Difference Set Transfers

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Difference Sets

Definition

A (v, k, λ)-*difference set* is a subset D of a group G such that

- $|G| = v$
- $|D| = k$
- each nonidentity element $g \in G$, can be represented as a "difference" $g = d_1 d_2^{-1}$ for exactly λ pairs $(d_1, d_2) \in D^2$.

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Small Example: $G = C_7 = \langle x \mid x^7 = 1 \rangle$, $D = \{x, x^2, x^4\}$.
Difference Table:

$d_1 \backslash d_2$	x	x^2	x^4
x	1	x^6	x^4
x^2	x	1	x^5
x^4	x^3	x^2	1

D is a (7,3,1)-difference set

Hadamard Difference Sets

Definition

A *Hadamard Difference Set* (HDS) is a (v, k, λ)-difference set such that $v = 4(k - \lambda)$.

Theorem

For any (v, k, λ)-HDS, $(v, k, \lambda) = (4m^2, 2m^2 \pm m, m^2 \pm m)$ for some $m \in \mathbb{Z}_{>0}$.

Because $(4m^2, 2m^2 - m, m^2 - m)$-difference sets and $(4m^2, 2m^2 + m, m^2 + m)$-difference sets are complementary, we will assume throughout that

$$(v, k, \lambda) = (4m^2, 2m^2 - m, m^2 - m).$$

Hadamard Difference Sets

Why study Hadamard Difference Sets?

- “The Hadamard parameters provide the richest source of known examples of difference sets."
- HDS’s are deeply connected to design theory, coding theory, and digital communications.

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Why the name “Hadamard"?

- The $(+1, -1)$ incidence matrix of the block design corresponding to an HDS is a regular Hadamard matrix.

The Group Ring

Let $G = \{g_1, ..., g_n\}$ be a group, and $D \subset G$ a difference set.

- We often work with the group ring $\mathbb{Z}[G]$ of formal sums

$$\sum_{i=1}^{n} c_i \cdot g_i \qquad\qquad c_1, ..., c_n \in \mathbb{Z},$$

  with addition and multiplication defined naturally.

- We often abuse notation to define elements $D, G \in \mathbb{Z}[G]$ by

$$G := \sum_{g \in G} g \qquad\qquad D := \sum_{d \in D} d.$$

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$$G := \sum_{g \in G} g \qquad D := \sum_{d \in D} d.$$

- Define

$$D^{(-1)} := \sum_{d \in D} d^{-1}.$$

- Under this notation, the condition for D to be a difference set is equivalent to D satisfying the equation

$$DD^{(-1)} = (k - \lambda) \cdot 1_G + \lambda \cdot G.$$

Example

$G = C_7 = \langle x \mid x^7 = 1 \rangle$, $D = \{x, x^2, x^4\}$ is a (7,3,1)-difference set

- In the group ring $\mathbb{Z}[G]$, we have $D = x + x^2 + x^4$, and $D^{(-1)} = x^6 + x^5 + x^3$. Thus

$$
\begin{aligned}
DD^{(-1)} &= (x + x^2 + x^4)(x^6 + x^5 + x^3) \\
&= (1 + x^6 + x^4) + (x + 1 + x^5) + (x^3 + x^2 + 1) \\
&= 3 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\
&= 2 + G.
\end{aligned}
$$

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$$\begin{aligned} DD^{(-1)} &= (x + x^2 + x^4)(x^6 + x^5 + x^3) \\ &= (1 + x^6 + x^4) + (x + 1 + x^5) + (x^3 + x^2 + 1) \\ &= 3 + x + x^2 + x^3 + x^4 + x^5 + x^6 \\ &= 2 + G. \end{aligned}$$

- Notice the relationship to the difference table from earlier:

d_1 \ d_2	x	x^2	x^4
x	1	x^6	x^4
x^2	x	1	x^5
x^4	x^3	x^2	1

Equivalence

Theorem

Let $D \in \mathbb{Z}[G]$ be a (v, k, λ)-difference set. If $g_0 \in G$ and $\phi \in \mathrm{Aut}(G)$, then $g_0\phi(D)$ is also a (v, k, λ)-difference set.

Equivalence

Theorem

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Definition

Two (v, k, λ)-difference sets $D_1, D_2 \in \mathbb{Z}[G]$ are equivalent if there exists $g_0 \in G$ and $\phi \in \mathrm{Aut}(G)$ such that $D_1 = g_0\phi(D_2)$.

Central Question

- Existence: Which groups contain a difference set and which do not.
- Enumeration: Find all difference sets up to equivalence in a group or set of groups.

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known results for Hadamard difference sets:

- $m = 1 \implies$ 2 out of 2 groups of size 4 have HDS, 2 total difference sets (trivial)
- $m = 2 \implies$ 12 out of 14 groups of size 16 have HDS, 27 total difference sets (Kibler)
- $m = 3 \implies$ 9 out of 14 groups of size 36 have HDS, 35 total difference sets (Kibler/Smith)
- $m = 4 \implies$ 259 out of 267 groups of size 64 have HDS (unpublished Davis/Smith)

GAP

In our research we made extensive use of the computer algebra system GAP. In GAP you can

- enumerate over all groups of a given size
- program algorithms and techniques
- construct cyclic groups, dihedral groups, elementary abelian groups, direct products, semidirect products, ...
- list all elements, subgroups, normal subgroups, automorphisms, quotient groups, irreducible representations, etc. of a given group
- construct homomorphisms, isomorphisms, ...
- unambiguously refer to certain groups and group elements
- much much much more

Our Results

During our 8 weeks in San Diego we

1. Programmed the construction techniques and special cases involved in determining the existence of difference sets in groups of order 64.
2. Developed an algorithm to exhaustively search groups of order 64 and enumerate all difference sets up to equivalence.
3. Discovered and explored the use of "difference set transfers" for groups of order 16, 36, 64, and 144.

Existence of Difference Sets in Groups of Order 64

- GAP has a catalog containing every group of order 64. Each group has a distinct entry, which allows us to unambiguously refer to any group and its elements.

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- GAP has a catalog containing every group of order 64. Each group has a distinct entry, which allows us to unambiguously refer to any group and its elements.
- In our first project, we determined that groups had a difference set by finding a construction that would produce a difference set.
- It is convenient to store these found difference sets as lists of GAP indices. For example, SmallGroup(64, 12) has the difference set [1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 14, 17, 18, 20, 26, 27, 31, 32, 33, 34, 35, 38, 39, 44, 50, 56, 60, 63] .

Finding Voodoo in Groups of Size 64

- After applying several different constructions and techniques to groups of order 64, we were left with four groups in which we could not produce a difference set.

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- It is very surprising that this works, and it doesn't only work occasionally. Many groups of order 64 “share" difference sets, and the same behavior can be found in some other orders.
- NEW PROJECT: Explain this voodoo!

Voodoo in Groups of Order 16

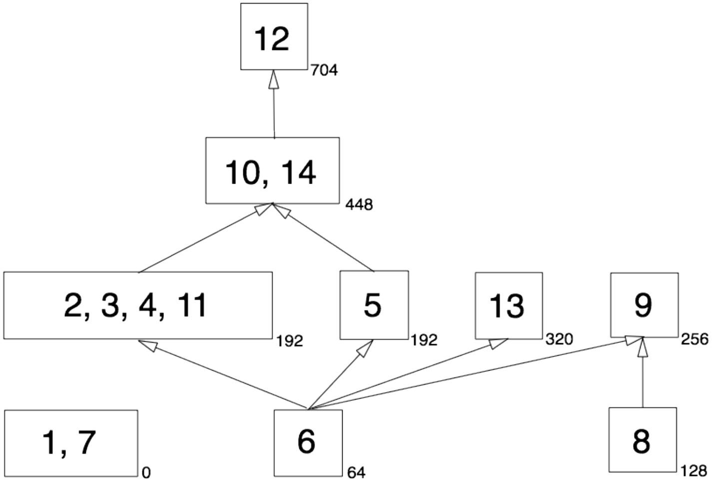
To approach the voodoo we moved back to groups of order 16, as in order 16

- we are still dealing with 2-groups,
- there are fewer total groups (14 compared to 267),
- similar voodoo behavior can be found (order 36 has some voodoo, but not as much),
- and we can easily find all difference sets and perform computations.

Chart - Voodoo in Groups of Order 16

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	192	192	192	64	64	0	64	128	192	192	192	128	192
3	0	192	192	192	64	64	0	64	128	192	192	192	128	192
4	0	192	192	192	64	64	0	64	128	192	192	192	128	192
5	0	64	64	64	192	64	0	0	64	192	64	192	64	192
6	0	64	64	64	64	64	0	0	64	64	64	64	64	64
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	64	64	64	0	0	0	128	128	64	64	64	64	64
9	0	128	128	128	64	64	0	128	256	128	128	128	128	128
10	0	192	192	192	192	64	0	64	128	448	192	448	192	448
11	0	192	192	192	64	64	0	64	128	192	192	192	128	192
12	0	192	192	192	192	64	0	64	128	448	192	704	256	448
13	0	128	128	128	64	64	0	64	128	192	128	256	320	192
14	0	192	192	192	192	64	0	64	128	448	192	448	192	448

Chart - Voodoo in Groups of Order 16



Power Commutator Presentations - The Voodoo Source

GAP stores p-groups using power-commutator presentations.

Definition

Given a group G of order p^n, a *power-commutator presentation* of G consists of a set of generators $\{f_1, f_2, \ldots, f_n\}$ with defining relations $f_i^p = \prod_{k=i+1}^{n} f_k^{\beta(i,k)}$ and $[f_j, f_i] = \prod_{k=j+1}^{n} f_k^{\beta(i,j,k)}$, where the value of β is in $\{1, \ldots, p-1\}$ and $1 \leq i < j \leq n$.

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The following theorem is due to Sylow.

Theorem

Every group of order p^n has a power-commutator presentation on n generators.

Power Commutator Presentations - The Voodoo Source

Example

We will look at a power-commutator presentation for D_8, the dihedral group of order 8. $D_8 = \langle f_1, f_2, f_3 | \; f_1^2 = f_2^2 = f_3^2 = 1,$ $[f_2, f_1] = f_3, [f_3, f_1] = [f_3, f_2] = 1\rangle$ Note that D_8 is generated by just f_1 and f_2, but these are not the traditional two generators of D_8. If we present $D_8 = \langle r, s | r^4 = s^2 = 1, [r, s] = r^2\rangle$, then $r = f_1 f_2$ and $s = f_2$.

Power Commutator Presentations - The Voodoo Source

Observation

If a group G of order 2^n has a power-commutator presentation over the set $\{f_1, \ldots, f_n\}$, then every element of G can be written as a product of f_is in increasing order, e.g. $f_1 f_3 f_4$.

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`gap:>Elements(G);` returns a list containing the elements of a group G ordered lexicographically as words on the basis of the power-commutator presentation. So, our equivalent lists of numbers are nothing more than equivalent words over power-commutator presentations.

Difference Set Transfers

Definition

Suppose we have two groups G and H with $|G| = |H| = p^n$ and their power commutator-presentations on generators $g_1, g_2, \ldots g_n$ and $h_1, h_2, \ldots h_n$. A *difference set transfer* occurs when a difference set in G can be converted to a difference set in H by writing the terms expressed on generators g_i as equivalently formed words on generators h_i.

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We want to find conditions on groups and their presentations so that difference set transfers occur. The hope is that studying transfers will lead to new construction techniques and existence results.

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First we would like to catalogue and prove difference set transfers in order 16.

The Spread Construction

Definition

Let G be a group of order 2^{2s+2} containing a normal elementary abelian subgroup $E = C_2^{s+1}$. E has $2^{s+1} - 1$ subgroups H_i of order 2^s (these subgroups are hyperplanes when viewing E as a vector space over $\mathbb{F}_2$) and E partitions G into 2^{s+1} cosets with coset representatives g_i. The set $D = g_1 H_1 + \cdots + g_{2^{s+1}-1} H_{2^{s+1}-1}$ is a difference set iff $g_i H_i g_i^{-1}$ is a permutation of the H_i.

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Key points:

- We need an elementary abelian subgroup of the appropriate size normal in G.
- The construction does not work for every set of coset representatives.
- However, if E is in the center of G then every set of coset representatives does work.

The Spread Construction - Example

$$G = C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = [x, y] = 1 \rangle$$

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x^2	x^2y	x^2y^2	x^2y^3
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$$E = C_2 \times C_2 = \langle x^2, y^2 \rangle = \{1, y^2, x^2, x^2y^2\}$$

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$$G = {\color{red}1E} + {\color{cyan}xE} + {\color{orange}yE} + {\color{green}xyE}$$

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The Spread Construction - Example

$$G = C_4 \times C_4 = \langle x, y \mid x^4 = y^4 = [x, y] = 1 \rangle$$

$\boxed{1}$	$\boxed{y}$	$\boxed{y^2}$	y^3
$\boxed{x}$	xy	xy^2	xy^3
x^2	x^2y	x^2y^2	$\boxed{x^2y^3}$
$\boxed{x^3}$	x^3y	x^3y^2	x^3y^3

$$E = C_2 \times C_2 = \langle x^2, y^2 \rangle = \{1, y^2, x^2, x^2y^2\}$$

$$G = 1E + xE + yE + xyE$$

$$H_1 = \{1, y^2\},\ H_2 = \{1, x^2\},\ H_3 = \{1, x^2y^2\}$$

$$D = 1H_1 + xH_2 + yH_3 = \{1, y^2, x, x^3, y, x^2y^3\}$$

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$$D = \{1, y^2, x, x^3, y, x^2y^3\}$$

	1	y^2	x	x^3	y	x^2y^3
1	1	y^2	x^3	x	y^3	x^2y
y^2	y^2	1	x^3y^2	xy^2	y	x^2y^3
x	x	xy^2	1	x^2	xy^3	x^3y
x^3	x^3	x^3y^2	x^2	1	x^3y^3	xy
y	y	y^3	x^3y	xy	1	x^2y^2
x^2y^3	x^2y^3	x^2y	xy^3	x^3y^3	x^2y^2	1

Counting Spread Constructions

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For a group G of order 2^{2s+2}, the spread construction generates $2^{s+1}!(2^{2^{s+1}-1})$ sets over any subgroup E isomorphic to C_2^{s+1}.

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Theorem

Given $|G| = 16$, if for $E \triangleleft G$, $E \cong C_2 \times C_2$, but $E \not\subset Z(G)$ then a spread construction over E generates at least 64 difference sets.

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Theorem

Given $|G| = 16$, if for $E \lhd G$, $E \cong C_2 \times C_2$, but $E \not\subset Z(G)$ then a spread construction over E generates at least 64 difference sets.

If $|G| = 16$, we can build $2^2!(2^3) = 192$ sets over a normal $C_2 \times C_2$. If our $C_2 \times C_2$ is in $Z(G)$, these are all difference sets. If our $C_2 \times C_2$ is not in $Z(G)$, 64 of these will be difference sets.

Theorem

Let G be a group of order 16 that does not contain a subgroup isomorphic to the quaternion group. If the socle of G has order 4, then every difference set in G can be generated via a spread construction over $soc(G)$.

Theorem

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A generalized version of this theorem would be very powerful, as it would reduce the search for all difference sets in a group to a search through all spread constructions. However, our proof relies to heavily on properties of difference sets in order 16 groups, and we have not been able to generalize it.

The Spread Construction - Explaining Transfers

- Let G_i denote the group with GAP ID $[16, i]$
- G_2, G_3, G_4, and G_{11} all have a socle of order 4, and do not contain a subgroup isomorphic to the quaternion group.
- In all of these groups $soc(G) = \langle f_3, f_4 \rangle$.
- Therefore, every single difference set in each of these groups is one of the 192 spread constructions over $\langle f_3, f_4 \rangle$, and we then see why all of them have the same difference sets.
- Furthermore, in G_{10}, G_{12}, and G_{14}, $\langle f_3, f_4 \rangle \subset Z(G)$, so every spread over $\langle f_3, f_4 \rangle$ is a difference set in all of these groups. Thus, we see why all of the difference sets in G_2, G_3, G_4, and G_{11} are difference sets in G_{10}, G_{12}, and G_{14}.

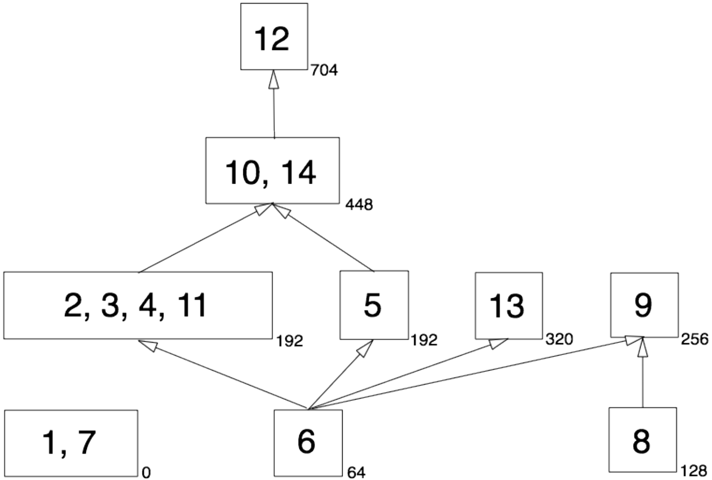
The Spread Construction - Explaining Transfers

- G_5 also has a socle of order 4 and no subgroups isomorphic to the quaternion group.
- However, $soc(G_5) = \langle f_2, f_4 \rangle$.
- So, the difference sets in G_5 are exactly the spread constructions over $\langle f_2, f_4 \rangle$.
- Because G_{10} and G_{14} all also have $\langle f_2, f_4 \rangle$ in their center, every difference set in G_5 is also a difference set in these two groups.
- Also, we can show using other methods that every difference set in G_6 is a spread over $\langle f_2, f_4 \rangle$, so every difference set in G_6 is a difference set over G_5.

Other Results

- We used basic algebra to explain why G_{10} and G_{14} share all of their difference sets. This proof can then be generalized to show that all difference sets in C_2^{2s+2} are difference sets in $C_4 \times C_2^{2s}$, and vice versa.
- As alluded to above, we used basic group theory to show that all difference sets in G_6 can be built with a spread construction over the unique normal $C_2 \times C_2$.
- We have found a way to classify every difference set in G_8 using subgroups isomorphic to the quaternions, and show that all of these difference sets are also difference sets in G_9.

Chart



Final Remarks

- There are still some unexplained transfers in groups of order 16.
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- More data from order 64 would be useful for forming conjectures about transfers.
- Some of our results on transfers were used to answer existence questions in order 64 and order 144 groups.

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