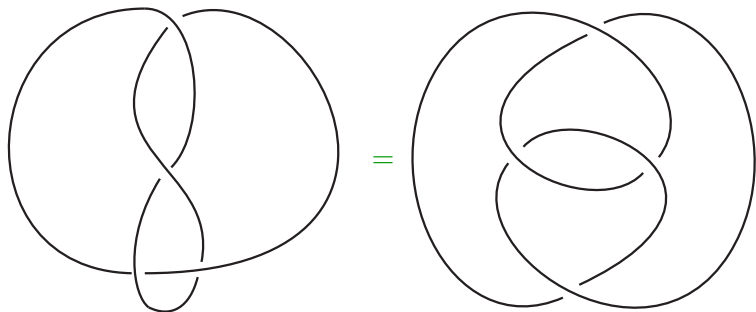
The Fundamental Problem of Knot Theory

The main problem in knot theory is determining when two knots are equivalent. This is nontrivial! Example:



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These are different diagrams of the **same knot**.

Invariants

So how do you tell things apart?

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- ▶ other common people invariants are hair color, eye color, and DNA sequence

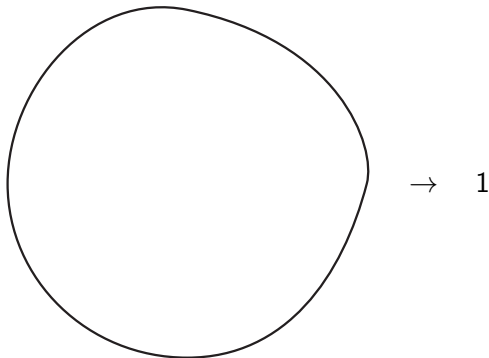
Knot Invariants

We need knot invariants:

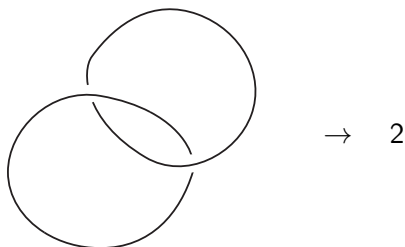
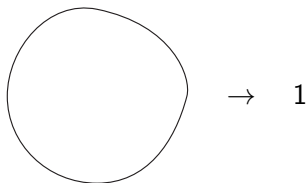
- ▶ should help us tell knots apart
- ▶ will probably involve characteristics of knots
- ▶ should not depend on a particular representation

Connected Components

One simple invariant of links is the number of connected components.



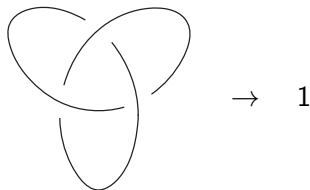
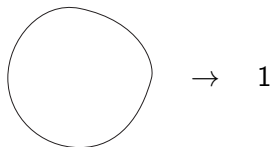
Connected Components



These two links have a different value for this invariant, so they are different links.

Connected Components

Two **different** links may have the same number of connected components, e.g.:



The trefoil and the unknot both have 1 connected component.

Completeness and Computability of Invariants

Invariants have varying degrees of **completeness** and **computability**.

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The knot complement and the Kontsevich integral are very **complete** but incredibly difficult to **compute**.

Kontsevich Integral is Scary

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{t_{\min} < t_1 < \dots < t_m < t_{\max} \\ t_j \text{ noncritical}}} \sum_{P=\{(z_j, z'_j)\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^m \frac{dz_j - dz'_j}{z_j - z'_j}$$

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We can, however, split the difference.

Section II

The Kauffman Bracket Polynomial

Let's Make a Polynomial

We can build a **polynomial** invariant to correspond to every knot.

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Different polynomials will mean we have different knots.

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1. We assign some polynomial to the simplest knot, the unknot.
2. We can “break crossings” one by one while storing information about those crossings in a polynomial equation.
3. After breaking all crossings of a complicated knot we are left with unknots, and then we recursively build up.

What in the world are you talking about?

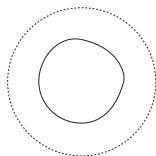
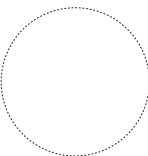
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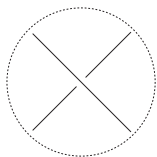
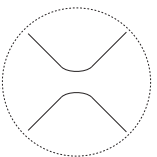
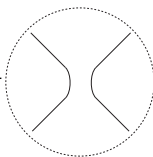
Stay with us here.

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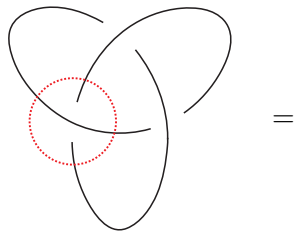
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Define:

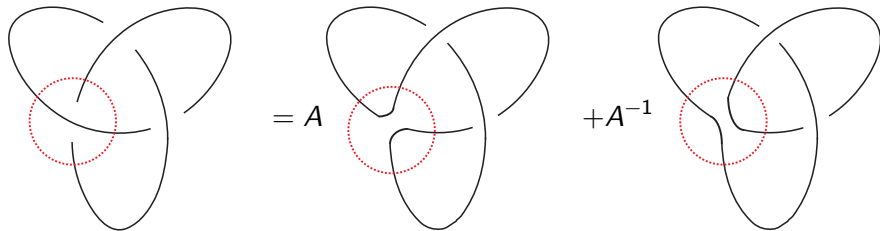
 $= (-A^2 - A^{-2})$ 

 $= A$  $+ A^{-1}$ 

Computation of Kauffman Bracket for Trefoil



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$$\text{Trefoil} = A \text{ (crossing resolved)} + A^{-1} \text{ (crossing resolved)}$$

Computation of Kauffman Bracket for Trefoil

$$\begin{aligned} \text{Trefoil} &= A \text{Trefoil}_1 + A^{-1} \text{Trefoil}_2 \\ &= A \left(A \text{Trefoil}_1 + A^{-1} \text{Trefoil}_2 \right) + A^{-1} \left(A \text{Trefoil}_1 + A^{-1} \text{Trefoil}_2 \right) \end{aligned}$$

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Computation of Kauffman Bracket for Trefoil

$$\begin{aligned}
 \text{Trefoil} &= AAA \text{ (Trefoil)} + AAA^{-1} \text{ (Trefoil)} \\
 &+ AA^{-1}A \text{ (Two loops)} + AA^{-1}A^{-1} \text{ (Trefoil)} \\
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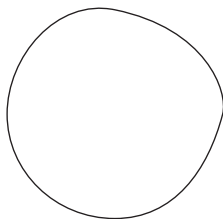
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 &= A^3(-A^2 - A^{-2})^3 + A(-A^2 - A^{-2})^2 \\
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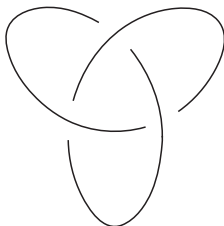
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 &\quad + A^{-1}(-A^2 - A^{-2}) + A^{-3}(-A^2 - A^{-2})^2 \\
 &= -A^9 + A + A^{-3} + A^{-7}
 \end{aligned}$$

The Trefoil is not an Unknot!



$$\rightarrow -A^2 - A^{-2}$$



$$\rightarrow -A^9 + A + A^{-3} + A^{-7}$$

As the Kauffman bracket polynomial is different, these are actually different knots!

Do you believe us?

Do you believe us? (Hint: you shouldn't)

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-
- ▶ we would like to understand three-dimensional spaces

Section III

Generalizing the Kauffman Bracket

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(things are going to get weird)

Three-dimensional spaces?

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Three-dimensional spaces?

- ▶ as topologists, we want to understand 3-dimensional spaces
- ▶ this is hard
- ▶ it hurts our brains
- ▶ so we start simple

Simple 3-Dimensional Spaces

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- ▶ \mathbb{R}^3 is a very simple 3-dimensional space

Simple 3-Dimensional Spaces

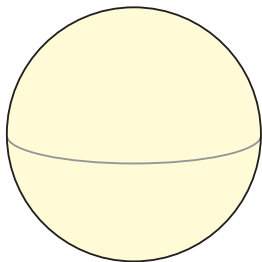
- ▶ \mathbb{R}^3 is a very simple 3-dimensional space
- ▶ we are going to instead consider **thickened surfaces**

Surfaces

A **surface** is a 2-dimensional space.

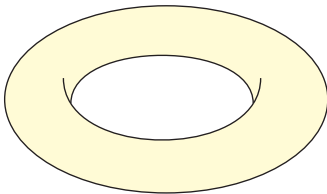
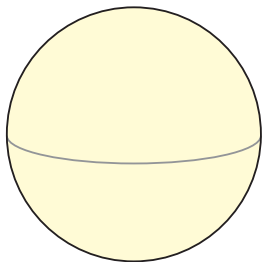
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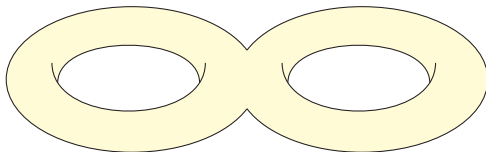
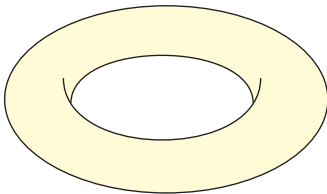
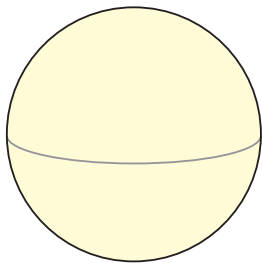
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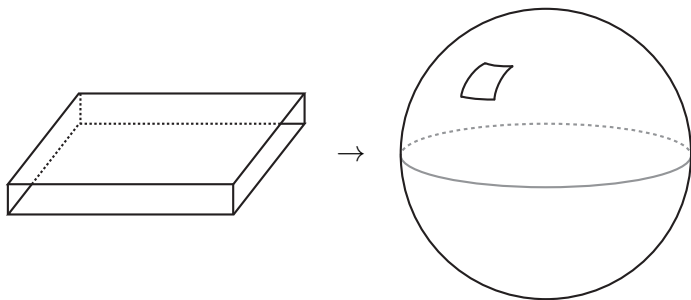


Thickened Surfaces

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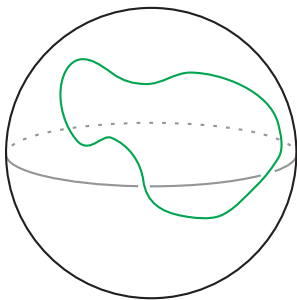


I thought this was a talk about knots...

Studying the knots that live in thickened surfaces can give us information about the structure of these spaces.

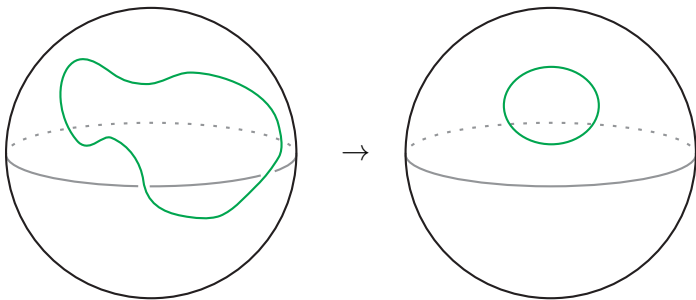
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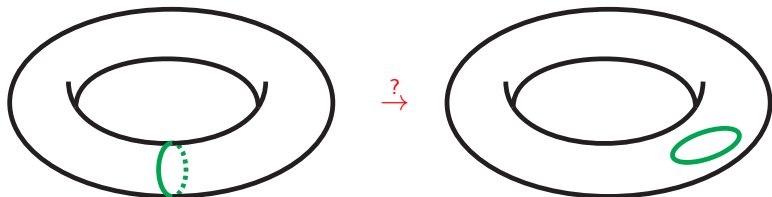
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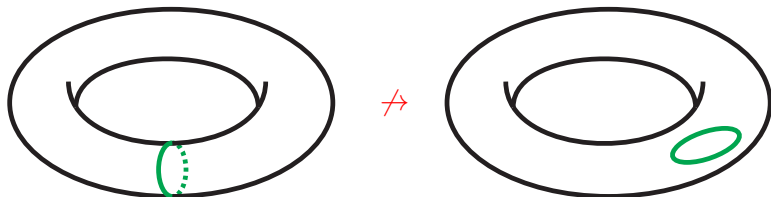
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Knots on a Thickened Surface

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- ▶ but there are a lot of them, and it's hard to tell them apart

Knots on a Thickened Surface

- ▶ we would like to study knots that live in thickened surfaces
- ▶ but there are a lot of them, and it's hard to tell them apart
- ▶ have we seen this problem before?

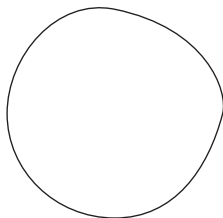
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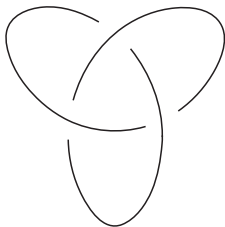
Recall that in \mathbb{R}^3 the Kauffman bracket produced a polynomial when given a knot.

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$$\rightarrow -A^2 - A^{-2}$$



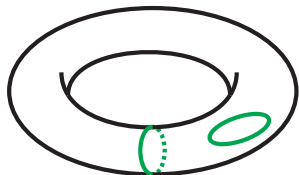
$$\rightarrow -A^9 + A + A^{-3} + A^{-7}$$

The Kauffman Bracket Returns

What does the Kauffman bracket do to knots on thickened surfaces?

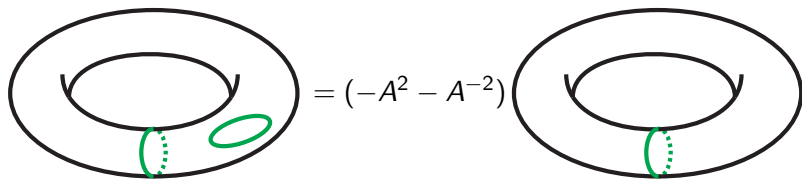
The Kauffman Bracket Returns

What does the Kauffman bracket do to knots on thickened surfaces? Example:

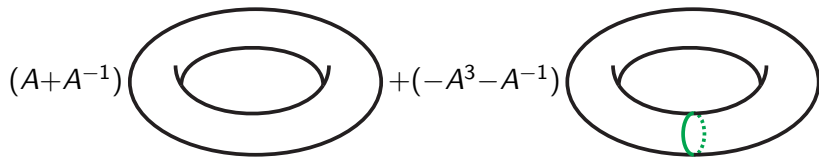


The Kauffman Bracket Returns

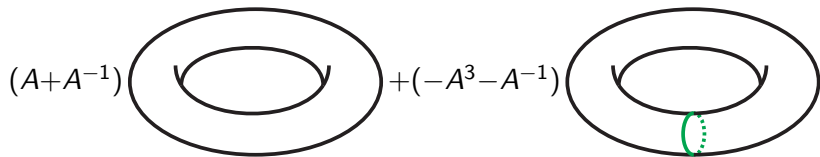
What does the Kauffman bracket do to knots on thickened surfaces? Example:



Structure of the Kauffman Bracket

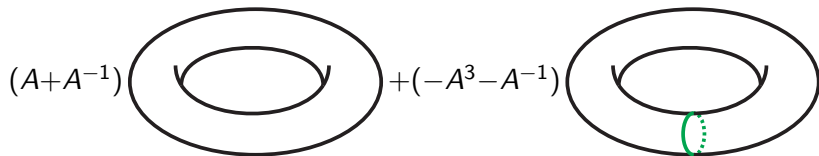


Structure of the Kauffman Bracket



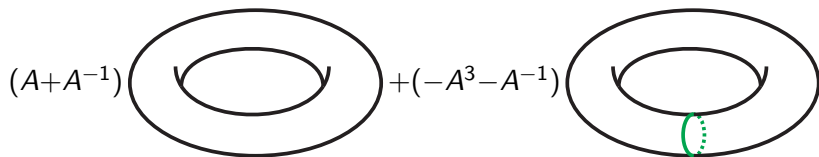
- We have **objects**

Structure of the Kauffman Bracket



- ▶ We have **objects**
- ▶ with polynomial **coefficients**

Structure of the Kauffman Bracket



- ▶ We have **objects**
- ▶ with polynomial **coefficients**
- ▶ that we can **add** together.

What is this structure?

IT'S A VECTOR SPACE!

What is this structure?

IT'S A ~~VECTOR SPACE~~!

What is this structure?

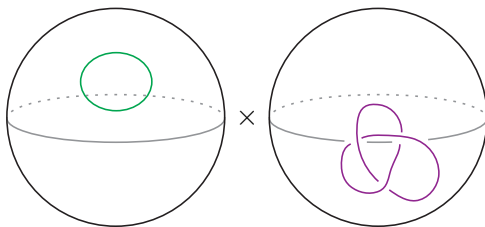
IT'S A ~~VECTOR SPACE~~!

(Actually a **module**, as polynomials form a ring, not a field.)

But there's one more thing...

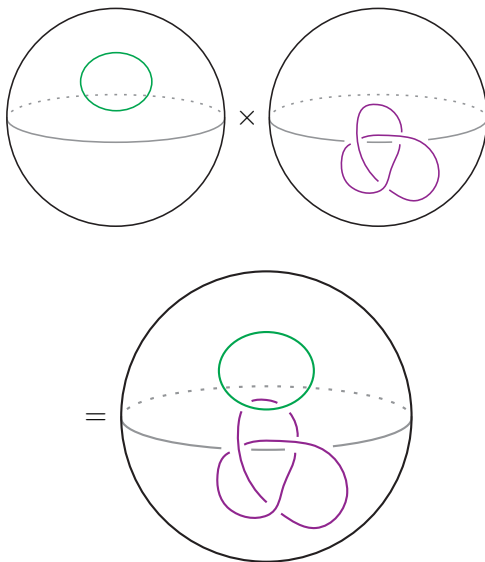
But there's one more thing...

Multiplication

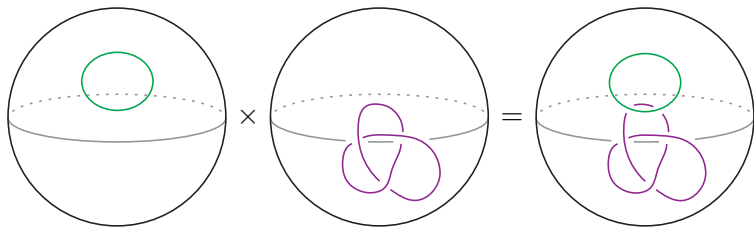


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Multiplication



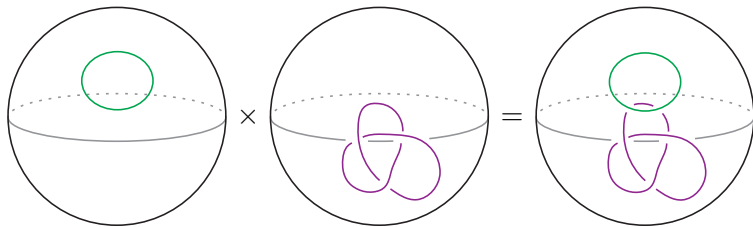
But there's one more thing...



“This is not your grandfather’s multiplication sign.”

–Joe Silverman, Ph.D.

But there's one more thing...



“This is [knot] your grandfather’s multiplication sign.”
–Joe Silverman, Ph.D.

Summary

In order to better understand thickened surfaces we have created a module with multiplication, which is an **algebra**.

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This algebra stores geometric information about the thickened surface in a more computationally approachable structure.

Section IV

Generalizing the Generalization of the Kauffman Bracket

Section IV

Generalizing the Generalization of the Kauffman Bracket
(things are going to get weirder)

Our Project

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The algebra defined in Section III has been known and studied since around 1990.

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Our project involved a generalization of this algebra to punctured surfaces developed by Julien Roger and Tian Yang in 2011.

Punctured Thickened Surfaces

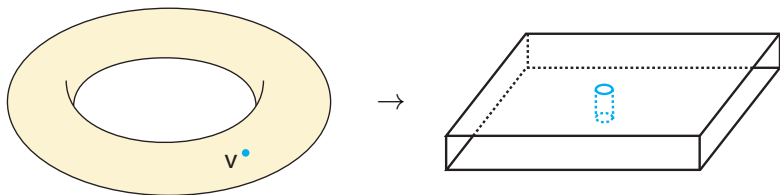
Take a thickened surface and punch a hole in it

Punctured Thickened Surfaces

Take a thickened surface and punch a hole in it (like a pin-hole in a balloon).

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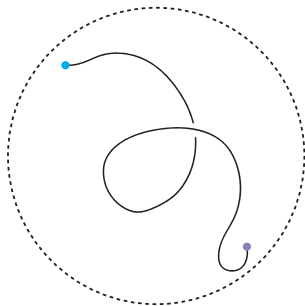


Arcs

In addition to **knots** on punctured surfaces, we have **arcs**.

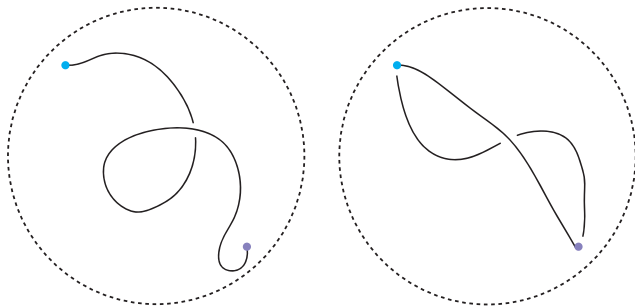
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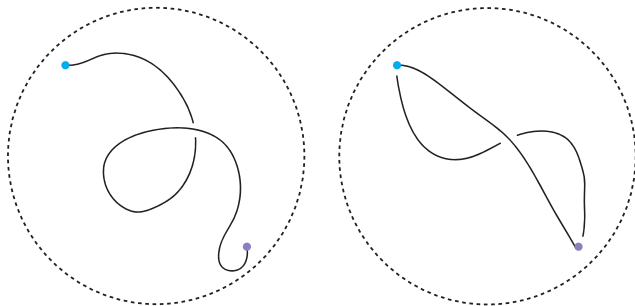
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Arcs

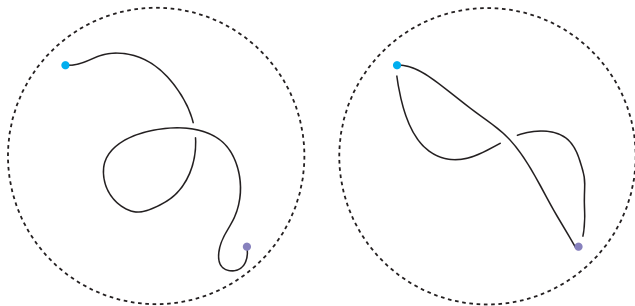
In addition to **knots** on punctured surfaces, we have **arcs**.



knots with two endpoints (thumbtacked at punctures)

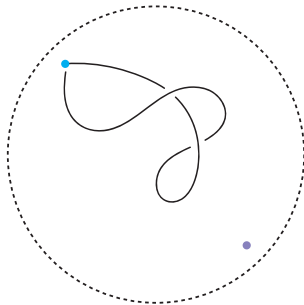
Arcs

In addition to **knots** on punctured surfaces, we have **arcs**.



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Arcs

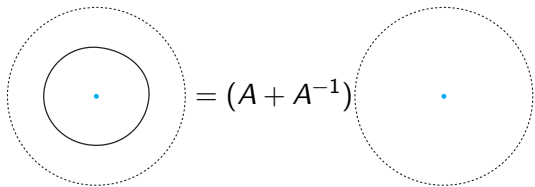


Puncture Relations

We need two more relations for punctured surfaces:

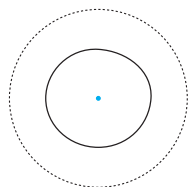
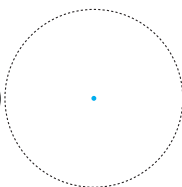
Puncture Relations

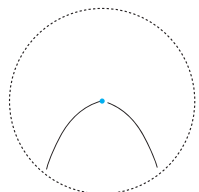
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Puncture Relations

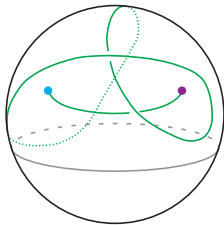
We need two more relations for punctured surfaces:


$$= (A + A^{-1})$$



$$= v^{-1} \left[A^{\frac{1}{2}} \right. \img alt="Diagram of a punctured disk with a dashed outer circle and a blue dot on the boundary. A solid arc connects two points on the boundary, passing through the dot, with the dot at the peak of the arc." data-bbox="425 600 595 835"/>
$$+ A^{-\frac{1}{2}} \img alt="Diagram of a punctured disk with a dashed outer circle and a blue dot on the boundary. A solid arc connects two points on the boundary, passing through the dot, with the dot at the midpoint of the arc." data-bbox="685 600 855 835"/>$$$$

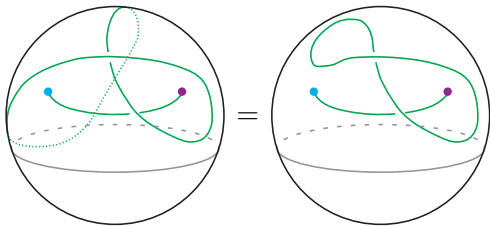
Let's see what happens

Let's use our rules on this...



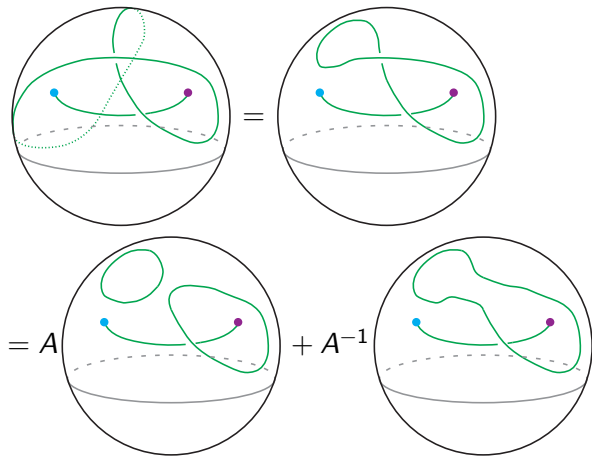
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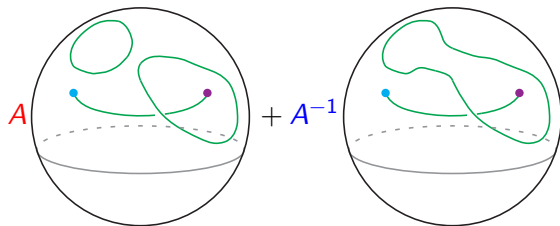


Let's see what happens

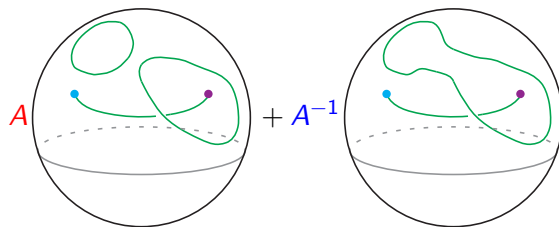
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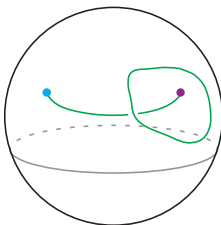
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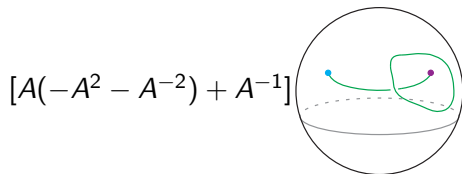
Let's see what happens



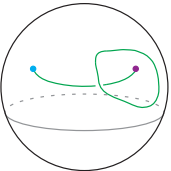
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

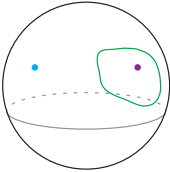
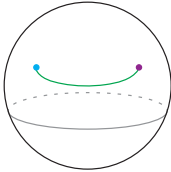


Let's see what happens

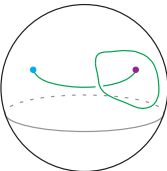


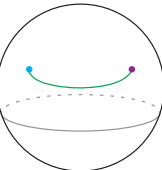
Let's see what happens

$$[A(-A^2 - A^{-2}) + A^{-1}]$$


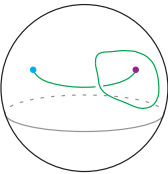
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

 \times


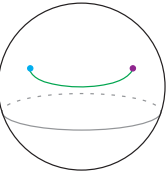
Let's see what happens

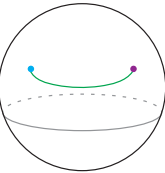
$$[A(-A^2 - A^{-2}) + A^{-1}]$$


$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


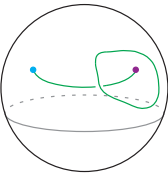
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$$[A(-A^2 - A^{-2}) + A^{-1}]$$


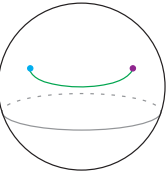
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


$$= (-A^4 - A^2)$$


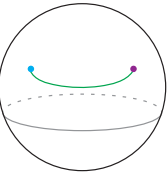
Let's see what happens

$$[A(-A^2 - A^{-2}) + A^{-1}]$$


A sphere with a green curve connecting a blue dot on the left to a purple dot on the right. The curve starts with a small loop on the right side.

$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$


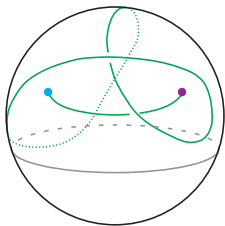
A sphere with a green curve connecting a blue dot on the left to a purple dot on the right. The curve is a simple arc.

$$= (-A^4 - A^2)$$


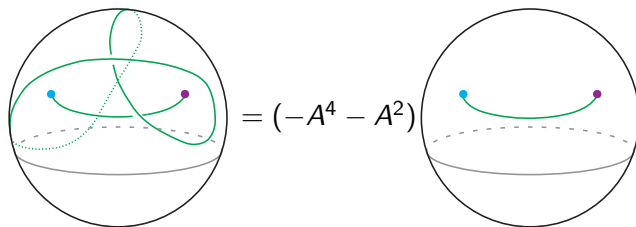
A sphere with a green curve connecting a blue dot on the left to a purple dot on the right. The curve is a simple arc.

There is nothing left to do.

Reviewing Calculations

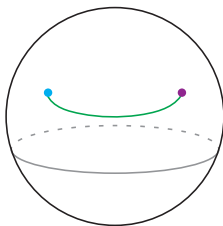


Reviewing Calculations



Generating Arc Algebras

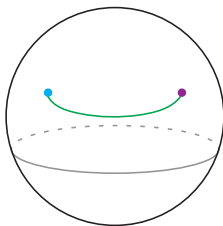
This arc:



is special.

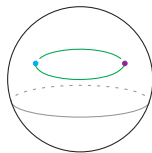
Generating Arc Algebras

This arc:

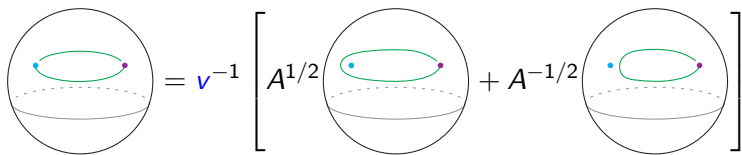


is special.
We'll come back to that...

Another Example



Another Example


$$\text{Sphere with ellipse and points} = v^{-1} \left[A^{1/2} \text{Sphere with ellipse and points} + A^{-1/2} \text{Sphere with ellipse and points} \right]$$

Another Example

$$\begin{aligned}
 & \left(\text{Diagram 1} \right) = v^{-1} \left[A^{1/2} \left(\text{Diagram 2} \right) + A^{-1/2} \left(\text{Diagram 3} \right) \right] = \\
 & (vv)^{-1} \left[A \left(\text{Diagram 4} \right) + \left(\text{Diagram 5} \right) + \left(\text{Diagram 6} \right) + A^{-1} \left(\text{Diagram 7} \right) \right]
 \end{aligned}$$

The diagrams are spheres with a horizontal equator (solid line in front, dashed in back). Each sphere contains a blue dot on the left and a purple dot on the right.

- Diagram 1: A large green ellipse encircling the equator, with an arrow pointing from the purple dot to the blue dot.
- Diagram 2: A large green ellipse encircling the equator, with an arrow pointing from the purple dot to the blue dot.
- Diagram 3: A large green ellipse encircling the equator, with an arrow pointing from the purple dot to the blue dot.
- Diagram 4: A small green circle encircling the blue dot.
- Diagram 5: A large green ellipse encircling the equator, with an arrow pointing from the purple dot to the blue dot.
- Diagram 6: A small green circle encircling the blue dot.
- Diagram 7: A small green circle encircling the purple dot.

Another Example

$$\begin{aligned}
 & \text{Diagram 1} = v^{-1} \left[A^{1/2} \text{Diagram 2} + A^{-1/2} \text{Diagram 3} \right] = \\
 & (vv)^{-1} \left[A \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + A^{-1} \text{Diagram 7} \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})]
 \end{aligned}$$

The diagrams are spheres with a horizontal equator (solid line in front, dashed in back). Each contains a blue dot on the left and a purple dot on the right. The diagrams are:

- Diagram 1:** A large green ellipse encircling the equator.
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- Diagram 4:** A small green circle encircling the blue dot.
- Diagram 5:** A large green ellipse encircling the equator.
- Diagram 6:** A small green circle encircling the purple dot.
- Diagram 7:** A small green circle encircling the purple dot.

Another Example

$$\begin{aligned}
 & \text{Diagram} = v^{-1} \left[A^{1/2} \text{Diagram} + A^{-1/2} \text{Diagram} \right] = \\
 & (vv)^{-1} \left[A \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + A^{-1} \text{Diagram}_4 \right] \\
 & = (vv)^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \\
 & = (vv)^{-1} (-A^2 + 2 - A^{-2})
 \end{aligned}$$

The diagrams are spheres with a horizontal line and a dashed back line. Each contains a blue dot and a purple dot. The green loops are as follows:

- Top row, right: A large horizontal ellipse connecting the blue and purple dots.
- Second row, left: A small vertical ellipse around the blue dot.
- Second row, second: A large horizontal ellipse connecting the blue and purple dots.
- Second row, third: A small vertical ellipse around the purple dot.
- Second row, right: A small vertical ellipse around the purple dot.

Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.

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4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.

Algebra of the Twice-Punctured Sphere

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3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
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5. What is left?

Algebra of the Twice-Punctured Sphere

1. There's no crossings (we removed them).

Algebra of the Twice-Punctured Sphere

1. There's no crossings (we removed them).
2. There's no unknots (we removed them).

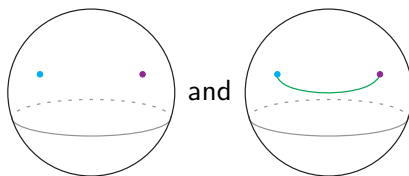
Algebra of the Twice-Punctured Sphere

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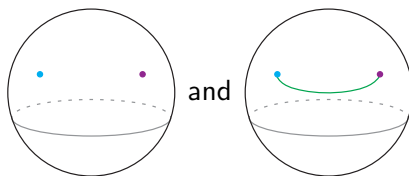
So we are left with:



Algebra of the Twice-Punctured Sphere

1. There's no crossings (we removed them).
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So we are left with:



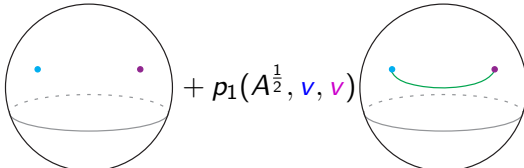
With polynomial coefficients in $A^{\frac{1}{2}}, v, v$.

Generators of the Arc Algebra

Every element k of the arc algebra for the twice-punctured sphere can be written:

Generators of the Arc Algebra

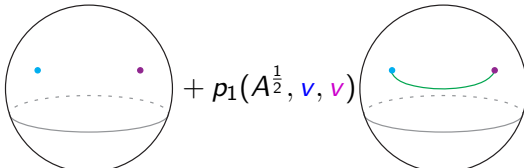
Every element k of the arc algebra for the twice-punctured sphere can be written:

$$k = p_0(A^{\frac{1}{2}}, v, v) \left(\text{Diagram 1} \right) + p_1(A^{\frac{1}{2}}, v, v) \left(\text{Diagram 2} \right)$$


The equation shows that any element k in the arc algebra for a twice-punctured sphere can be expressed as a linear combination of two specific arcs. The first term, $p_0(A^{\frac{1}{2}}, v, v)$, is multiplied by a diagram of a sphere with two punctures (blue and purple dots) on the upper hemisphere. The second term, $p_1(A^{\frac{1}{2}}, v, v)$, is multiplied by a diagram of a sphere with the same two punctures, but with an additional green arc connecting them on the upper hemisphere.

Generators of the Arc Algebra

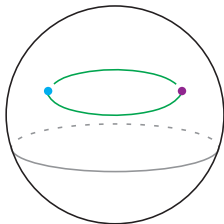
Every element k of the arc algebra for the twice-punctured sphere can be written:

$$k = p_0(A^{\frac{1}{2}}, v, v) \left(\text{Diagram 1} \right) + p_1(A^{\frac{1}{2}}, v, v) \left(\text{Diagram 2} \right)$$


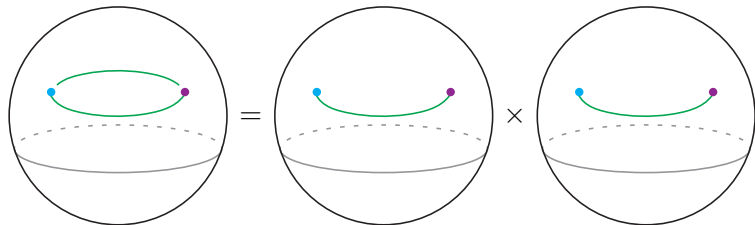
The equation shows the decomposition of an element k in the arc algebra. The first term is $p_0(A^{\frac{1}{2}}, v, v)$ multiplied by a diagram of a sphere with two punctures (blue and purple dots) and no arc. The second term is $p_1(A^{\frac{1}{2}}, v, v)$ multiplied by a diagram of a sphere with two punctures (blue and purple dots) and a green arc connecting them.

We say that the two diagrams in this sum **generate** this arc algebra.

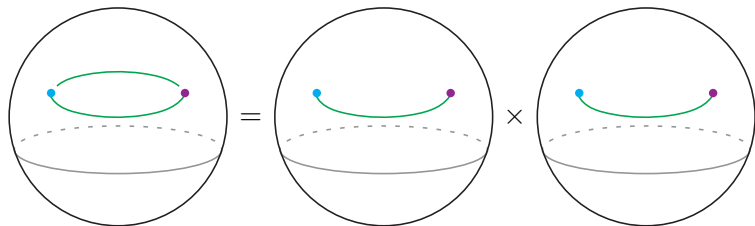
Relation on the Generator



Relation on the Generator

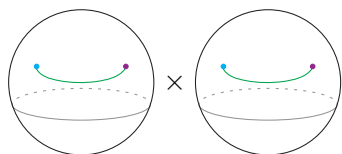


Relation on the Generator



$$= (vv)^{-1}(-A^2 + 2 - A^{-2})$$

Relation on the Generator



The diagram shows two identical spheres, each represented by a circle with a horizontal line across its middle. The back half of the sphere is indicated by a dashed line. On the upper hemisphere of each sphere, there is a green arc connecting a blue dot on the left to a purple dot on the right. The two spheres are separated by a multiplication symbol (\times).

$$\times = (vv)^{-1}(-A^2 + 2 - A^{-2})$$

Our Project

Our project was to find generators (and relations) for the arc algebras of various surfaces.

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and we hope to achieve a general description.

References

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Thanks

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Thanks to you, as well, for coming to our talk.

Questions

We are prepared to answer any and all of your questions to the best of our ability.