Perfectoid rings, almost mathematics, and the cotangent complex

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Scholze introduced perfectoid fields (and more generally, perfectoid spaces) in his paper [Sch12], in which he proved a wide range of special cases of Deligne's Weight Mondromy Conjecture for p-adic fields.

1 Perfectoid fields

Recall that a *valued field* is a field k together with a homomorphism $|\cdot| : k^\times \to \Gamma$ for some totally ordered abelian group Γ (whose operation we will write multiplicatively). One requires $|\cdot|$ to satisfy the *ultrametric inequality*:

$$|x+y| \leqslant \max\{|x|, |y|\}.$$

As is standard, write 1 for the unit in Γ, and introduce an extra element 0, stipulating that $0 < \gamma$ for all $\gamma \in \Gamma$. We define $|0| = 0$, thus extending $|\cdot|$ to a function $k \to \Gamma \cup \{0\}$. One calls the *rank* of $|\cdot|$ the dimension $\dim_{\mathbf{Q}}(|k^\times| \otimes \mathbf{Q})$. We will say that the valuation $|\cdot|$ is *non-discrete* if $|k^\times| \not\simeq \mathbf{Z}$.

Put

$$k^\circ = \{x \in k : |x| \leqslant 1\}.$$

This is called the *ring of integers* of k. It is a valuation ring with (unique) maximal ideal

$$k^+ = \{x \in k : |x| < 1\}.$$

We call k° / k^+ the *residue field* of k.

Definition 1.1. *A* perfectoid field *is a complete valued field k with respect to a non-discrete rank-one valuation, with residue characteristic $p > 0$, such that the Frobenius* $\mathrm{Fr} : k^\circ / p \to k^\circ / p$ *is surjective.*

A typical example of a perfectoid field is $\mathbf{Q}_p(\zeta_{p^\infty})^\wedge$, the completion of $\mathbf{Q}_p(\zeta_{p^n} : n \geqslant 1)$ with respect to the p-adic topology. Similarly, $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ and $\mathbf{C}_p = \left(\overline{\mathbf{Q}_p}\right)^\wedge$ are perfectoid. An example in characteristic p is the t-adic completion of $\mathbf{F}_p(t^{1/p^\infty})$. For a perfectoid field k of residue characteristic p, choose $\pi \in k^\times$ with $|p| \leqslant |\pi| < 1$. Note that the Frobenius map $a \mapsto a^p$ is defined on A/π for any k°-algebra A.

Definition 1.2. *Let k be a perfectoid field. A* perfectoid k-algebra *is a Banach k-algebra A such that $A^\circ = \{x \in A : |x| \leqslant 1\}$ is open and for which* $\mathrm{Fr} : A^\circ / \pi \to A^\circ / \pi$ *is surjective.*

If k is a perfectoid field, let $\mathsf{Perf}(k)$ denote the category of perfectoid k-algebras, with continous k-maps as morphisms. We will construct, for any perfectoid field k with residue characteristic p, a perfectoid field $k^\flat$ of characteristic p. Start by defining

$$k^{\flat\circ} = \varprojlim_{\mathrm{Fr}} (k^\circ/\pi) = \left\{ (x_i) \in \prod_{i \geqslant 0} k^\circ/\pi : x_{i+1}^p = x_i \right\}.$$

It is not too difficult to check directly that $k^{\flat\circ}$ is a valuation ring, and we put $k^\flat = \mathrm{Frac}(k^{\flat\circ})$. There is a canonical map $(-)^\sharp : k^{\flat\circ} \to k^\circ$ defined by

$$(x_0, x_1, \dots)^\sharp = \lim_{n\to\infty} \widetilde{x_n}^{p^n},$$

where $\widetilde{x_n}$ is an arbitrary lift of $x_n \in k^\circ/\pi$ to k°. The map $(-)^\sharp$ is *not* additive unless k already has characteristic p, in which case $k = k^\flat$. In general, $(-)^\sharp$ extends to an isomorphism of multiplicative groups $k^{\flat\times} \to k^\times$, and we can use this to define a valuation on $k^\flat$ by $|x|_{k^\flat} = |x^\sharp|_k$. See Lemma 3.4 of [Sch12] for a proof that $(-)^\sharp$ has the claimed properties, and that $k^\flat$ is a perfectoid field with the same value group as k.

Example 1.3. Let N be a lattice (i.e. a finite free $\mathbf{Z}$-module) and let $\sigma \subset N_{\mathbf{R}}$ be a strongly convex polyhedral cone. Let $\sigma^\wedge \subset N_{\mathbf{R}}^\vee$ be its dual. If we put $M = N^\vee$, then the spectra of algebras of the form $k[\sigma] = k[\sigma^\vee \cap M]$ form affine charts for toric varieties over k. There is a "perfectoid version" of this. Write $k\langle\sigma\rangle = k\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$ for the ring

$$\left(k^\circ[\sigma^\vee \cap M \otimes \mathbf{Z}[\frac{1}{p}]]^\wedge\right) \otimes k.$$

Then $k\langle\sigma\rangle$ is a perfectoid algebra over k. (*Note*: $k\langle\sigma\rangle$ is *not*, in this context, a non-commutative polynomial algebra over k.

If A is a perfectoid k-algebra, define $A^\flat$ in much the same way, via

$$A^\flat = \left(\varprojlim_{\mathrm{Fr}} A^\circ/\pi\right) \otimes_{k^{\flat\circ}} k^\flat.$$

It turns out that if A is perfectoid, then $A^\flat$ is also perfectoid, and we have the following deep theorem:

Theorem 1.4 (Scholze)**.** *The functor $(-)^\flat : \mathsf{Perf}(k) \to \mathsf{Perf}(k^\flat)$ is an equivalence of categories.*

In fact, much more can be shown, e.g. $(-)^\flat$ induces equivalences of categories between $\mathsf{FEt}(A)$ and $\mathsf{FEt}(A^\flat)$ for all A, where $\mathsf{FEt}(A)$ is the category of finite étale algebras over A (it turns out that such algebras are perfectoid).

Example 1.5. If $A = k\langle\sigma\rangle = k\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$ as in Example 1.3, then $A^\flat = k^\flat\langle\sigma\rangle = k^\flat\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$.

Theorem 1.4 is proved without introducing much heavy machinery in Section 3.6 of [KL]. The basic idea is that the map $(-)^\sharp : k^{\flat\circ} \to k^\circ$ induces a ring homomorphism $\theta : W(k^{\flat\circ}) \to k^\circ$, where $W(-)$ is the ring of p-typical Witt vectors. The inverse to the functor $(-)^\flat$ is $A^\sharp = W(A^\circ) \otimes_{W(k^{\flat\circ})} k$. Scholze's proof is more conceptual, and passes through a diagram

$$\begin{tikzcd}
\mathsf{Perf}(k) \arrow[r, "\sim"] \arrow[d, "(-)^\flat"] & \mathsf{Perf}(k^{\circ a}) \arrow[r, "\sim"] & \mathsf{Perf}(k^{\circ a}/\pi) \arrow[d, equal] \\
\mathsf{Perf}(k^\flat) \arrow[r, "\sim"] & \mathsf{Perf}(k^{\flat\circ a}) \arrow[r, "\sim"] & \mathsf{Perf}(k^{\flat\circ a}/\pi^\flat)
\end{tikzcd}$$

in which most of the categories have yet to be defined. The superscript $(-)^{\mathrm{a}}$ should be read "almost," following the "almost mathematics" initially created by Faltings, and developed systematically in [GR03].

2 Almost mathematics

Almost mathematics was first introduced by Faltings in [Fal88], where he proved a deep conjecture of Fontaine on the étale cohomology of varieties over p-adic fields. We follow the treatment in Section 2.2 of [GR03].

Let V be a valuation ring with maximal ideal $\mathfrak{m}$. Throughout this section, we assume that *the value group of V is non-discrete*. This implies $\mathfrak{m}^2 = \mathfrak{m}$. In fact, much of the theory works for any unital ring V with idempotent two-sided ideal $\mathfrak{m}$, but we have no need to work at this level of generality. The reader should keep in mind the example $V = k^\circ$ for a perfectoid field k.

Let $\mathsf{Mod}(V)$ be the category of all V-modules, and let $\mathsf{Ann}(\mathfrak{m})$ be the full subcategory consisting of those modules killed by $\mathfrak{m}$.

Lemma 2.1. $\mathsf{Ann}(\mathfrak{m})$ *is a Serre subcategory of* $\mathsf{Mod}(V)$.

Proof. We need to show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of V-modules for which M' and M'' are killed by $\mathfrak{m}$, then M is also killed by $\mathfrak{m}$. We trivially have $\mathfrak{m}^2 M = 0$, but $\mathfrak{m}^2 = \mathfrak{m}$, whence the result. □

Definition 2.2. *The category of* V^{a}-modules *is the Serre quotient* $\mathsf{Mod}(V^{\mathrm{a}}) = \mathsf{Mod}(V) / \mathsf{Ann}(\mathfrak{m})$.

Write $(-)^{\mathrm{a}} : \mathsf{Mod}(V) \to \mathsf{Mod}(V^{\mathrm{a}})$ for the quotient functor. Even though "V^{a}" does not exist as a ring we will write $\hom_{V^{\mathrm{a}}}(-,-)$ for hom-sets in $\mathsf{Mod}(V^{\mathrm{a}})$. While in general, hom-sets in quotient categories can be difficult to describe, the category $\mathsf{Mod}(V^{\mathrm{a}})$ is relatively easy to understand via the following theorem.

Theorem 2.3. *There is a natural isomorphism* $\hom_{V^{\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}}) = \hom_V(\mathfrak{m} \otimes M, N)$.

It follows that $\mathsf{Mod}(V^{\mathrm{a}})$ is naturally a $\mathsf{Mod}(V)$-enriched category. We define two functors $(-)_!, (-)_* : \mathsf{Mod}(V^{\mathrm{a}}) \to \mathsf{Mod}(V)$:

$$
\begin{aligned}
M_* &= \hom_{V^{\mathrm{a}}}(V^{\mathrm{a}}, M) \\
M_! &= \mathfrak{m} \otimes M_*.
\end{aligned}
$$

Theorem 2.4. *The triple* $((-)_!, (-)^{\mathrm{a}}, (-)_*)$ *is adjoint. Moreover, these adjunctions induce natural isomorphisms* $(M_*)^{\mathrm{a}} = M = (M_!)^{\mathrm{a}}$.

This is suggestive of the situation in which $j : U \hookrightarrow X$ is an open embedding of topological spaces. The restriction functor $j^* : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ fits into an adjoint triple $(j_!, j^*, j_*)$ in which $j^* j_* = 1 = j^* j_!$. So we should think of $\mathsf{Mod}(V^{\mathrm{a}})$ as the category of quasi-coherent sheaves on some subscheme of $\operatorname{Spec} V$, even though there is no such subscheme. In fact, since $\mathsf{Mod}(V^{\mathrm{a}})$ does not contain enough projectives, it should be thought of as some kind of "non-affine" object.

The category $\mathsf{Mod}(V^{\mathrm{a}})$ inherits the structure of a tensor category from $\mathsf{Mod}(V)$. In fact, we have internal tensor and hom defined by

$$
\begin{aligned}
M^{\mathrm{a}} \otimes N^{\mathrm{a}} &= (M \otimes N)^{\mathrm{a}} \\
\hom^{\mathrm{a}}(M^{\mathrm{a}}, N^{\mathrm{a}}) &= \hom(M, N)^{\mathrm{a}}
\end{aligned}
$$

There is a tensor-hom adjunction $\hom(L \otimes N, M) = \hom(L, \hom^{\mathrm{a}}(M, N))$. This allows us to speak of algebra objects in $\mathsf{Mod}(V^{\mathrm{a}})$ as commutative unital monoid objects in the tensor category $(\mathsf{Mod}(V^{\mathrm{a}}), \otimes)$. We call such objects V^{a}*-algebras*. It is easy to check that they are all of the form A^{a}, for A some V-algebra.

If A is a V^{a}-algebra, we can form the category $\mathsf{Mod}(A)$ of A-modules in the obvious way. This is also an abelian tensor category, so it makes sense to speak of "flat objects" in the usual way, i.e. an A-module M is *flat* if the functor $M \otimes_A -$ is exact.

Definition 2.5. *Let k be a perfectoid field. A* perfectoid $k^{\circ\mathrm{a}}$-algebra *is a flat, π-adically complete $k^{\circ\mathrm{a}}$-algebra A for which* $\mathrm{Fr} : A/\pi^{1/p} \to A/\pi$ *is an isomorphism.*

Note that even though $\pi^{1/p}$ may not exist as an actual element of k°, the ideal $(\pi^{1/p})$ is well-defined, so it makes sense to write $M/\pi^{1/p}$ if M is any k°- (or $k^{\circ a}$)-module. Write $\mathsf{Perf}(k^{\circ a})$ for the category of perfectoid $k^{\circ a}$-algebras.

Theorem 2.6. *The functor $A \mapsto A^{\circ a}$ induces an equivalence of categories $\mathsf{Perf}(k) \xrightarrow{\sim} \mathsf{Perf}(k^{\circ a})$.*

Idea of proof. See the first part of Section 5 in [Sch12]. Besides some technicalities, one has the existence of an inverse functor $A \mapsto A_* \otimes_{k^\circ} k$. ☐

3 The cotangent complex

We have seen in the last section that localization induces an equivalence between the category of perfectoid k-algebras and the category of perfectoid $k^{\circ a}$-algebras. Since $k^\times$ and $k^{\flat\times}$ are canonically isomorphic via $(-)^\sharp$, also write π for the element of $k^\flat$ corresponding to $\pi \in k$. It is easy to check that $k^\circ/\pi = k^{\flat\circ}/\pi$. It easily follows that $\mathsf{Mod}(k^{\circ a}/\pi) = \mathsf{Mod}(k^{\flat\circ a}/\pi)$. The idea of this section is to pass from $\mathsf{Perf}(k^{\circ a})$ to a suitable category of "perfectoid $k^{\circ a}/\pi$-algebras."

Definition 3.1. *Let k be a perfectoid field. A $k^{\circ a}/\pi$-algebra A is* perfectoid *if it is flat and* $\mathrm{Fr} : A/\pi^{1/p} \to A$ *is an isomorphism.*

Let $\mathsf{Perf}(k^{\circ a}/\pi)$ denote the category of perfectoid $k^{\circ a}/\pi$-algebras. We will show that the functor $A \mapsto A/\pi$ from $\mathsf{Perf}(k^{\circ a})$ to $\mathsf{Perf}(k^{\circ a}/\pi)$ is an equivalence of categories by introducing an "almost version" of the cotangent complex. Let's start by recalling the classical theory:

Theorem 3.2. *There is a functorial way of assigning to a flat map $A \to B$ of (commutative, unital) rings an object (the* cotangent complex*) $\boldsymbol{L}_{B/A}$ of $\mathsf{D}^{\leqslant 0}(B)$. This complex satisfies the following properties:*

1. *There is a functorial way of assigning to a square-zero extension $0 \to I \to \widetilde{A} \twoheadrightarrow A$ is a square-zero extension an* obstruction class
$$\boldsymbol{o}(I) \in \mathrm{Ext}^2(\boldsymbol{L}_{B/A}, I_B)$$
such that $\boldsymbol{o}(I) = 0$ if and only if there is a flat $\widetilde{A}$-algebra $\widetilde{B}$ such that $\widetilde{B} \otimes_{\widetilde{A}} A = B$. (One calls such a $\widetilde{B}$ a deformation *of B to $\widetilde{A}$.)*

2. *If a deformation of B to $\widetilde{A}$ exists, then the set $\mathrm{Def}_A^{\widetilde{A}}(B)$ of deformations of B to $\widetilde{A}$ is a $\mathrm{Ext}^1(\boldsymbol{L}_{B/A}, I_B)$-torsor.*

3. *If $\widetilde{B}$ and $\widetilde{B}'$ are deformations of A-algebras B, B' to $\widetilde{A}$, and if $f : B \to B'$ is an A-algebra map, then there is a functorial way of assigning to f an* obstruction class
$$\boldsymbol{o}(f) \in \mathrm{Ext}^1(\boldsymbol{L}_{B/A}, I_{B'})$$
such that the set $\mathrm{Def}_A^{\widetilde{A}}(f)$ of isomorphism classes of lifts of f to $\widetilde{f} : \widetilde{B} \to \widetilde{B}'$ is nonempty if and only if $\boldsymbol{o}(f) \neq 0$.

4. *If a lift of f to $\widetilde{A}$ exists, then $\mathrm{Def}_A^{\widetilde{A}}(f)$ is a $\hom(\boldsymbol{L}_{B/A}, I_{B'})$-torsor.*

Proof. See Proposition 2.1.2.3 in Chapter III of [Ill71] for parts 1 and 2. ☐

Gabber and Ramero were able to generalize the "classical" theory of the cotangent complex to an almost setting. To be precise, Theorem 3.2 remains true if we work V^{a}-algebras, for any non-discrete valuation ring V. In other words, there is a canonical object $\boldsymbol{L}^{\mathrm{a}}_{B/A} \in \mathsf{D}^{\leqslant 0}(B)$ such that the theore still works.

Theorem 3.3 (Scholze)**.** *Let k be a perfectoid field. If A is a perfectoid $k^{\circ a}/\pi$-algebra, then $\boldsymbol{L}^{\mathrm{a}}_{A/(k^{\circ a}/\pi)} = 0$ as an object of $\mathsf{D}(A)$.*

Idea of proof. This is a deep result, but at the heart of its proof is the fact that if R is a smooth perfect $\mathbf{F}_p$-algebra, then $\boldsymbol{L}_{R/\mathbf{F}_p} = 0$. Smoothness yields $\boldsymbol{L}_{R/\mathbf{F}_p} = \Omega^1_{R/\mathbf{F}_p}[0]$, and from $d(r^p) = pdr^{p-1} = 0$ we see that $\Omega^1_{R/\mathbf{F}_p} = 0$. $\square$

Corollary 3.4 (Scholze)**.** *The functor $A \mapsto A/\pi$ induces an equivalence of categories* $\mathsf{Perf}(k^{\circ a}) \xrightarrow{\sim} \mathsf{Perf}(k^{\circ a}/\pi)$.

Proof sketch. Let $A_n = k^{\circ a}/\pi^{n+1}$. We content ourselves with showing that objects and morphisms in $\mathsf{Alg}(A_0)$ lift uniquely to each $\mathsf{Alg}(A_n)$. Let B_0 be a perfectoid A_0-algebra. Theorem 3.3 tells us that $\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} = 0$, so B_0 and any morphisms from B_0 to other perfectoid A_0-algebras lift uniquely to A_1 by the almost version of Theorem 3.2. All that remains for the induction to work is to show that $\boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} = 0$ implies $\boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} = 0$. There is an exact sequence

$$0 \longrightarrow B_0 \xrightarrow{\pi^n} B_{n+1} \longrightarrow B_n \longrightarrow 0,$$

and general theory gives us an exact triangle in $\mathsf{D}(A_{n+1})$:

$$\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} \longrightarrow \boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} \longrightarrow \boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} \longrightarrow .$$

Since $\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} = 0$ by Theorem 3.3 and $\boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} = 0$ by assumption, we get that $\boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} = 0$. $\square$

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