Algebra 3

Def: An S-algebra is a ring A with the structure of an S-module (an action of S on A).

Representations are models of objects

e.g.
- bijections $X \to X$ are actions of the symmetric group
- G a group $\Rightarrow$ End(G) is a ring
- k-modules are vector spaces for k a field
- $\mathrm{End}(G) = \mathrm{Hom}_k(G, G)$

Representation Theory: relate abstract object to concrete model

e.g.
- G a group, X a set

  $G \to S(X)$ is a group action "permutation representation"
- R a ring, M an abelian group

  a map $R \to \mathrm{End}(M)$ defines an R-module

Notation: R-mod is the category of R-modules.

- A any k-algebra, V a k-vector space

  $A \to \mathrm{End}_k(V)$ a k-algebra morphism.

  Choose a basis, then $\mathrm{End}_k(V) = M_{n\times n}(k)$

Def: Let G be a group, R a ring

RG is the free R-module with basis G, such that the basis elements obey the group multiplication law.

Called the group-ring.

e.g.

- $\mathbb{Z}G$ is just a ring
- $\mathbb{C}G$ is a $\mathbb{C}$-algebra
- a representation of G is a $\mathbb{C}G$-module

Note that a typical element of $\mathbb{C}G$ looks like

$$\sum_{i=1}^{N} \alpha_i g_i \quad \text{with } \alpha_i \in \mathbb{C},\ g_i \in G$$

alternatively, a function $f: G \to \mathbb{C}$ if G is finite

If G is infinite, replace $\mathbb{C}G$ by $L^2(G)$, $L^p(G)$, $C^\infty(G)$, etc.

8/27/14

Artin - Wedderburn Theory

Convention: Rings are unital, not necessarily commutative.

Notation: R-mod: left R-modules
mod-R: right R-modules
R-fgmod: finitely generated left R-modules
$_RR$: R as a left R-module
R_R : R as a right R-module
$I \leq_R R$ left ideal I
$I \leq R_R$ right ideal I
$I \trianglelefteq R$ two sided ideal

Zorn's Lemma: If the poset $(S, \subseteq)$ is inductive, then S has a maximal element.

Krull's Lemma: Let $M \in$ R-fgmod, let $S = \{N < M \text{ submodule}\}$. Then S is inductive under $\subseteq$. Hence, $\exists$ maximal proper submodules of M.

Def: $M \in$ R-mod is simple if it has no proper submodules.

Def: $M \in$ R-mod is semisimple if it is isomorphic ~~if it is~~ to a direct sum of simples.

Def: A ring R is semisimple if it is semisimple as an R-mod.

Def: A ring R is simple if it does not have any proper two-sided ideals.

__Def:__ M is Artinian if it satisfies the descending chain condition on submodules

__Def:__ M is Noetherian if it satisfies the ascending chain condition on submodules.

__Goal:__ Classification of Simple Rings.

__Remark:__ By a theorem of Wedderburn, finite division rings are fields. Also, finite domains are finite division rings.

e.g. • $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is a division $\mathbb{R}$-algebra

$$i^2 = j^2 = k^2 = ijk = -1$$

- division rings are simple
- Matrix ring over a division ring is simple.

__Exercise:__ Two-sided ideals of R are of the form $M_n(I)$ for $I \trianglelefteq R$.

- Weyl Algebra $\mathbb{C}\left[x, \frac{\partial}{\partial x} \;\middle|\; -x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x = 1\right]$

__Schur's Lemma:__ Let M be a simple R-mod. Then $\mathrm{End}_R(M)$ is a division ring.

Proof of Schur's Lemma: Let $f: M \to M$, $f \neq 0$, $(f \in \mathrm{End}_R(M))$

$$M \text{ simple} \Rightarrow \left\{ \begin{array}{l} \operatorname{im} f \leq M \Longrightarrow \operatorname{im} f = M \\ \ker f \leq M \Longrightarrow \ker f = 0 \end{array} \right\} \longrightarrow f \text{ bijective}$$

Hence f has an inverse, so $\mathrm{End}_R(M)$ is a division ring. ▨

Remark: ${}_R R$ semisimple $\Longrightarrow R \cong \bigoplus\limits_{\substack{I \leq_R R \\ \text{minimal}}} I$ as an R-mod.

Brauer's Lemma: Let $I \leq_R R$ be minimal

Then either $I^2 = 0$ or $I = Re$ and $e^2 = e$

Proof: I is simple, so by Schur $\mathrm{End}_R(I)$ is a division ring.

$\forall a \in I$, let $\phi_a : I \longrightarrow I$
$x \longmapsto xa$

$\phi_a \in \mathrm{End}_R(I) \Longrightarrow \phi_a$ is either invertible or zero.

If $\phi_a = 0$, then $\forall a \in I$, $a \neq 1 \Longrightarrow Ia = 0 \Longrightarrow I^2 = 0$

If $\phi_a \neq 0$, then $I = Ia$ because ϕ_a is surjective.

But $a \in I$, so $Ra \subseteq I = Ia$. But I is minimal, so $Ra = Ia = I$

Thus $\exists\, e \in I$ s.t. $a = ea \Longrightarrow a = e^2 a \Longrightarrow (e^2 - e)a = 0$

but $(e^2 - e)a = \phi_a(e^2 - e) = 0$. However, ϕ_a is injective, so $e^2 = e$. Furthermore, $e \neq 0$ because $\phi_a \neq 0$.

$Re \subseteq I$, but by minimality $Re = I$.

▨

Semisimple Modules:

Fact: M semisimple $\iff M = \sum_{i \in I} N_i$, $N_i \leq M$ simple for some I.

Proof: $(\Rightarrow)$ easy

$(\Leftarrow)$ Define $S = \{ J \subseteq I \mid \sum_{j \in J} N_j \text{ is direct} \}$

We want to show $(S, \subseteq)$ is inductive. Let $(J_\lambda)_{\lambda \in \Lambda}$ be a chain, $K = \bigcup_{\lambda \in \Lambda} J_\lambda$. Claim K is a majorant.

A typical element of $\sum_{k \in K} N_k$ is of the form

$$0 = n_{\lambda_1} + \cdots + n_{\lambda_s} \in \underbrace{\sum_{j \in J_{\max\{\lambda_1, \dots, \lambda_s\}}} N_j}_{} \quad \Big\} \text{— direct, because } J_\lambda \in S \text{ for all } \lambda \in \Lambda.$$

Hence $n_{\lambda_1} + \cdots + n_{\lambda_s} = 0$ in $\sum_{k \in K} N_k$ as well, so the sum is direct. By Zorn's Lemma, there is a maximal element in S, call it J. Then for all $i \in I$, define

$$P_i = N_i \cap \left(\bigoplus_{j \in J} N_j \right) \leq N_i .$$

If $P_i = 0$, then $J \cup \{i\} \in S$, and it contradicts maximality. So $P_i \neq 0$, $P_i \leq N_i$ and N_i is simple, so $P_i = N_i$.

$\Longrightarrow \quad N_i \subseteq \bigoplus_{j \in J} N_j$ for all i. Hence, $M = \bigoplus_{i \in I} N_i = \bigoplus_{j \in J} N_j$.

■

Def: M is said to be completely reducible if $\forall P \leq M$, $\exists Q \leq M$ such that $M = P \oplus Q$. Q need not be unique.

e.g. Let M be completely reducible, $N \leq M$. Then N is also completely reducible. Likewise for M/N

Remark: Let M be completely reducible. Then it has a simple submodule.

Proof: If M is simple, done.

Otherwise, if $\exists P \neq 0$, $P \subsetneq M$, then take $x \in M \setminus P$, $x \neq 0$.

Consider $S = \{Q \leq M \mid x \notin Q\}$. $P \in S \Rightarrow S \neq \emptyset$.

S is clearly inductive, so take Q maximal.

Now let T be such that $M = Q \oplus T$, $x = q + t$ for $q \in Q, t \in T$.
Note $t \neq 0$ because $x \notin Q$

Claim T is simple. If $0 \neq S \leq T$, for some submodule $S \leq M$, then $Q \leq Q \oplus S \Rightarrow x \in Q \oplus S \Rightarrow t \in S$.

Since $T \leq M$, T is completely reducible, say $T = S \oplus W$, $W \neq 0$.
The above argument also shows $t \in S \cap W$, but $t \neq 0$, so we get a contradiction.

Hence T has no nontrivial submodules.

Proposition: M semi-simple $\Longleftrightarrow$ M completely reducible

Proof: $(\Longrightarrow)$ $M = \bigoplus_{i \in I} N_i$, N_i simple. Given $P \leq M$,

Let $S = \{ J \subseteq I \mid P \cap \bigoplus_{j \in J} N_j = 0 \}$. As before, S is inductive, so choose $J \in S$ maximal.

Claim: $P \oplus \left(\bigoplus_{j \in J} N_j \right) = M$.

pf: $\forall i \in I$, $N_i \cap \left(P \oplus \bigoplus_{j \in J} N_j\right) \neq 0$ because J max'l; as before $N_i = \left(P \oplus \bigoplus_{j \in J} N_j\right) \cap N_i \Longrightarrow N_i \subseteq P \oplus \bigoplus_{j \in J} N_j$.

Thus, $M = P \oplus \left(\bigoplus_{j \in J} N_j \right)$, and we have a compliment of P.

$(\Longleftarrow)$ $0 \neq P := \sum_{\substack{N \subseteq M \\ \text{simple}}} N$. Because M is completely reducible, $M = P \oplus Q$. $Q \leq M$, so Q is also completely reducible, so there is $K \leq Q$ simple, $k \neq 0$. But necessarily $K \subseteq P$, so contradiction.

Hence $Q = 0 \Longrightarrow M = P \Longrightarrow M$ semisimple. ■

Corollary: If $N \leq M$, M semisimple, then N, M/N are semisimple.

Remark: If R is left semisimple, $R = \bigoplus_{\lambda \in \Lambda} I_\lambda$, I_λ minimal (simple)

$1 = e_{\lambda_1} + \dots + e_{\lambda_N}$ $\quad R \ni x = x \cdot 1 = x e_{\lambda_1} + \dots + x e_{\lambda_N} \in I_{\lambda_1} \oplus \dots \oplus I_{\lambda_N}$

$\Longrightarrow \Lambda$ is finite.

__Def:__ Let M be semi-simple. $\hat{M} := \{N \leq M \text{ simple}\}/_{\text{iso}}$

Let $\pi \in \hat{M}$; $\quad M(\pi) := \sum\limits_{\substack{N \leq M \\ N \cong \pi}} N$ is the __π-isotypic__ __component__ of M.

__Remark:__ • $M = \bigoplus\limits_{\pi \in \hat{M}} M(\pi)$

• M, N simple, nonisomorphic. Then $\mathrm{Hom}_R(M, N) = 0$

e.g. let D be a division ring, $R = M_n(D)$. Then R is simple.

__Caution:__ semisimple depends on left / right (for rings)
simple is no two-sided ideals (for rings)
Simple $\not\Rightarrow$ Semisimple (for rings)

$D^n = \left\{\begin{matrix}\text{column}\\ \text{vectors over } D\end{matrix}\right\} \in R\text{-mod} \qquad D^n$ is simple

The j^{th} column ideal $\quad I_j = \left\{ \underset{\uparrow_{j^{th}\text{ column}}}{\left[\begin{matrix} & * & \\ 0 \cdots 0 & * & 0 \cdots 0 \\ & \vdots & \\ & * & \end{matrix}\right]} : * \in D \right\} \leq_R R = M_n(D)$

$I_j \cong D^n$ as R-mod, hence simple.

Thus, I_j is minimal. $\quad R = \bigoplus\limits_{j=1}^{n} I_j \Longrightarrow R$ left semisimple.

Similarly, R is also right semisimple.

$$\begin{tikzcd} I \arrow[r, hook, "f"] \arrow[dr, "\pi_j \circ f"'] & R \arrow[d, "\pi_j"] \\ & I_j \cong D^n \end{tikzcd}$$

Now let $I \leq_R R$ be minimal, $I \neq 0$.

By Schur's lemma, $\pi_j \circ f = 0$
or $\pi_j \circ f$ is an isomorphism.

Cannot have $\pi_j \circ f = 0 \ \forall j$, because then $I = 0$

So $\exists j$ such that $\pi_j \circ f$ is an IM $\Longrightarrow I \cong D^n$.

this shows that R has more minimal ideals than the j^{th} column ideals $R = \bigoplus_{j=1}^{n} I_j$, but all are isomorphic to D^n.

Let $M \in R\text{-mod}$ be simple. Let $x \in M \setminus \{0\}$. $(R = M_n(D)$ still$)$

$$\begin{tikzcd} R \arrow[r, "f"] & M & R\text{-mod map} \\ r \arrow[r, mapsto] & r\cdot x & \end{tikzcd}$$

Since M is simple, f is surjective. $R/_J \cong M$ as R-mod, for some $J \trianglelefteq_R R$.

Since R is semisimple, then R is completely reducible, so $R \cong I \oplus J$ for some $I \trianglelefteq_R R$. So $I \cong R/_J \cong M$ therefore I is simple, so a minimal left ideal. Hence $I \cong D^n \implies M \cong D^n$.

So all simple R-modules are D^n!

$M_n(D)$-mod :
simple up to iso $= \{D^n\}$
all modules semisimple
indecomposable $=$ simple $\cong D^n$.
finitely generated $=$ finite direct sum of simple
$=$ Artinian, Noetherian

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e.g. $\mathbb{C}[x]/_{(x^2)}$ is indecomposable but not simple (as $\mathbb{C}$-mod)
the ideal (x) has no compliment.

<u>Def</u>: A module is <u>finite length</u> if it is both Artinian and Noetherian.

Theorem (Skolem - Noether):

Let $M_n(D) \underset{g}{\overset{f}{\rightrightarrows}} M_n(D)$ be two ring morphisms.

Then $\exists\ S \in GL_n(D)$ such that $f(a) = S g(a) S^{-1}$ for all $a \in M_n(D)$

Corollary: if $g = id$, $f(a) = SaS^{-1}$. All ring morphisms are conjugation.

Proof: Consider
$$\phi : M_n(D) \times D^n \longrightarrow D^n$$
$$(a, \vec{v}) \longmapsto f(a)\vec{v}.$$
$$\psi : M_n(D) \times D^n \longrightarrow D^n$$
$$(a, \vec{v}) \longmapsto g(a)\vec{v}$$

Gives two $M_n(D)$-mod structures on D^n, say D^n_f, D^n_g.

Since D^n_g, D^n_f semisimple, $D^n_f \cong D^n \cong D^n_g$.

So there is $D^n_f \overset{S}{\longleftrightarrow} D^n_g$ and $S \in GL(D)$, $M_n(D)$-linear,

$$\begin{tikzcd}
\vec{v} \arrow[r, phantom, "\in"] \arrow[d, mapsto] & D^n_f \arrow[r, "\overset{S}{\sim}"] \arrow[d, "\phi(a,\cdot)"] & D^n_g \arrow[r, phantom, "\ni"] \arrow[d, "\psi(a,\cdot)"] & S\vec{v} \arrow[d, mapsto] \\
f(a)\vec{v} \arrow[r, phantom, "\in"] \arrow[rrr, mapsto, bend right] & D^n_f \arrow[r] & D^n_g \arrow[r, phantom, "\ni"] & Sf(a)\vec{v} = g(a)S\vec{v}
\end{tikzcd}$$

True for any $\vec{v} \implies Sf(a) = g(a)S$ ∎

Question: Can you figure out D given just the category $M_n(D)$-mod?

$$\begin{tikzcd} D^{op} \arrow[r, "\phi"] & \mathrm{End}(D^n) & \text{ring morphism} \\ d \arrow[r, mapsto] & \phi_d & \phi_d(\vec{v}) = \vec{v} \cdot d \end{tikzcd}$$

$$\phi_{d_1 \cdot^{op} d_2}(\vec{v}) = \phi_{d_2 d_1}(\vec{v}) = (\vec{v} \cdot d_2) \cdot d_1 = \phi_{d_1} \circ \phi_{d_2}(\vec{v})$$

Note ϕ is injective. Claim ϕ also surjective.

Let $f \in \mathrm{End}(D^n)$. Completely determined by behavior on a single elt of D^n b/c D^n is simple. So only need to know $f\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$.

$$f\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} f\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } d = \phi_d\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right).$$

$\Longrightarrow f = \phi_d$. Thus, $D^{op} \cong \mathrm{End}(D^n)$

So in $M_n(D)$-mod, D is intrinsic.

$D \cong \mathrm{End}_{M_n(D)}(\text{simple})^{op}$, and simples are $\mathrm{End}_{M_n(D)}(\text{simple})$-modules.

n = dimension of a simple module over $\mathrm{End}_{M_n(D)}(\text{simple})$.

Recall: Let R be a ring.

- Free module is $\cong \bigoplus R$
- Spanning set / generating set exists $\Big\}$ in $M \in R\text{-mod}$
- notion of R-linear independence
- Basis is spanning & independent
- Invariant Basis Number property: all bases have same cardinality for a ring with this property.
  - commutative rings
  - division rings
  - left-Noetherian rings.

Proposition: Let R be left-semisimple. Then

(1) All R-modules are semisimple

(2) any simple R-module is isomorphic to a minimal ideal

(3) If $R = \bigoplus_{j=1}^{n} I_j$, I_j minimal $\leq_R R$, then any minimal ideal is isomorphic to some I_j

(4) $\hat{R} = \left\{ \begin{matrix} \text{simple} \\ R\text{-mod} \end{matrix} \right\} /_{\cong}$ is finite

(5) $$\mathrm{Hom}_R(M;N) \cong \bigoplus_{\pi \in \hat{R}} \mathrm{Hom}(M(\pi);N(\pi))$$

Proof of (5):

$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R\Big(\bigoplus_{\pi \in \hat{R}} M(\pi) ; \bigoplus_{\rho \in \hat{R}} N(\rho)\Big)$$
$$\cong \bigoplus_{\pi,\rho} \mathrm{Hom}_R(M(\pi);N(\rho)) \cong \bigoplus_{\pi} \mathrm{Hom}(M(\pi),N(\pi))$$

Proposition: Let R be left semisimple. Then TFAE:

1. $M \cong \bigoplus$ simples
2. M is finitely generated
3. M Artinian
4. M Noetherian
5. M finite length

Recall: Composition series:

$$(0) \subseteq M_0 \subseteq \cdots \subseteq M_n = M \qquad M_{i+1}/M_i \text{ simple}$$

M_i is composition factor

n is length.

Theorem: (Artinian & Noetherian) $\Longleftrightarrow$ finite composition series

Theorem (Jordan-Holder): If M has two finite composition series, then they have the same length, and factors are the same up to labels.

The common length is the length of M.

In the category of R-mod for R semisimple, we have

- $\hat{R}$ finite
- $R \cong \bigoplus_{\pi \in \hat{R}} n_\pi I_\pi$
- R-modules are all semisimple
- indecomposable = simple.

"semisimple category"

Recall: $M_n(D)$ semisimple.

${}_RR$ semisimple $= \bigoplus_{i=1}^{n} I_i$, I_i are simple, minimal ideals

Theorem (Wedderburn Structure Theorem):

Let ${}_RR \cong \bigoplus_{\pi \in \hat{R}} n_\pi I_\pi$ be semisimple. (Pairs (n_π, D_π) unique up to iso)

Let $D_\pi = \mathrm{End}_R(I_\pi)^{op}$

Then $R \simeq \bigoplus_{\pi \in \hat{R}} M_{n_\pi}(D_\pi)$ ← Ring Isomorphism!!

and the simple modules are $\{D_\pi^{n_\pi} \mid \pi \in \hat{R}\}$.

Proof: Consider $\mathrm{End}_R(R)$

$$\left.\begin{array}{l} R^{op} \xrightarrow{\sim} \mathrm{End}_R R \\ a \longmapsto \phi_a(x) = x \cdot a \end{array}\right\}$$ Ring Morphism; proved a few lectures ago.

Note
$$\mathrm{End}_R(R) \cong \mathrm{Hom}_R\Big(\bigoplus_{\pi} n_\pi I_\pi ; \bigoplus_{\rho} n_\rho I_\rho\Big)$$

$$\cong \mathrm{Hom}_R\Big(\bigoplus_{\pi} n_\pi I_\pi ; \bigoplus_{\pi} n_\pi I_\pi\Big) \cong \bigoplus_{\pi} \mathrm{Hom}(n_\pi I_\pi ; n_\pi I_\pi)$$

$$\cong \bigoplus_{\pi \in \hat{R}} M_{n_\pi}(\mathrm{End}_R(I_\pi))$$

$$\cong \bigoplus_{\pi \in \hat{R}} M_{n_\pi}(D_\pi{}^{op})$$

Apply the opposite functor again to get

$$R \cong \mathrm{End}_R(R)^{op} \cong \Big(\bigoplus_{\pi \in \hat{R}} M_{n_\pi}(D_\pi{}^{op})\Big)^{op} \cong \bigoplus_{\pi \in \hat{R}} M_{n_\pi}(D_\pi).$$

∎

Corollary left semisimple and right semisimple are the same, because $M_n(D)$ is both left/right semisimple.

Remark: If R is k-algebra, same theorem applies with D_π division k-algebras.

If K an algebraically closed field and D_π finite dim $/K$, then let $a \in D_\pi$. $D_\pi \supseteq K(a) \supseteq K$, $K(a)/_K$ finite $\Rightarrow a \in K$.

Hence, $D_\pi = K$. So if R is a finite dimensional K-algebra, then $R \simeq \bigoplus_{\pi \in \hat{R}} M_{n_\pi}(K)$ and $\dim_K R = \sum_{\pi \in \hat{R}} n_\pi^2$.

Proposition: Let ${}_R R$ be Artinian, simple. Then R is semisimple.

Proof: R has a minimal left ideal because Artinian, say I.

$$\sum_{a \in R} Ia \trianglelefteq R \Rightarrow R = \sum_{a \in R} Ia \quad \text{sum of simple modules.} \quad ■$$

$\uparrow$ each isomorphic to I hence minimal.

Corollary: If ${}_R R$ is artinian and simple, then $R \simeq M_n(D)$, D a division ring.

Corollary: R a finite dimensional k-algebra. Then $R \cong M_n(D)$ because finite-dimensional $\Rightarrow$ Artinian.

Theorem (Artin): Let R be simple. then TFAE

(1) R semisimple
(2) R Artinian
(3) $\exists$ a minimal left ideal
(4) $R \simeq M_n(D)$, $D^{op} \cong \mathrm{End}_R(I)$ is a division ring.

Prop: Let M be an R-module. TFAE
(1) M is Noetherian
(2) $\forall \Sigma \subseteq \{N \leq M\}$, $(\Sigma, \leq)$ is inductive
(3) $\forall N \leq M$, N is finitely generated.

Prop: Let $M \in R$-mod. TFAE
(1) M Artinian
(2) $\forall \Sigma \subseteq \{N \leq M\}$, $(\Sigma, \geq)$ inductive

Prop: $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ exact
M Noetherian/Artinian iff M', M'' Noetherian/Artinian.

Corollary: $\bigoplus_{i \in I} M_i$ is Noetherian/Artinian $\Longleftrightarrow \forall i$, M_i is Noetherian/Artinian

Corollary: Let ${}_R R$ be Noetherian/Artinian, M any fg module over R. Then M is Noetherian/Artinian.

Wedderburn - Artin Radical

Defn $I \leq_R R$ is nilpotent if $I^N = 0$ for some $N > 0$

$I \leq_R R$ is nil if $\forall x \in I, \exists N$ such that $x^N = 0$

Fact: Let $I \leq_R R$ and M a simple R-module. Then if I is nilpotent, then $IM = 0$.

Proof: $IM \leq M$. If $IM \neq 0$, then $IM = M$ because M is simple. Iterate, so then $I^N M = 0$, contradiction. ■

Def: R a ring, $\mathrm{rad}(R)$ is the largest ideal that acts trivially on all simple R-modules. The Radical of R.

Prop: Let ${}_R R$ be Artinian and $I \leq_R R$ a nil ideal. Then I is nilpotent.

Proof: $I \supseteq I^2 \supseteq I^3 \supseteq \dots \supseteq I^N = I^{N+1} = \dots$ because Artinian, this chain stabilizes at N for some N. Let $J = I^N = I^{N+1} = \cdots$

If $J = 0$, done. Otherwise, $J \cdot I = J \neq 0$.

Consider $S = \{K \leq_R R \mid J \cdot K \neq 0\}$. This set contains I, and has a minimal element K.

$$JK \neq 0 \Rightarrow \exists a \in K, \ Ja \neq 0.$$

$$Ja \subseteq Ra \subseteq K \Rightarrow J(Ja) = Ja \neq 0$$

By minimality, $Ja = K \Rightarrow \exists e \in J, a = ea$. Iterate to get $a = e^M a \ \forall M$.

But $e \in J \subseteq I \Rightarrow e$ nilpotent, so $e^M = 0$ for large M, hence $a = 0$. Thus $J = 0$. ■

Remark: I_1, I_2 nilpotent $\Longrightarrow I_1 + I_2$ nilpotent.

$\sum_{\text{infinite}} I_j$ is nil if each I_j is nilpotent.

Remark: Let $I \leq_R R$ be nilpotent. $\sum_{a \in R} Ia \trianglelefteq R$ is nil.

Proposition: Let ${}_R R$ be Artinian, $J = \sum_{\substack{I \leq_R R \\ \text{nilpotent}}} I$.

Then J is the largest two-sided nilpotent ideal of R, and moreover

$$J = \sum_{\substack{K \leq R_R \\ \text{nilpotent}}} K.$$

Terminology: $J = W(R)$ is the Wedderburn–Artin radical of R.

Proof: $J \leq_R R$ is nil, and b/c R is Artinian then J is nilpotent too.

$Ja = \sum_{\substack{I \leq_R R \\ \text{nilpotent}}} Ia$, each Ia is a nilpotent left ideal of R

Hence, $Ja \subseteq J$, so $J \trianglelefteq R$. J is necessarily now the largest nilpotent b/c any nilpotent must participate in the sum. ■

Prop: If R is left-Artinian with $W(R) = 0$, then R is semisimple.

Proof: Take a minimal left ideal I. By Brauer's Lemma, either $I^2 = 0$ or $I = Re$ and $e^2 = e$.

We know $I^2 \neq 0$, because $I \neq 0$, and if $I^2 = 0$ then $I \subseteq W(R) = 0 \Longrightarrow I = 0$, contradiction.

Hence, $R = Re + R(1-e)$.

$\longrightarrow$

Proof (continued): Claim that the sum $Re \oplus R(1-e)$ is direct. If $x = ae = b(1-e)$, then $xe = ae^2 = ae = x$

$$\Longrightarrow x = xe = b(1-e)e = b(e-e^2) = b(e-e) = 0.$$

If $R(1-e)$ is not minimal, then take $J \leq R(1-e)$ minimal, $J = R(e')$. Then $R = Re \oplus Re' \oplus R(1-e-e')$. Repeat, and the process stops b/c R is Artinian. Get $\blacksquare$

$$R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_N,$$ so R is Semisimple.

Proposition: Let ${}_RR$ be Artinian. Then $R/W(R)$ is semisimple.

Proof; Show $W(R/W(R)) = 0$, then use the previous proposition. Hence want to show $R/W(R)$ has no nilpotent ideals.

If $I \trianglelefteq R/W(R)$ nilpotent, then there is $J \trianglelefteq R$ containing $W(R)$.

Some power of J is in $W(R)$, say $J^N \subseteq W(R)$

But $W(R)$ is nilpotent $\Longrightarrow$ J nilpotent $\Longrightarrow J \subseteq W(R)$

Hence $J = W(R) \Rightarrow I = 0$. $\blacksquare$

Corollary 1: ${}_RR$ Artinian $\Longrightarrow$ $W(R) = \mathrm{rad}(R)$.

proof requires another corollary

Corollary 2: ${}_RR$ Artinian $\Longrightarrow$ $\hat{R} = \widehat{R/W(R)}$

in other words,
$$\begin{aligned} R/W(R)\text{-mod} &\longrightarrow R\text{-mod} \\ (M, \lambda) &\longmapsto (M, \lambda \circ \pi) \end{aligned}$$
restricts to a bijection between simple $R/W(R)$ modules and simple R-modules. In particular, $\hat{R}$ is finite.

Proof of Corollary 2:

$$\begin{tikzcd} R/W(R) \arrow[r, "\lambda"] & \mathrm{End}(M,+) \\ R \arrow[u, "\pi"] \arrow[ru] & \end{tikzcd}$$

(M,λ) simple $\iff$ $(M,\lambda\circ\pi)$ simple

Want to show that all simple R-modules are of the form $(M,\lambda\circ\pi)$ where M is a simple $R/W(R)$-module.

Let $N \in R$-mod, simple. $W(R) \subseteq \mathrm{rad}(R) \Rightarrow W(R)\cdot N = 0$

$\Rightarrow W(R) \subseteq \ker \Lambda$

$\Rightarrow \exists\, \lambda$ s.t. $\Lambda = \lambda\circ\pi$.

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\Lambda"'] & R/W(R) \arrow[ld, dashed, "\text{universal property of quotient } \exists\lambda"] \\ \mathrm{End}(N,+) & \end{tikzcd}$$

■

proof of corollary 1:

We have $W(R) \subseteq \mathrm{rad}(R)$

Since $\mathrm{rad}(R) \trianglelefteq R$, we have that $\mathrm{rad}(R)/W(R) \trianglelefteq R/W(R)$ and so acts trivially on simple $R/W(R)$ modules by corollary 2.

$R/W(R) \cong \bigoplus M_{n_i}(D_i)$ b/c semisimple, and $D_i^{n_i}$ are simple $R/W(R)$-modules. Now let $\sum a_i \in \bigoplus M_i(D_i)$ acting trivially on all $D_i^{n_i}$.

$$0 = \left(\sum a_i\right)\cdot v_j = a_j \cdot v_j \Longrightarrow a_j = 0 \quad \text{for all } j.$$

$$\Rightarrow \sum a_j = 0.$$

Therefore, $\mathrm{rad}(R/W(R)) = 0 \Longrightarrow \dfrac{\mathrm{rad}(R)}{W(R)} = 0 \Longrightarrow \mathrm{rad}(R) = W(R)$. ■

Corollary 3: Let R be a finite dimensional k-algebra. Then $\hat{R}$ is finite and simple R-modules are finite dimensional over k.

Theorem (Wedderburn): Let R be a finite dimensional k-algebra where k has characteristic zero. Then $R \cong S \oplus W(R)$ as a k-vector space with $S \subseteq R$ a semisimple subalgebra, $S \cong R/W(R)$.

e.g. $\mathbb{C}[x \mid x^2 = 0]$ has decomposition $\mathbb{C} \oplus \mathbb{C}x$, $\mathbb{C}$ is the only simple module. $\mathbb{C}x$ is indecomposable

Prop: Let ${}_RR$ be Artinian, $M \in R$-mod. TFAE

(1) M Artinian
(2) M Noetherian
(3) M finite length
(4) M finitely generated

Proof: Let $I = W(R)$. $M \supseteq IM \supseteq I^2M \supseteq \dots \supseteq 0$ ← stops b/c I is ~~Noetherian~~ nilpotent. $I^jM/I^{j+1}M \in R$-mod, also R/I-mod.

Hence (1), (2), (3), (4) are equivalent for $I^jM/I^{j+1}M$.

This yields an exact sequence

$$0 \longrightarrow I^{j+1}M \hookrightarrow I^jM \twoheadrightarrow I^jM/I^{j+1}M \longrightarrow 0$$

So if $I^{j+1}M$ has equivalence of properties, then so does I^jM. Iterate to get to M. ■

Corollary (Hopkins): ${}_RR$ Artinian $\Longrightarrow$ ${}_RR$ Noetherian.

09/15/14

ALGEBRA 3

Recall: $\text{rad}(R) \trianglelefteq R$

If ${}_RR$ artinian, $W(R) \triangleleft R$ is the maximal nilpotent ideal, called Wedderburn - Artin Radical.

$$W(R) = \text{rad}(R) \quad , \quad W(R) = 0 \Rightarrow R \text{ semisimple.}$$

$R/W(R)$ semisimple, and $\widehat{R} \cong \widehat{R/W(R)}$

To study R-mod, look for indecomposables. Simple modules are not enough to understand R-mod in the case that $\text{rad}(R) \neq 0$. (See example $\frac{\mathbb{C}[x]}{\langle x^2 \rangle}$).

Indecomposable Modules

Def: R is local if $R/\text{rad}(R)$ is a division ring.

Remark: If R is local then $R \setminus \text{Rad}(R)$ = units of R. b/c quotient is division ring.

Proof: If $a \notin \text{rad}(R)$ then $\exists\, b \in R$ $ab - 1 \in \text{Rad}(R)$
$ba - 1 \in \text{rad}(R)$

If $J \leq_R R$ is maximal, then R/J has as ideals those which contain J, of which there are none, so R/J is a simple R-mod.

$ba-1$ acts by multiplying by zero on R/J, since R/J simple,

$$\Rightarrow ba - 1 \in J \ \forall\, J \leq {}_RR \text{ max'l.}$$

$Ra \lneq {}_RR \Rightarrow Ra \subseteq J$ for some maxl J

in particular $ba \in Ra$, $\Rightarrow 1 \in J$ ✱.

Hence $Ra = R$, $\Rightarrow \exists\, c$ s.t. $ca = 1$. ⟶

proof ctd: Hence $c \notin \mathrm{rad}(R)$. Repeat for c, so the element c has a left inverse. Thus, c has left and right inverses, so c is a unit.

$\Rightarrow$ a is a unit. ■

Prop: Let $M \in R$-mod. (1) If $\mathrm{End}_R(M)$ is local, then M is indecomposable. (2) If M is Artinian, Noetherian, Indecomposable then $\mathrm{End}_R(M)$ local.

Analogue of Shur's Lemma for indecomposables.

Proof (1): Let $M = P \oplus Q$

$$\begin{tikzcd}
 & M \arrow[dl, "\pi_P"'] \arrow[dr, "\pi_Q"] & \\
P \arrow[d, hook] & & Q \arrow[d, hook] \\
M & & M
\end{tikzcd}$$

$$\begin{array}{cccccc}
\mathrm{id}_M = & \pi_P & \oplus & \pi_Q & \text{is an equality in} & \mathrm{End}_R(M) \\
\downarrow & \in & & \in & & \uparrow \\
1 \in \mathrm{End}_R(M) & & & & & \text{local}
\end{array}$$

Since $\mathrm{End}_R(M)$ is local, then π_P, π_Q are either units or in the radical. Cannot both be in radical, b/c then $1 \in \mathrm{rad}(\mathrm{End}_R(M))$

So say π_P is invertible.

$$\pi_P \circ \pi_Q = 0 \Longrightarrow \pi_Q = 0 \Longrightarrow Q = 0. \quad ■$$

Proof of (2): Uses the fitting lemma.

Fitting Lemma: If M is Artinian and Noetherian, $f \in \mathrm{End}_R(M) \Rightarrow \exists\, M = P \oplus Q$ such that $f\restriction_P$ is an isomorphism, and $f\restriction_Q$ is nilpotent, and P, Q are f-stable submodules.

Proof of (2): If M is in addition indecomposable, then f is either an IM or nilpotent, by ~~fitting~~ lemma.

if f is nilpotent, $f \in \mathrm{rad}(\mathrm{End}_R(M))$.

Hence $\mathrm{End}_R(M)/\mathrm{rad}(\mathrm{End}_R(M))$ is a division ring.

Hence $\mathrm{End}_R(M)$ is local. $\blacksquare$

Remark: Let $M_1 \oplus M_2 \overset{\phi}{\cong} N_1 \oplus N_2$ such that $\phi_{11} : M_1 \to N_1$ is an IM. Then $\phi_{22} - \phi_{21} \circ \phi_{11}^{-1} \circ \phi_{12} : M_2 \longrightarrow N_2$ is an isomorphism.

Know ϕ is IM, so $\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$ is IM.

So are $\begin{bmatrix} id_{N_1} & 0 \\ -\phi_{21}\phi_{11}^{-1} & id_{N_2} \end{bmatrix}$ and $\begin{bmatrix} id_{M_1} & -\phi_{11}^{-1}\phi_{12} \\ 0 & id_{M_2} \end{bmatrix}$. So,

$$\begin{bmatrix} id_{N_1} & 0 \\ -\phi_{21}\phi_{11}^{-1} & id_{N_2} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} id_{M_1} & -\phi_{11}^{-1}\phi_{12} \\ 0 & id_{M_2} \end{bmatrix} \quad \text{is IM}$$

$$= \begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} - \phi_{21}\phi_{11}^{-1}\phi_{12} \end{bmatrix} \begin{bmatrix} id_{M_1} & -\phi_{11}^{-1}\phi_{12} \\ 0 & id_{M_2} \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} - \phi_{21}\phi_{11}^{-1}\phi_{12} \end{bmatrix} \quad \text{is also IM.} \quad \blacksquare$$

Splitting Lemma for R-mod:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] & N \arrow[l, dashed, bend left, "\alpha"] \arrow[r, "g"] & P \arrow[l, dashed, bend left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$ exact

TFAE: (1) $\exists\, \alpha : \alpha \circ f = id_M$

(2) $\exists\, \beta : g \circ \beta = id_P$

(3) $N = im\, f \oplus ker\, \alpha$

(4) $N = im\, \beta \oplus ker\, g$

Say the sequence "splits."

Prop: Let $M = M_1 \oplus \dots \oplus M_t \overset{\text{equal, not IM'ic}}{\equiv} N_1 \oplus \dots \oplus N_s$, such that M_i, N_j are indecomposable and $End_R(M_i)$, $End_R(N_j)$ are local. Then $s=t$ and $M_i \cong N_j$ up to re-ordering.

Proof:

Note that $End_R(M_i)$ local $\Longrightarrow$ M_i local, so had extra conditions.

By induction on $t \geq 1$. Let $M = \bigoplus_{i=1}^{t} M_i$.

$id = \sum_{j=1}^{s} \pi_{N_j}$ ($t=1 \Longrightarrow$ M indecomposable $\Longrightarrow s=1$ and $N_1 = M$.)

If $t>1$, then $id = \sum_{j=1}^{s} \pi_{N_j}$

$$\pi_{M_1} = \sum_{j=1}^{s} \pi_{M_1} \circ \pi_{N_j} \Longrightarrow \underbrace{\pi_{M_1}\restriction_{M_1}}_{= id_{M_1}} = \sum_{j=1}^{s} \pi_{M_1} \circ \pi_{N_j} \restriction_{M_1}$$

is an identity in $End_R(M_1)$, which is local.

By same logic as before, one of $\pi_{M_1} \circ \pi_{N_j} \restriction_{M_1} : M_1 \longrightarrow M_1$ is ~~local~~. invertible. Say $j=1$ is invertible.

Proof ctd.

$$\begin{tikzcd}
M_1 \arrow[r, "\phi", "\sim"'] \arrow[d, "\pi_{N_1}|_{M_1}"'] & M_1 \arrow[dl, bend right=80, "\beta"'] \\
N_1 \arrow[ur, "\pi_{M_1}|_{N_1}=\alpha"'] &
\end{tikzcd}$$

$\pi_{N_1}|_{M_1}$ injective

$\alpha = \pi_{M_1}|_{N_1}$ surjective (so $\exists$ right inverse β)

$\alpha \circ \beta = id_M$

Gives exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow N_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} M_1 \longrightarrow 0$$

$$\left.\begin{array}{l} \Longrightarrow \quad N_1 = \operatorname{im}\beta \oplus \ker\alpha \\ \qquad\quad N_1 = \text{idecomposable} \end{array}\right\} \Longrightarrow \ker\alpha = 0 \Longrightarrow \alpha \text{ injective} \Longrightarrow \alpha \text{ IM.}$$

$\downarrow$

(if $\ker\alpha = N_1$, then $\alpha = 0 \Longrightarrow M_1 = 0$ ⨳)

So we have $\alpha: M_1 \to N_1$ is an isomorphism.

$$\begin{tikzcd}
M \arrow[rr, "id"] \arrow[d, equal] & & M \arrow[d, equal] \\
M_1 \oplus (M_2 \oplus \dots \oplus M_t) & & N_1 \oplus (N_2 \oplus \dots \oplus N_s)
\end{tikzcd}$$

$id_{11} = \alpha$ is an IM, so by previous lemma, we get that $M_2 \oplus \dots \oplus M_t \cong N_2 \oplus \dots \oplus N_s$. And then apply induction hypothesis. ▨

Theorem (Krull-Schmidt): Let M be Artinian and Noetherian. Then $M \cong \bigoplus_{\text{finite}}$ (indecomposables) and the decomposition is unique up to IM.

Proof: Use previous theorems / propositions.

Proof ctd: Decomposition is unique b/c Artinian / Noetherian
$\Rightarrow$ M_i are Artinian and Noetherian, and indecomposable gives $End_R(M_i)$ is local.

Decomposition exists b/c if M indecomposable done.
Else $M = M_1 \oplus M_2$, M both Artinian and Noetherian
$\Rightarrow$ construction matches decomposition into indecomposables (?).

Remark: Simple $\Rightarrow$ indecomposable.
Indecomposable $\not\Rightarrow$ Simple. e.g. $\mathbb{Z}/(4)$ is not simple yet indecomposable.

Thm holds if

- ${}_R R$ Artinian, $M \in R$-mod finitely generated
- R is a finite dimensional k-algebra, $M \in R$-mod finitely gen.
- R-semisimple, $M \in R$-mod is f.g.

Questions: (1) Understand indecomposable R-modules
(2) For given M understand its decomposition.

Finite Dimensional Central simple algebras

Def: A k-algebra R is central if $Z(R) = k \cdot 1$

By considering a k-algebra A as a $Z(A)$-algebra, everything reduces to this case.

Now suppose $\mathbb{k}$ is a field.

E.g. $A \otimes_k B$ and B central. Then $Z(A \otimes_k B) = Z(A) \otimes_k 1$

If A simple and B central simple then $A \otimes_k B$ is simple

If A, B central simple then $A \otimes_k B$ is central simple.

$\mathbb{C}$ is a simple $\mathbb{R}$-algebra. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a 4-dimensional $\mathbb{R}$-algebra, and commutative. If it's simple, $\cong \mathbb{H}$ or ~~otherwise~~ $\cong M_2(\mathbb{R})$

So it cannot be simple, b/c neither $\mathbb{H}$ nor $M_2(\mathbb{R})$ are commutative.

Remark: Let R be a finite dimensional, central, simple $\mathbb{k}$-algebra. Then $R \cong M_n(D)$; $Z(D) = k \cdot 1$. So D must be a central division k-algebra. (also $\dim < \infty$).

Classification reduces to classification of central f.d. division $\mathbb{k}$-algebras. Let D_1, D_2 be two such things.

$D_1 \otimes_k D_2$ is another such k-algebra,

$$D_1 \otimes_k D_2 \cong M_n(E)$$

↑ also f.d. central, simple.

↙ isomorphism class of D

Def: $\mathcal{B}(k) = \{ [D] \mid D \text{ f.d. central division } k\text{-algebra} \}$

Define $[D_1] \boxtimes [D_2] = [E]$. Then with the operation $\boxtimes$, $\mathcal{B}(k)$ is called the *Brauer group* of k.

Proposition: Let A be an n-dim central simple k-algebra. Then $A \otimes_k A^{op} \cong M_n(k)$

Corollary: In the Brauer group, inverses are $[D]^{-1} = [D^{op}]$.

Proof:

$$A \longrightarrow \mathrm{End}_k(A) \cong M_n(A) \qquad \text{left } A\text{-module}$$

$$a \longmapsto \ell(a) \qquad \ell(a)(x) = a \cdot x$$

ring morphism

$$A^{op} \longrightarrow \mathrm{End}_k(A) \cong M_n(A) \qquad \text{right } A\text{-module}$$

$$a \longmapsto r(a) \qquad r(a)(x) = x \cdot a$$

Both ℓ and r are ring morphisms. Gives

$$A \otimes A^{op} \xrightarrow{\ell \otimes r} \mathrm{End}_k(A)$$

$$a \otimes b \longmapsto \ell(a) r(b)$$

! make sure to check that ℓ and r commute!

central simple $\Rightarrow \mathrm{Ker}(\ell \otimes r) = 0 \Rightarrow \ell \otimes r$ injective

But $\dim(A \otimes A^{op}) = n^2 = \dim M_n(A) \Longrightarrow \ell \otimes r$ is an isomorphism. ■

Theorem (Skolem–Noether II):

Let A, B be f.d. simple k-alg

B central

$A \underset{g}{\overset{f}{\rightrightarrows}} B$ k-alg morphisms

Then $\exists X \in \mathcal{U}(B)$ $f(a) = X^{-1} g(a) X \ \forall a \in A$.

Proof: $$\underbrace{A \otimes B^{op}}_{\text{simple } k\text{-alg}} \underset{g \otimes id}{\overset{f \otimes id}{\rightrightarrows}} B \otimes B^{op} \cong M_n(k) \circlearrowright k^n.$$

So there are two different $A \otimes B^{op}$ module structures on k^n, to distinguish, call them k^n_f and k^n_g.

Since $A \otimes B^{op}$ simple, any module over it is simple, and There is only one simple module up to isomorphism.

The isomorphism type is identified by dimension.

$\dim k^n_f = \dim k^n_g \Rightarrow k^n_f \cong k^n_g$ as $A \otimes B^{op}$ modules.

Since $M_n(k) \cong B \otimes B^{op}$, we have an invertible matrix corresponds to ~~Hence there is~~ $X \in \mathcal{U}(B \otimes B^{op})$ s.t. $f(a) \otimes b = X^{-1}(g(a) \otimes b) X$ for all $a \in A$. Now for $a = 1$, $X(1 \otimes b) = (1 \otimes b)X \ \forall b \in B$.

$$\Downarrow$$

$$X \in B \otimes 1$$

So therefore, $f(a) = X^{-1} g(a) X$.

■

$$\begin{tikzcd}
k^n_f \arrow[r, "X", "\sim"'] \arrow[d, "a\otimes b \text{ acts}"'] & k^n_g \arrow[d, "a\otimes b \text{ acts}"] \\
k^n_f \arrow[r, "X", "\sim"'] & k^n_g
\end{tikzcd}$$

$$X((a \otimes b) \cdot \vec{v}) = (a \otimes b) \cdot X\vec{v}$$

$$\Downarrow$$

$$X(f(a) \otimes b) \cdot v) = (g(a) \otimes b) \cdot X\vec{v}.$$

Remark: $\mathrm{End}_{k\text{-alg}}(M_n(k)) = \mathrm{Aut}_{k\text{-alg}}(M_n(k)) \cong \mathcal{U}(M_n(k)) = GL_n(k)$.

$$\mathrm{End}(A) = \mathrm{Aut}(A) \cong \mathcal{U}(A)$$

Weil (1960): f.d. central simple algebra w/ involution

$$\Updownarrow$$

algebraic groups over k.

Theorem: (Double Centralizer):

Let A = f.dim central, simple k-alg.
$B \subseteq A$ simple subalgebra
$C = Z_A(B)$

Then (1) C is simple,
(2) $\dim_k(A) = \dim_k(B) \cdot \dim_k(C)$
(3) $Z_A(C) = B$
(4) B central $\Longrightarrow B \otimes C \cong A$

Proof:
$$\begin{tikzcd} \underbrace{A \otimes \mathrm{End}_k(B)}_{\text{central simple}} & B \arrow[l, shift right, "i\otimes 1"'] \arrow[l, shift left, "1\otimes \ell"] \end{tikzcd}$$
ℓ is left multiplication
i is inclusion.

By Skolem-Noether, there is $X \in \mathcal{U}(A \otimes \mathrm{End}_k(B))$

$$b \otimes 1 = X^{-1}(1 \otimes \ell(b))X \quad \forall b \in B.$$

$Z(B \otimes 1)$ ~~$C \otimes \mathrm{End}_k(B)$~~ $= C \otimes \mathrm{End}_k(B)$

$Z(1 \otimes \ell(B)) = A \otimes r(B)$ ↙ by Skolem-Noether, related by X.

Hence $\dim(C \otimes \mathrm{End}(B)) = \dim(A \otimes r(B)) \Longrightarrow \dim C \dim B^2 = \dim A \dim B$

$\Longrightarrow \dim A = \dim B \dim C$.

Proof continued:

$A \otimes r(B)$ is simple, so $C \otimes \mathrm{End}_k(B) \cong A \otimes r(B)$ is simple too.

Hence, C must be simple.

Now begin with $C \subseteq A$ and conclude

$\dim Z_A(C) \cdot \dim C = \dim A. \implies \dim Z_A(C) = \dim B.$

But also $B \subseteq Z_A(C) \implies B = Z_A(C).$

If B is central, then $B \otimes C$ simple.

$$\begin{tikzcd} B \otimes C \arrow[r, "\phi"] & A \\ b \otimes c \arrow[r, mapsto] & bc \end{tikzcd}$$

simplicity $\implies \phi$ injective.

$\dim (B \otimes C) = \dim A \implies B \otimes C \overset{\phi}{\cong} A.$ ■

Remark: A a f.dim central simple k-algebra

$A \otimes_k \bar{k}$ is a simple k-algebra.

but also f.dim ~~and~~ as a $\bar{k}$-algebra

So $A \otimes_k \bar{k} \cong M_n(D)$

$\uparrow$ f.dim division $\bar{k}$-algebra must be $\bar{k}$.

$\Downarrow$

$A \otimes_k \bar{k} \cong M_n(\bar{k}). \implies \dim_k A = \dim_{\bar{k}} (A \otimes_k \bar{k}) = n^2$

Example: D f.dim central simple k-algebra, division

$\cup|$

K maxl subfield $\qquad Z_D(K) = K$ by maximality

$\cup|$

k $\qquad$ Double centralizer thm $\implies$

$$\begin{array}{c} \dim_k D \\ \| \\ (\dim_k K)^2. \end{array}$$

Similarly for A not a division algebra.

Example: $\mathcal{B}(\mathbb{F}_q) = \{\mathbb{F}_q\}$

$\mathcal{B}(\mathbb{C}) = \{\mathbb{C}\}$

Theorem (Frobenius): $\mathcal{B}(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$.

Proof: Let D = fin.dim central $\mathbb{R}$-algebra

$$D \supseteq \underset{\substack{\uparrow \\ \text{maxl} \\ \text{subfield}}}{K} \overbrace{}^{<\infty}\!\!\!\! \supseteq \mathbb{R}$$

If $k = \mathbb{R}$, then $\dim_{\mathbb{R}} D = 1 \Longrightarrow D = \mathbb{R}$

If $k = \mathbb{C}$, $\dim_{\mathbb{R}} D = 4$

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "i"] \arrow[r, shift right, "\text{conjugation}"'] & D \end{tikzcd}$$

By Skolem–Noether,

$z = X\bar{z}X^{-1}$ $\forall z \in \mathbb{C}$, $X \in D \setminus \{0\}$.

But of course X commutes w/ reals,

so $i = X(-i)X^{-1}$ and

(*) $Xi = -iX \Longrightarrow X^2 i = iX^2$.

But $X^2 \in Z_D(\mathbb{C}) \Longrightarrow X^2 \in \mathbb{C}$.

If $X^2 \in \mathbb{C} \setminus \mathbb{R}$, then X commutes with $\underbrace{1, X^2}_{\to \text{span } \mathbb{C}} \Longrightarrow X \in \mathbb{C}$, but that contradicts (*).

So $X^2 \in \mathbb{R}$, and from (*) $X \notin \mathbb{R}$, so must have $X^2 < 0$. by scaling, may assume $X^2 = -1$. Let $j = X$ and $k = ij$.

By this, D must be quaternions. ∎

09/22/14

REPRESENTATION THEORY OF FINITE GROUPS

Work over $\mathbb{C}$

Remark: Over $\mathbb{R}$-algebras, V a simple module, f.dim by Schur, $\mathrm{End}_{\mathbb{R}}(V)$ must be a $\mathbb{R}$-division algebra.

So $\mathrm{End}_{\mathbb{R}}(V) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
if $\mathbb{R}$, a real representation
if $\mathbb{C}$, a complex rep
if $\mathbb{H}$, quaternionic

V is an $\mathrm{End}_{\mathbb{R}}(V)$- module

Notation: G a finite group, $n = |G|$.

$\mathbb{C}G$ is the group-ring, an n-dim $\mathbb{C}$-VS.
also a $\mathbb{C}$-algebra

A $\mathbb{C}G$-module is a ring morphism $\mathbb{C}G \to \mathrm{End}_{\mathbb{C}}(V)$.

$$\begin{tikzcd} \mathbb{C}G \arrow[r, "(\mathbb{C}G\text{-mod})"] & \mathrm{End}_{\mathbb{C}}(V) \\ G \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, "\substack{\text{group}\\ \text{morphism}}"', "(G\text{-rep})"] & \end{tikzcd}$$

$\mathbb{C}G$-modules are G-representations!

Remark: Let V, W be $\mathbb{C}G$-modules, $f: V \to W$ $\mathbb{C}$-linear.

define: $F: V \to W$ by $F(v) = \frac{1}{n}\sum_{g \in G} g \cdot f(g^{-1} \cdot v)$

~~Is it $\mathbb{C}$-linear?~~

$$F(h \cdot v) = \frac{1}{n}\sum_{g \in G} g \cdot f(\underbrace{g^{-1}h}_{=x^{-1}} \cdot v)$$

$$= \frac{1}{n}\sum_{x \in G} g \cdot f(x^{-1} \cdot v) = \frac{1}{n}\sum_{x \in G} \cancel{h}\, x\, f(x^{-1} \cdot v) = hF(v)$$

So F is $\mathbb{C}G$-linear.

Theorem: (Mascke): $\mathbb{C}G$ is semisimple.

Proof: $\mathbb{C}G$ semisimple $\iff \forall V \in \mathbb{C}G$-mod, V semisimple

$\iff \forall V \in \mathbb{C}G$-mod, V completely-reducible.

Let $V \in \mathbb{C}G$-mod, $W \leq V$ ~~semisimple~~ submodule.

$$0 \to W \xhookrightarrow{i} V \twoheadrightarrow V/W \to 0$$ is $\mathbb{C}G$ linear, exact.

i is also $\mathbb{C}$-linear, ~~as~~ a $\mathbb{C}$-VS V has a compliment, and $\exists\, f: V \to W$, $f \circ i = id_W$. Form $F(v) = \frac{1}{n}\sum_{g\in G} g^{-1} f(gv)$

$\uparrow$ $\mathbb{C}G$-linear

$$F \circ i(w) = F(w) = \frac{1}{n}\sum_{g \in G} g \cdot f(g^{-1}w)$$

by defn, $f(g^{-1}w) = f(i(g^{-1}w)) = g^{-1}w$

and hence $F(w) = \frac{1}{n}\sum_{g\in G} g g^{-1} w = w$.

Thus have a $\mathbb{C}G$-linear map $F: V \to W$ so the sequence splits, and $V = W \oplus \ker F$. ∎

Remark: $V, W \in \mathbb{C}G$-mod.

$V \times W \xrightarrow{(\cdot,\cdot)} \mathbb{C}$ bilinear

Then $\langle v, w\rangle := \frac{1}{n}\sum_{g\in G} (g\cdot v, g\cdot w)$ is $\mathbb{C}$-bilinear and also g-invariant, as before.

In particular, $(\cdot,\cdot)$ pos, def $\Rightarrow \langle \cdot,\cdot\rangle$ pos def.

Remark: Let (π, V) be a G-module.

$$\pi : G \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$$

V admits a G-invariant, Hermition, pos. def. bilinear form.

W.R.T. $\langle \cdot , \cdot \rangle$, each $\pi(g)$ is a unitary matrix.

Corollary: (1) $\mathbb{C}G \cong \bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C})$ an algebra morphism

(2) $n = d_1^2 + d_2^2 + \dots + d_N^2$

(3) $\{\mathbb{C}^{d_i}\}_{i=1..N} = \widehat{\mathbb{C}G} \ (= \widehat{G})$ are the simple modules.

(4) $\mathbb{C}G \cong \bigoplus_{i=1}^{N} d_i \mathbb{C}^{d_i}$ as a G-mod

(5) For any $V \in \mathbb{C}G$-mod, $V \cong \bigoplus_{i=1}^{N} n_i \mathbb{C}^{d_i}$

Example:

$$\begin{tikzcd} G \text{ is abelian} \arrow[r, Leftrightarrow] & \mathbb{C}G \text{ abelian} \arrow[r, Leftrightarrow] & \bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C}) \text{ abelian} \arrow[d, Leftrightarrow] \\ & & d_i = 1 \ \forall i \end{tikzcd}$$

Hence $N = n$, and

$$\mathbb{C}G \cong \mathbb{C} \oplus \mathbb{C} \oplus \dots \oplus \mathbb{C}$$

but action on each copy of $\mathbb{C}$ may not be the same, more like

$$\mathbb{C}G \cong (\pi_1, \mathbb{C}) \oplus \dots \oplus (\pi_n, \mathbb{C})$$

$\{\pi_i\}_{i=1..N} = \widehat{G}$ $\qquad \pi_i : G \longrightarrow \mathbb{C}$

$\pi_i(g)$ may as well be assumed to be unitary, so has an inverse $\forall g \in G$.

$$\pi_i : G \xrightarrow{\text{group morphism}} \mathbb{C}^*$$

"multiplicative character"

Suppose $\pi_1, \pi_2 : G \to \mathbb{C}^*$ are group morphisms. The 1 dim $\mathbb{C}G$-modules are isomorphic if and only if $\pi_1 = \pi_2$. All 1-d representations are determined uniquely by their characters.

Given $G \underset{\rho}{\overset{\pi}{\rightrightarrows}} M_d(\mathbb{C}) \cong End_{\mathbb{C}}(\mathbb{C}^d)$

$(\pi, \mathbb{C}) \xrightarrow[\sim]{X} (\rho, \mathbb{C})$ is an isomorphism of representations

iff ~~such that~~ $X(g \cdot v) = g X(v)$

$\pi(g) = X^{-1} \rho(g) X$, $\forall g \in G$.

<u>Example</u>: [redacted] $(\mathbb{Z}/3\mathbb{Z}\, ; +, 0)$

$i = 0,1,2$ $\quad \mathbb{Z}/3\mathbb{Z} \xrightarrow{\pi_i} \mathbb{C}^* \qquad \xi = e^{2\pi i/3}$

$1 \longmapsto \xi^i$

And any representation is a direct sum of 1d representations.

Hence, Any acton of G on V is diagonalizable, if G is abelian.

$\exists$ basis so that $\pi(g)$ is diagonal $\forall g \in G$.

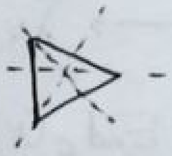
Corollary: Let (π, V) be a $\mathbb{C}G$-module, $g \in G$ abelian. Then, $\pi(g)$ is diagonalizable, and the eigenvalues are roots of 1 of order dividing $|G|$.

Example: S_3 has order **6**.

Thus, the dimensions of simple modules are $6 = 2^2 + 1^2 + 1^2$, (not $6 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$ b/c then it's abelian, but S_3 is not).

$$S_3 \xrightarrow{\text{trivial}} \mathbb{C}^* \qquad S_3 \xrightarrow{\text{sgn}} \mathbb{C}^*$$
$$g \longmapsto 1 \qquad \sigma \longmapsto \text{sgn}(\sigma)$$

$S_3 \cong D_6$, so the symmetries of an equilateral triangle



Hence S_3 acts on the plane by rotations and reflections.

$$\mathbb{C}S_3 \xrightarrow{\pi} M_2(\mathbb{C})$$
$$(1,2,3) \longmapsto \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix}$$
$$(2,3) \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

If $(\pi, \mathbb{C}^2)$ is not simple, then $(\pi, \mathbb{C}^2) = \oplus(1\text{d reps})$ that is, there is a basis of eigenvectors for each $\pi(g)$, $g \in S^3$

For $\pi(2,3)$, eigenvectors are $\binom{1}{0}$ and $\binom{0}{-1}$, but $\pi(123)$ doesn't share these eigenvectors. Hence, cannot be direct sum of 1d reps (b/c then would be diagonal).

Example: Let G be a simple group, nonabelian

Then G has no 1d reps except the trivial one

This is because $\phi : G \longrightarrow \mathbb{C}^*$ is either trivial, or injective. If injective, then an embedding, and $G \cong \operatorname{im}\phi$, and $\mathbb{C}^*$ abelian $\Rightarrow$ G abelian. $\divideontimes$.

Constructions:

(1) Always have trivial representation

$$\mathbb{C}_{triv} : \begin{array}{l} G \longrightarrow \mathbb{C}^* \\ g \longmapsto 1 \end{array}$$

(2) Let (π, V) be a rep. of G.

$$\overline{V} = V \quad \text{as a set with} \quad \begin{array}{l} \mathbb{C} \times \overline{V} \longrightarrow \overline{V} \\ (\lambda, v) \longmapsto \overline{\lambda} v. \end{array}$$

Make $G \xrightarrow{\overline{\pi}} \operatorname{End}_{\mathbb{C}}(\overline{V})$ given $G \xrightarrow{\pi} \operatorname{End}_{\mathbb{C}}(V)$ with $\overline{\pi}(g) := \pi(g)$, extend by $\overline{V}$-linearity thing.

$(\overline{\pi}, \overline{V})$ is a G-module

(3) $V^* = \{ f : V \to \mathbb{C} \mid \mathbb{C}\text{-linear} \}$ $\quad \big(\pi^*(g)(f)\big)(v) = f(\pi(g^{-1})\, v)$

(π^*, V^*) is the dual representation.

(4) $(\overline{\pi}^*, \overline{V}^*)$ is another representation of G, but

$$(\overline{\pi}^*, \overline{V}^*) \cong (\pi, V) \quad \text{and} \quad (\pi^*, V^*) \cong (\overline{\pi}, \overline{V}).$$

Proof: Suffices to show $(\overline{\pi}^*, \overline{V}^*) \cong (\pi, V)$, since $V^{**} \cong V$. (π, V) is a unitary representation of G w.r.t. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$, $\langle \cdot, \cdot \rangle$ induces isomorphism $V \to \overline{V}^*$ $v \mapsto \langle v, \cdot \rangle$. ■ Check that it's G-invariant.

09/24/14

<u>Recall:</u> $\mathbb{C}G$ is a semisimple ring, $\simeq \bigoplus_{i=1}^{N} M_{d_i}(\mathbb{C})$

$\hat{G} = \left\{ (\pi_i, \mathbb{C}^{d_i}) \mid i=1,\dots,N \right\}$ where π_i is the action of G

$\pi_i : G \to GL_{d_i}(\mathbb{C})$

$\pi_i : \mathbb{C}G \to M_{d_i}(\mathbb{C})$.

The Fourier Transform

G is finite and abelian; $d_i = 1$, $N = n$

$\hat{G} = \{\pi : G \to \mathbb{C}^* \text{ group morphism}\}$ are all the irreps

Let $\pi, \rho \in \hat{G}$, gives $(\pi \cdot \rho) : G \to \mathbb{C}^*$ group morphism

$(\pi \cdot \rho)(g) : G \to \pi(g)\rho(g)$

Note that

$\pi \cdot \mathrm{triv} = \mathrm{triv} \cdot \pi = \pi$

$(\pi \cdot \pi^*)(g) = \pi(g)\pi^*(g) = \pi(g)\pi(g^{-1}) = \pi(e) = 1 \implies \pi \cdot \pi^* = \mathrm{triv}.$

So $\hat{G}$ is a group under $\cdot$ w/ triv the identity.

"Pontryagin dual of G".

<u>Exercise:</u> (1) $\hat{\hat{G}} \cong G$

(2) $\mathbb{C}G \simeq \mathcal{F}(G) = \{f : G \to \mathbb{C}\}$

$$\begin{array}{ccc} \mathbb{C}G & = & \bigoplus_{\pi \in \hat{G}} \mathbb{C}_\pi \\ \cong & & \cong \\ \mathcal{F}(G) & = & \bigoplus \mathbb{C} \cdot \pi \end{array}$$

$\uparrow$ any function on G is a linear combination of characters.

Let $f \in \mathcal{F}(G)$. then $f = \sum_{\pi \in \hat{G}} \hat{f}(\pi)\pi$

$\hat{f} : \hat{G} \to \mathbb{C}$, $\hat{f} \in \mathcal{F}(\hat{G})$ is the Fourier Transform!

<u>Recall:</u> (π, V) a G-module. (π^*, V^*) is defined by

$(\pi^*(g) \cdot f)(v) = f(\pi^{-1}(g) \cdot v)$

$(\overline{\pi}, \overline{V}) \cong (\pi^*, V^*)$ as a G-module.

$\overline{V}$ is like V but with $\mathbb{C}$ acting on $w \in \overline{V}$ by $\lambda \cdot w = \overline{\lambda} w$.

Let $g \in G$. There is a basis for V such that $\pi(g) \in \mathrm{End}(V)$ is diagonal ($\langle g \rangle \leq G$ is an abelian group, hence as a $\langle g \rangle$-module, V is just $\oplus \mathbb{C}$).

$$\pi(g) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ In this basis, $\overline{\pi}(g) \cdot v = \pi(g)(v) = \underbrace{\overline{\lambda} \cdot v}_{\in \overline{V}} = \underbrace{\lambda v}_{\in V}$

because $\pi(g)(v) = \lambda v$.

In fact, $g^{\mathrm{ord}(g)} = 1 \Longrightarrow \lambda^{\mathrm{ord}(g)} = 1 \Longrightarrow \overline{\lambda} = \lambda^{-1}$.

Corollary: $\mathrm{tr}_V(\pi(g)) = \overline{\mathrm{tr}_{\overline{V}}(\overline{\pi}(g))}$. One is conjugate of the other.

~~If dim V = 1, then~~ Also $\mathrm{tr}_V(\pi(g)) = \overline{\mathrm{tr}_V(\pi(g^{-1}))}$.

Characters:

"class functions" b/c constant on ~~con~~ conjugacy classes

Defn: $\mathcal{F}(G)_{\text{class}} = \{ f : G \to \mathbb{C} \mid f(xgx^{-1}) = f(g) \ \forall x, g \in G \}$.
$$\begin{matrix} \mathcal{F}(G)_{\text{class}} \\ \cap \\ \mathcal{F}(G) \end{matrix}$$

Defn: the convolution of two functions $f_1, f_2 \in \mathcal{F}(G)$ is

~~[illegible]~~ $$(f_1 * f_2)g = \sum_{x \in G} f_1(gx^{-1}) f_2(x)$$

Let $\delta_1 : G \to \mathbb{C}$ be defined as $\delta_1(g) = \begin{cases} 0 & g \neq 1 \\ 1 & g = 1 \end{cases}$.

$(f * \delta_1)(g) = f(g)$, $(\delta_1 * f)(g) = f(g)$.

$(\mathcal{F}(G), +, 0; *, \delta_1)$ is a $\mathbb{C}$-algebra!

Def: $$G \times \mathcal{F}(G) \xrightarrow{\pi_\ell} \mathcal{F}(G) \qquad G \times \mathcal{F}(G) \xrightarrow{\pi_r} \mathcal{F}(G)$$
$$(g, f) \longmapsto (\pi_\ell(g) \cdot f)(x) = f(g^{-1} \cdot x) \qquad (g, f) \longmapsto f(xg)$$

Exercise: π_ℓ, π_r are $\mathbb{C}$-linear G-actions on $\mathcal{F}(G)$

Def: $G \times \mathbb{C}G \xrightarrow{L} \mathbb{C}G$ $\qquad$ $G \times \mathbb{C}G \xrightarrow{R} \mathbb{C}G$

$(g, x) \longmapsto gx$ $\qquad$ $(g, x) \longmapsto xg^{-1}$ $\quad$ still a left action

$$g_1 \cdot (g_2 \cdot x) = g_2 \cdot (x g_1^{-1})$$
$$= x(g_1 g_2)^{-1} = (g_1 g_2) \cdot x$$

Remark: $\mathcal{F}(G) \cong \mathbb{C}G$

$$f \longleftrightarrow \sum_{x \in G} f(x) x.$$

$$\pi_\ell(g) f \longleftrightarrow \sum_x (\pi_\ell(g) f)(x)\, x = \sum_x f(g^{-1}x)\, x = \sum_y f(y) g y = L(g)\Big(\sum_x f(x) x\Big)$$

$$\pi_r(g) f \longleftrightarrow R(g) \sum_x f(x) x$$

$$\delta_1 \longleftrightarrow 1$$

$$f_1 * f_2 \longleftrightarrow \sum_{x \in G} (f_1 * f_2)(x)\, x = \sum_{x, y \in G} f_1(xy^{-1}) f_2(y) \quad (xy^{-1}) \cdot y$$
$$= \sum_Y \Big(\sum_X f_1(xy^{-1})\, xy^{-1}\Big) f_2(y) y$$
$$= \sum_Y \Big(\sum_z f_1(z) z\Big) f_2(y) y = \Big(\sum_Y f_2(y) y\Big)\Big(\sum_z f_1(z) z\Big)$$

multiplication in the ring $\mathbb{C}G$

[redacted] $\bar{f} \longleftrightarrow$ conjugation

$$f \in \mathcal{F}(G)_{class} \longleftrightarrow \sum_{x \in G} f(x) x$$

constant on conjugacy classes, in the center of $\mathbb{C}G$

$$\sum_x f(x) x = \sum_x f(\underbrace{g x g^{-1}}_{Y}) x = \sum_{Y \in G} f(y)\, g^{-1} y g$$
$$= g^{-1} \Big(\sum_Y f(y) y\Big) g$$

$$\mathcal{F}(G)_{class} \longleftrightarrow Z(\mathbb{C}G)$$

Corollary $\dim \mathcal{F}(G)_{\text{class}} = \dim Z(G)$

$\nearrow$ # of conjugacy classes of G

$\uparrow$ # of summands in $\mathbb{C}G = \bigoplus_{i=1}^{n} M_{d_i}(\mathbb{C})$

$=$ # of central primitive idempotents

$= |\hat{G}|$

Def: Let (π, V) be a finite dimensional G-module.
$\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) = \mathrm{tr}(\pi(g))$ called the "character of (π, V)."

Remark: If $\dim V = 1$, $\chi_\pi = \pi$ is also multiplicative (HM under mult.)

$$\chi_\pi(xgx^{-1}) = \mathrm{tr}_V(\pi(xgx^{-1})) = \mathrm{tr}(\pi(x)\pi(g)\pi(x^{-1})) = \mathrm{tr}(\pi(g)) = \chi_\pi(g).$$

So characters are class functions.

Def: $R(G) = \mathrm{Span}_{\mathbb{C}} \{ [V] \mid V \text{ a } G\text{-module}, [V] \text{ it's IM class} \} / [V]+[W]=[V \oplus W]$
"Grothendieck group of the G-modules."

Called the "Representation ring of G". Why also a ring?

Define $[V] \cdot [W] = [V \otimes W]$, where $V \otimes W$ is a G-module by $g \cdot (v \otimes w) = (gv) \otimes (gw)$

$[\mathbb{C}] = 1$ and $[0] = 0$.

Remark: A basis for $R(G)$ is the set of irreducible representations.

$R(G)_{\mathbb{Z}}$ is the same thing, but the span over $\mathbb{Z}$ instead.

This has basis $\{ (\pi_i, \mathbb{C}^{d_i}) \}_{i=1 \dots N}$.

09/29/14

Recall:

$\mathbb{C}G$ is semisimple

How is it a sum of matrix algebras?

Since we work over $\mathbb{C}$, enough to know their dimensions?

$$\begin{tikzcd}
\mathbb{C}G \arrow[r, "\oplus\pi", "\sim"'] \arrow[rd, bend right, "\pi"'] & \bigoplus_{\pi\in\widehat{G}} M_{d_\pi}(\mathbb{C}) \curvearrowright \mathbb{C}^{d_\pi} \arrow[d, "\mathrm{proj}_\pi"] \\
& M_{d_\pi}(\mathbb{C})
\end{tikzcd}$$

the representation

Simple modules are $\mathbb{C}^{d_\pi}$

$M_{d_\pi}(\mathbb{C}) \curvearrowright \mathbb{C}^{d_\pi}$ by mult.

So the module is acted on by $\bigoplus_{\pi} M_{d_\pi}(\mathbb{C})$ after proj onto π^{th} coordinate

In practice, need to explicitly know the reps themselves!

Another point of view:

$$\begin{tikzcd}
\mathcal{F}(G) \arrow[r, leftrightarrow, "\sim", "\substack{\text{IM of } *\text{-algebras} \\ \text{IM of Hermitian IP space}}"'] & \mathbb{C}G \arrow[r, leftrightarrow, "\oplus\pi", "\sim"'] & \bigoplus_{\tau\in\widehat{G}} M_{d_\pi}(\mathbb{C})
\end{tikzcd}$$

identity

$\delta_1 = \begin{cases} 1 & \text{at identity} \\ 0 & \text{else} \end{cases}$

Def: Let, $f_1, f_2 \in \mathcal{F}(G)$. $\langle f, g\rangle := \frac{1}{n} \sum_{g\in G} f_1(g)\overline{f_2(g)}$.

This is a hermitian inner product.

Def: Let $g, h \in G$. $\langle g, h\rangle = \delta_{gh/n}$ Also a hermitian inner product when we extend to $\mathbb{C}G$.

$\langle\cdot,\cdot\rangle$ on $\mathcal{F}(G)$ corresponds to $\langle g, h\rangle$ on $\mathbb{C}G$, and to the inner product $tr(AB^{\dagger})$ on matrices.

$$\begin{tikzcd}
\mathcal{F}(G) \arrow[r, leftrightarrow, "\sim"] & \mathbb{C}G \arrow[r, leftrightarrow, "\sim"] & \bigoplus_{\pi} M_{d_\pi}(\mathbb{C}) \\
\mathcal{F}(G)_{class} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow] & Z(\mathbb{C}G) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow] & \bigoplus_{\pi} \mathbb{C} I_{d_\pi} = Z\left(\bigoplus_{\pi} M_{d_\pi}(\mathbb{C})\right) \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

$|G/\!/G| = \dim \mathcal{F}(G)_{class} = \dim Z(\mathbb{C}G) = |\hat{G}| = \#$ of irreps.

Notation: $G/\!/G$ is the set of equivalence classes in G under the action $(h,g) \longmapsto hgh^{-1}$.

representation ring of G

$$\begin{tikzcd}
R(G) \arrow[r] & \mathcal{F}(G)_{class} \\
{[V]} \arrow[r, mapsto] & \chi_V
\end{tikzcd}$$

$\{[V_\pi] \mid \pi \in \hat{G}\}$ is a basis for $R(G)$

$\{\chi_\pi \mid \pi \in \hat{G}\}$ is a basis for $\mathcal{F}(G)_{class}$.

$$\begin{aligned}
\mathcal{F}(G)_{class} &\cong \{f : \mathbb{C}G \to \mathbb{C} \text{ linear} \mid f(xgx^{-1}) = f(g)\ \forall g, x \in G\} \\
&= \{f : \mathbb{C}G \to \mathbb{C} \text{ linear} \mid f(xg) = f(gx)\ \forall g, x \in G\} \\
&\cong \{f : \bigoplus_{\pi} M_{d_\pi}(\mathbb{C}) \mid f(ab - ba) = 0\ \forall a, b \in \bigoplus_{\pi} M_{d_\pi}(\mathbb{C})\} \\
&\cong \left(\bigoplus_{\pi \in \hat{G}} M_{d_\pi}(\mathbb{C}) / \mathfrak{sl}_{d_\pi}(\mathbb{C}) \right)^{*} = \bigoplus_{\pi \in \hat{G}} \mathbb{C} \{ \mathrm{trace}_{\mathbb{C}^{d_\pi}}(\cdot) \} \\
&= \bigoplus_{\pi \in \hat{G}} \mathbb{C} \chi_\pi .
\end{aligned}$$

only one linear map on M_{d_π} which vanishes on $\mathfrak{sl}_{d_\pi}$: the trace

Therefore, the map $R(G) \longrightarrow \mathcal{F}(G)_{\text{class}}$ sends a basis

$$[V] \longrightarrow \chi_V$$

to a **basis**! We also know that it's linear, so it's an isomorphism!

<u>Properties</u> <u>of Characters:</u>

(1) $\chi_V \in \mathcal{F}(G)_{\text{class}}$

(2) $\chi_V(1) = \dim V$

(3) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$

(4) $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$

(5) $\chi_V(g^{-1}) = \overline{\chi_V(g)}$

(6) $\chi_{\bar{V}} = \chi_{V^*} = \overline{\chi_V}$.

What is the hermitian structure on $R(G)$?

<u>Def:</u> $\overline{[V]} := [\bar{V}]$, involution on $R(G)$.

Hermition product on $R(G)$

$$\langle [V], [W] \rangle := \dim \underbrace{\mathrm{Hom}_{\mathbb{C}G}(V, W)}_{\substack{\text{HM's which} \\ \text{respect the } \mathbb{C}G \\ \text{structure of } V, W.}}$$

<u>Corollary</u>: A finite dimensional representation is uniquely determined by its character (up to Isomorphism), b/c we have the IM between $\mathcal{F}(G)_{\text{class}}$ and $R(G)$.

<u>Theorem</u> (first orthogonality): $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_{\mathbb{C}G}(V, W)$

<u>Proof:</u> $\langle \chi_V, \chi_W \rangle = \frac{1}{n} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \frac{1}{n} \sum_{g \in G} \chi_V(g) \chi_{W^*}(g)$

$$= \chi_{V \otimes W^*} \left(\frac{1}{n} \sum_{g \in G} g \right)$$

But also $V \otimes W^* \cong \oplus n_i V_i$ for some $n_i \in \mathbb{Z}$, V_i simple $\mathbb{C}G$ modules.

$$= \sum_i n_i \chi_{V_i} \left(\frac{1}{n} \sum_{g \in G} g \right)$$

$$= \underset{\substack{\uparrow \\ \text{the multiplicity of trivial rep.}}}{n_{\text{triv}}}$$

$P: V_i \longrightarrow V_i$

$P: v \longmapsto \frac{1}{n} \sum_g g \cdot v$

$P(V_i) \subseteq V_i$, but V_i simple, so either $P(V_i) = 0$ or $P(V_i) = \mathrm{id}_{V_i}$.

Proof ctd:

Hence
$$\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}_{triv}; V \otimes W^*)$$
$$= \dim \operatorname{Hom}_{\mathbb{C}G}(V, W)$$

Corollary: $\{\chi_\pi \mid \pi \in \widehat{G}\}$ is orthonormal basis for $\mathcal{F}(G)_{class}$.

10/01/14

Recall:

$$\begin{tikzcd}
 & \mathcal{F}(G) \arrow[r, leftrightarrow, "\sim"] & \mathbb{C}G \arrow[r, leftrightarrow, "\overset{\oplus\pi}{\sim}"] & \bigoplus_{\pi \in \widehat{G}} M_{d_\pi}(\mathbb{C}) \\
R(G)_{\mathbb{Z}} & \cup\!| & \cup\!| & \cup\!| \\
\cap\!| & & & \\
R(G) \arrow[r, leftrightarrow, "\sim"] & \mathcal{F}(G)_{class} \arrow[r, leftrightarrow, "\sim"] & Z(\mathbb{C}G) \arrow[r, leftrightarrow, "\sim"] & \bigoplus_{\pi \in \widehat{G}} \mathbb{C} I_{d_\pi}
\end{tikzcd}$$

Orthogonality: $\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}G}(V, W)$

Corollary: $\{\chi_\pi \mid \pi \in \widehat{G}\}$ is an orthonormal basis for $\mathcal{F}(G)_{class}$.

We're really interested in $R(G)_{\mathbb{Z}}$ instead of $R(G)$. What is the corresponding subring of $\mathcal{F}(G)_{class}$?

ON Bases:

$R(G)$ $\{[V_\pi] \mid \pi \in \widehat{G}\}$

$\mathcal{F}(G)_{class}$ $\{\chi_\pi \mid \pi \in \widehat{G}\}$

$Z(\mathbb{C}G)$ $\{e_\pi\}$ ↖ primitive central idempotents

$\bigoplus_\pi \mathbb{C} I_{d_\pi}$ $\{I_{d_\pi} \mid \pi \in \widehat{G}\}$

In fact, $\pm\chi_\pi$ are the unique norm 1 elements in $\mathcal{F}(G)_{\text{class},\mathbb{Z}}$

$f \in \mathcal{F}(G)_{\text{class},\mathbb{Z}}$, $f = \sum_{\pi\in\widehat{G}} n_\pi \chi_\pi$. If $\|f\| = 1$, then

$$1 = \|f\|^2 = \sum_{\pi\in\widehat{G}} n_\pi^2 \Longrightarrow f = \pm\chi_{\pi_0} \text{ for some } \pi_0 \in \widehat{G}.$$

<u>Exercise</u>: Let V be a simple R-module, f. dim
W a simple S-module, f. dim

Then $V \otimes W$ is a simple $R \otimes S$-module

<u>Remark</u>:

$$\begin{aligned} G \times \mathcal{F}(G) &\xrightarrow{\pi_\ell} \mathcal{F}(G) \\ (g, f) &\longmapsto (\pi_\ell(g)f^{-1})(x)\cancel{\}} = f(g^{-1}x) \end{aligned}$$

$$\begin{aligned} G \times \mathcal{F}(G) &\xrightarrow{\pi_r} \mathcal{F}(G) \\ (g, f) &\longmapsto (\pi_r(g)f)(x) = f(xg) \end{aligned}$$

Note that π_ℓ and π_r commute, so can make

$$\begin{aligned} (G \times G) \times \mathcal{F}(G) &\xrightarrow{\pi_\ell \times \pi_r} \mathcal{F}(G) \\ (g_1, g_2; f) &\longmapsto \pi_\ell(g)\pi_r(g)f \end{aligned}$$

Similarly,

$$\begin{aligned} (G \times G) \times \mathbb{C}G &\xrightarrow{L \times R} \mathbb{C}G \\ (g_1, g_2); x &\longmapsto g_1 x g_2^{-1} \end{aligned}$$

} gives $G \times G$-module structure to $\mathcal{F}(G)$ and $\mathbb{C}G$, and they will be isomorphic as $G \times G$ modules.

equivalently, a $G \times G^{op}$ module by

$$(g_1, g_2), x \longmapsto g_1 x g_2$$

or a $\mathbb{C}G$-mod-$\mathbb{C}G$ bimodule.

<u>Aside</u>:

$$R^{op}\text{-mod} \xrightarrow{\sim} \text{mod-}R$$

is an isomorphism of categories.

As a $\mathbb{C}G$-bimodule, $\mathbb{C}G$ is **semisimple**.

So breaks up as direct sum of minimal two-sided ideals, which are exactly $M_{d_\pi}(\mathbb{C})$.

<u>Remark:</u> Let (π, V) be a G-module. Then $\mathrm{End}_{\mathbb{C}}(V)$ becomes a $\mathbb{C}G$-bimodule by the action

$$G \times G \times \mathrm{End}_{\mathbb{C}}(V) \xrightarrow{\rho} \mathrm{End}_{\mathbb{C}} V$$

$$(g_1, g_2, f) \longmapsto \big(\rho(g_1 g_2) f\big)(v) = g_1 \cdot f(g_2^{-1} v)$$

this is a linear $G \times G$-action

<u>Remark</u>: Let (π, V) be a finite dimensional G-module.

$$\begin{tikzcd} V \otimes V^* \arrow[loop left, "\pi\otimes\pi^*"', "G\times G"] \arrow[r, "\sim"] & \mathrm{End}_{\mathbb{C}} V \arrow[loop right, "\rho", "G\times G"'] \end{tikzcd}$$ ~~IS a G IM.~~

$$(v, f) \longrightarrow f_v(\omega) = f(\omega) v$$

Realize the $G \times G$ action on $\mathrm{End}_{\mathbb{C}} V$ by $\pi \otimes \pi^*$

$$\begin{tikzcd}
(v, f) \arrow[r, mapsto] \arrow[d, "\pi\otimes\pi^*"'] & f_v(\omega) \arrow[d, "\rho(g,h)"] \\
(\pi(g)v, \pi^*(h)f) \arrow[r, mapsto] & (\pi^*(h)f)_{\pi(g)v})(\omega) \arrow[d, equal] \\
& (\pi^*(h)f)(\omega)(\pi(g)v) \arrow[d, equal] \\
& g(f(h^{-1}\cdot w)v) \arrow[d, equal] \\
& (\rho(g h) f_v)(\omega)
\end{tikzcd}$$

The diagram commutes!

Prop: $\mathbb{C}G \cong \bigoplus_{\pi \in \widehat{G}} (\pi \otimes \pi^*)$ as a $G \times G$ module.

decomposition into simples (proof follows from previous page)

Notation: $G/\!/G$ is set of conjugacy classes, $[g]$ is conjugacy class of $g \in G$.

Theorem (Orthogonality 2): Let $g, h \in G$. Then,

$$\frac{1}{n} \sum_{\pi \in \widehat{G}} \chi_\pi(g) \overline{\chi_\pi(h)} = \begin{cases} \frac{1}{|[g]|} & \text{if } h \in [g] \\ 0 & \text{otherwise} \end{cases}$$

Proof:

$$\frac{1}{n} \sum_{\pi \in \widehat{G}} \chi_\pi(g) \overline{\chi_\pi(h)} = \frac{1}{n} \sum_{\pi \in \widehat{G}} \chi_\pi(g) \chi_{\pi^*}(h)$$

$$= \frac{1}{n} \sum_{\pi \in \widehat{G}} \chi_{\pi \otimes \pi^*}(g,h) = \frac{1}{n} \chi_{\bigoplus_{\pi \in \widehat{G}} \pi \otimes \pi^*}(g,h)$$

$$= \frac{1}{n} \chi_{\mathbb{C}G}(g,h) = \frac{1}{n} \operatorname{tr}(x \longmapsto gxh^{-1})$$

We can compute the trace of this map!

Only contributes to trace when $x = gxh^{-1} \iff g = xhx^{-1}$

$$= \frac{1}{n} \sum_{\substack{x \in G \\ x = gxh^{-1}}} 1 = \frac{1}{n} \#\{x : g = xhx^{-1}\} = \begin{cases} \frac{1}{n} |Z_G(g)| \\ 0 & \text{if } h \notin [g] \end{cases}$$

but by orbit-stabilizer, $|Z_G(g)| = \frac{|G|}{|[g]|}$, so

$$= \begin{cases} \frac{1}{|[g]|} & \text{if } g \in [h] \\ 0 & \text{otherwise.} \end{cases}$$

■

Corollary: $e_\pi \cdot \mathbb{C}G \simeq M_{d_\pi}(\mathbb{C})$ and $\mathbb{C}G \xrightarrow{\pi} e_\pi \mathbb{C}G$.

and $\chi_\pi = tr(\pi)$ when realized in this way.

Corollary: $e_\pi = \sum_{g \in G} \overline{\chi_\pi(g)}\, g$

Proof: $x e_\pi x^{-1} = \sum_{g \in G} \overline{\chi_\pi(g)}\, x g k^{-1} = \sum_{h \in G} \overline{\chi_\pi(g)}\, h$

χ_π is a class function!

$$= \sum_{h \in G} \overline{\chi_\pi(h)}\, h = e_\pi.$$

Let $\rho \in \hat{G}$

$$\begin{array}{ccc} V_\rho & \xrightarrow{F_{\pi,\rho}} & V_\rho \\ v & \longmapsto & \rho(e_\pi)v \\ \rho(x)v & \longmapsto & \rho(e_\pi)\rho(x)v = \rho(x)\rho(e_\pi)v \end{array}$$

$$F_{\pi,\rho}(\rho(x)\cdot v) = \rho(x)\, F_{\pi,\rho}(v)$$

$$F_{\pi,\rho} \in End_{\mathbb{C}G}(V_\rho) \underset{\text{Schur's Lemma}}{\Longrightarrow} F_{\pi,\rho} = C_{\pi\rho}\, Id_{V_\rho}$$

$$\begin{aligned} \chi_\rho(e_\pi) &= trace(F_{\pi,\rho}) = C_{\pi,\rho}\, d_\rho \\ &= \frac{d_\pi}{n} \sum_{g \in G} \overline{\chi_\pi(g)}\, \chi_\rho(g) \\ &= d_\pi \langle \chi_\pi, \chi_\rho \rangle \\ &= d_\pi \delta_{\pi,\rho} \end{aligned}$$

So $C_{\pi,\rho} = \delta_{\pi,\rho}$

Hence, multiplication by e_π is 0 on V_ρ for $\rho \neq \pi$, but identity on V_π. So e_π corresponds to I_{d_π}, proj. onto π-component.

Character Tables:

conjugacy classes

$$\begin{array}{c|cccc} G & [1] & [g] & \cdots & \\ \hline \text{triv} & 1 & 1 & \cdots & \\ \pi & d_\pi & \chi_\pi(g) & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \end{array}$$

irreducible representations

Note that
$|\hat{G}| = |G/\!/G|$
because $\mathcal{F}(G)_{\text{class}} \cong R(G)$,
$\{\chi_\pi \mid \pi \in \hat{G}\}$ is basis for $\mathcal{F}(G)_{\text{class}}$,
but $\{\delta_{[g]}\}$ is also a basis for $\mathcal{F}(G)_{\text{class}}$.

rules:

- $n = \sum_{\pi \in \hat{G}} d_\pi^2$
- $\chi_\pi(g) =$ sum of $|g|^{\text{th}}$ roots of unity $\leftarrow$ algebraic integers $\chi_\pi(g) \in \overline{\mathbb{Z}} \subseteq \overline{\mathbb{Q}}$
  d_π many of them $\leftarrow$ $\notin \mathbb{Q} \setminus \mathbb{Z}$
- $\frac{1}{n} \sum_{g \in G} \chi_\pi(g) \overline{\chi_\rho(g)} = \delta_{\pi,\rho}$

  (row orthogonality) $\hookrightarrow \frac{1}{n} \sum_{[g] \in G/\!/G} |[g]| \, \chi_\pi(g) \overline{\chi_\rho(g)} = \delta_{\pi,\rho}$
- $\frac{[g]}{n} \sum_{\pi \in \hat{G}} \chi_\pi(g) \overline{\chi_\pi(h)} = \delta_{[g];[h]}$

  (column orthogonality) $\hookrightarrow \frac{[g]}{n} \sum_{\pi \in \hat{G}} \chi_\pi([g]) \overline{\chi_\pi([h])}$

Does every possible character table correspond to a group?

No, there are even categories that look like categories of representations but do not correspond to groups. (fusion categories)

6	1	3	2
S_3	[(1)]	[(12)]	[(123)]
triv	1	1	1
sgn	1	−1	1
$\mathbb{C}^2$	2	0	−1

Notation:

cycle type for permutations

essentially where the dots go under permutation.

Also $\ydiagram{1,1,1}$ $\ydiagram{2,1}$ $\ydiagram{3}$

To find the way the group acts given the character table:

$$\chi_\pi \rightsquigarrow \underset{\mathbb{C}G}{\overset{\in}{e_\pi}} = \frac{d_\pi}{n} \sum_{g \in G} \overline{\chi_\pi(g)}\, g$$

characters correspond to primitive central idempotents.

Then $\mathbb{C}Ge_\pi \cong M_{d_\pi}(\mathbb{C})$, and so get action of G on $\mathbb{C}^{d_\pi}$. $M_{d_\pi}(\mathbb{C})$ G $\mathbb{C}^{d_\pi}$ in a natural way,

How a tensor product of representations breaks up into simples is unknown for S_n; Kronecker coefficients

	1	6	3	8	6
S_4	$\ydiagram{1,1,1,1}$	$\ydiagram{2,1,1}$	$\ydiagram{2,2}$	$\ydiagram{3,1}$	$\ydiagram{4}$
triv	1	1	1	1	1
sgn	1	−1	1	1	−1
$\mathbb{C}^2$	2	0	2	−1	0
$\mathbb{C}^3$	3	1	−1	0	−1
$\mathbb{C}^3 \otimes \mathrm{sgn}$	3	−1	−1	0	1

this column must be $\perp$ to column $\ydiagram{2,1,1}$

this column: only way to get $a^2+b^2+c^2 = 24-1^2-1^2$ is $a=2, b=c=3$.

get $\chi_{\mathbb{C}^3}(\ydiagram{2,1,1})$ by column orthogonality with itself

orthogonality relations w/ triv row and $\mathbb{C}^2$ row.

tensoring representations gives another rep, and if we tensor with 1D representations, a simple stays simple. So $\mathbb{C}^2 \otimes \mathrm{sgn} \cong \mathbb{C}^2$, thus has same character. Hence, $\chi_{\mathbb{C}^2}(\ydiagram{2,1,1})$ is

$$= \chi_{\mathbb{C}^2}(\ydiagram{2,1,1})\,\chi_{\mathrm{sgn}}(\ydiagram{2,1,1})$$
$$= -\chi_{\mathbb{C}^2}(\ydiagram{2,1,1})$$

$\chi_{\mathbb{C}^2}(\ydiagram{2,1,1}) = 0$.

$D_{10} \longrightarrow$ symmetries of ⬠, generated by a, b subject to

$$a^2=1,\ b^5=1,\ aba^{-1}=b^{-1}$$

Conjugacy classes

$[1]$ $\quad [a]=\{a, ab^3, ab, ab^2, ab^4\}$ $\quad [b]=\{b, b^4\}$

$[b^2]=\{b^2, b^3\}$

10	1	2	2	5
D_{10}	$[1]$	$b^{\pm 1}$	$b^{\pm 2}$	reflections
triv	1	1	1	1
χ_1	1	1	1	-1
[illegible] $=\chi_2$	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1+-\sqrt{5}}{2}$	0
$\overline{\chi_3}=\chi_1\otimes\chi_3=\chi_3$	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0

$x\bar{x}+z\bar{z}+1+1=5$

$x\bar{x}+z\bar{z}=3$

$y\bar{y}+w\bar{w}=3$

$2+2x+2y=0$

$x+y=-1$

$x=-1-y$

$4+2x\bar{x}+2y\bar{y}=10$

~~2+2x[illegible]~~

$x\bar{x}+y\bar{y}=3$

$$\left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{-1-\sqrt{5}}{2}\right)$$

$$\frac{1}{4}(1+\sqrt{5}-\sqrt{5}-5)=\frac{1}{4}(-4)=-1$$

χ_2 is action on the plane, $D_{10} \circlearrowright \mathbb{C}^2$

χ_3 is another action on the plane

$$\begin{aligned} D_{10} &\longrightarrow D_{10} \circlearrowright \mathbb{C}^2 \\ a &\longmapsto a \\ b &\longmapsto b^2 \end{aligned}$$

Integrality Properties

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \overline{\mathbb{Q}}$$

$$\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}.$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, "\cap" rotate=90] & \mathbb{Q} \arrow[d, phantom, "\cap" rotate=90] \\ \overline{\mathbb{Z}} \arrow[r, phantom, "\subseteq"] & \overline{\mathbb{Q}} \end{tikzcd}$$

$\uparrow$ algebraic integers

$\hookleftarrow$ algebraic numbers

Remark:

$$\begin{tikzcd} \mathcal{F}_{Glass}(G) \arrow[r, dash, "\sim"] & Z(\mathbb{C}G) \\ \chi_\pi \arrow[r, mapsto] & \sum_{g\in G} \overline{\chi_\pi(g)}\, g = \frac{n}{d_\pi} e_\pi \end{tikzcd}$$

If $C \in G/\!/G$, $e_C = \sum_{g\in C} g \in Z(\mathbb{C}G)$

Note that $\{e_C \mid C \in G/\!/G\}$ is a linearly independent set in $Z(G)$, and so since there are $|G/\!/G|$ of them, then this is a basis for $Z(G)$. $\{e_\pi \mid \pi \in \hat{G}\}$ is also a basis.

Change of basis matrix is just the character table!

Let $(\pi, V_\pi) \in \hat{G}$.

$$\begin{tikzcd} V_\pi \arrow[r, "F_C"] & V_\pi \\ \vec{v} \arrow[r, mapsto] & e_C \cdot \vec{v} \end{tikzcd}$$

let $x \in G$

$$F_C(x\cdot\vec{v}) = (e_C \cdot x)\cdot \vec{v} = x \cdot (e_C \cdot \vec{v})$$
$$= x \cdot F_C(\vec{v})$$

$$F_C \in \mathrm{End}_{\mathbb{C}G}(V_\pi) \Rightarrow F_C = k\, \mathrm{Id}_{V_\pi}$$

$\hookleftarrow$ Schur's Lemma.

$$|C| \cdot \chi_\pi(C) = \sum_{g\in C} \chi_\pi(g) = \mathrm{trace}(F_C) = k d_\pi$$

$$k = \frac{|C|}{d_\pi} \chi_\pi(C) \qquad e_C \cdot \vec{v} = k\vec{v}$$

$\hookleftarrow$ Eigenvalue of e_C.

$\hookrightarrow$ $k = k_{C,\pi}$ depends on C, π.

$\mathbb{C}G \xrightarrow{e_C} \mathbb{C}G$. In the basis $B = \{g \mid g \in G\}$, the matrix of e_C is a sum of permutation matrices ($e_C = \sum_{g \in G} g$) and in particular it has $\mathbb{Z}$-entries.

Thus, the characteristic polynomial of e_C is in $\mathbb{Z}[\lambda]$, monic. Hence, eigenvalues of e_C are algebraic integers.

$\Downarrow$

eigenvalues of e_C are the $k_{C,\pi} = \frac{|C|}{d_\pi} \chi_\pi(C) \in \overline{\mathbb{Z}}$.

~~Therefore, since $\chi_\pi(C)$ is an algebraic integer, $\frac{|C|}{d_\pi} \in \overline{\mathbb{Z}}$ too, but $|C|/d_\pi$ is an integer, so $|C|/d_\pi \in \mathbb{Z}$.~~

<u>Remark</u>:

$$1 = \frac{1}{n} \sum_{g \in G} \chi_\pi(g) \overline{\chi_\pi(g)}$$

$$= \frac{d_\pi}{n} \sum_{C \in G//G} \frac{|C|}{d_\pi} \chi_\pi(C) \overline{\chi_\pi(C)}$$

$$= \frac{d_\pi}{n} \sum_{C \in G//G} k_{\pi,C} \overline{\chi_\pi(C)}. \Rightarrow \frac{n}{d_\pi} = \underbrace{\sum_{C \in G//G} k_{\pi,C} \overline{\chi_\pi(C)}}_{\in \overline{\mathbb{Z}}}$$

<u>Theorem (Frobenius)</u>: $d_\pi \mid n$

<u>Proof</u>: $\frac{n}{d_\pi} \in \overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$. $\blacksquare$

Lemma 1: If $\overbrace{(|C|; d_\pi)}^{GCD} = 1$ then $\chi_\pi(C) = 0$

or $\pi(g) = \frac{\chi_\pi(g)}{d_\pi} Id_{V_\pi}$ $\forall g \in C$.

Proof: $1 = a|C| + b d_\pi$, $a, b \in \mathbb{Z}$.

$$\Longrightarrow \frac{1}{d_\pi}\chi_\pi(C) = a\frac{|C|}{d_\pi}\chi_\pi(C) + b\chi_\pi(C)$$

$$= a k_{C,\pi} + b\chi_\pi(C) \in \overline{\mathbb{Z}}.$$

$\chi_\pi(C) = \chi_\pi(g)$ $\forall g \in C$.

[redacted] $\chi_\pi(g)$ is sum of $|g|^{th}$ roots of unity.

Hence $\frac{1}{d_\pi}(\Sigma \text{ roots of unity}) \in \overline{\mathbb{Z}}$.

Call it α.

If $\chi_\pi(C) \neq 0$ then $\alpha \neq 0$.

α is a root of $f \in \mathbb{Z}[x]$ monic, irreducible.

A conjugate of α is (by defn) a root of its irreducible poly, f.

$\{\beta : \beta \text{ conjugate to } \alpha\} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \alpha$.

$\Longrightarrow$ conjugates of α are also of the form $\frac{1}{d_\pi}\left(\text{sum of roots of 1 of order } |g|\right)$.

$$\prod_{\substack{\text{conjugates} \\ \beta \text{ of } \alpha}} (x - \beta) = f \Longrightarrow \underset{\mathbb{Z}^*}{\underset{\cap}{\text{constant term of } f}} = \prod_{\substack{\beta \text{ conj.} \\ \text{of } \alpha}} \beta$$

By triangle inequality, $\frac{1}{d_\pi}\left(\text{sum of } |g|^{th} \text{ roots of } 1\right) \leq 1$, but it's a positive integer, so $\frac{1}{d_\pi}\left(\text{sum of } |g|^{th} \text{ roots of } 1\right) = 1$. $\Longrightarrow |\alpha| = 1$, so all roots of 1 that appear in sum are the same $\Longrightarrow \pi_g$ is diagonal. ■

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Lecture 15. ①

Algebra 3

Integrality

Recall.
- $\chi_\pi(g) \in \overline{\mathbb{Z}}$. ~~a~~ algebraic integers
- $\frac{|C|}{d_\pi} \chi_\pi(C) \in \overline{\mathbb{Z}}$: $C \in G/\!/G$. algebraic integers
- $\frac{n}{d_\pi} \in \overline{\mathbb{Z}} \Rightarrow d_\pi \mid n$.

algebraic integer but $\mathbb{Q} \not\subset \overline{\mathbb{Z}} \Rightarrow \frac{n}{d_\pi} \in \mathbb{Z}$

Lem 1. $(|C|, d_\pi) = 1$ then either $\begin{cases} \chi_\pi(C) = 0 \\ \text{or } \pi(g) = \frac{\chi_\pi(C)}{d_\pi} \mathrm{Id}_{V_\pi}, \forall g \in C. \end{cases}$

Lem 2. Let $|C| = p^k$ (p prime). Then $\underset{\pi \neq \text{triv}}{\exists \pi \in \widehat{G}}$ such that $\pi(g) = \frac{\chi_\pi(C)}{d_\pi} \mathrm{Id}_{V_\pi}, \forall g$

Proof. Need $\pi \in \widehat{G}$ such that $\begin{cases} p \nmid d_\pi. \\ \chi_\pi(C) \neq 0. \end{cases}$ $(\overset{\text{Lem 1}}{\Longrightarrow}$ QED).

Orthogonality 2 for C. ~~C~~ 1; ~~1~~

$$0 = \sum_{\pi \in \widehat{G}} \chi_\pi(C) \cdot d_\pi.$$

$$= 1 + \underbrace{p \sum_{\substack{\text{triv} \neq \pi \in \widehat{G} \\ p \mid d_\pi}} \overbrace{\chi_\pi(C)}^{\in \overline{\mathbb{Z}}} \cdot \overset{\in \mathbb{Z}}{\frac{d_\pi}{p}}}_{\in \overline{\mathbb{Z}}} + \underbrace{\sum_{\substack{\text{triv} \neq \pi \in \widehat{G} \\ p \nmid d_\pi}} \chi_\pi(C) d_\pi}_{X}$$

If $X = 0$, $-\frac{1}{p} = \sum_{\substack{\text{triv} \neq \pi \in \widehat{G} \\ p \mid d_\pi}} \chi_\pi(C) \frac{d_\pi}{p} \in \overline{\mathbb{Z}}$ Contradiction!

So $X \neq 0$. $\Rightarrow \exists \pi \neq \text{triv}$, such that $\begin{cases} p \nmid d_\pi \\ \chi_\pi(C) \neq 0. \end{cases}$ □

Lem 3. Let $|C| = p^k$ (p prime, $k \neq 1$), then $\exists \pi \in \widehat{G}$. $\pi \neq \text{triv}$, such that $\{1\} \neq \ker(\pi) \leq$

Proof. Let π as in Lemma 2.

$$\{1\} \neq \langle gh^{-1} \mid g, h \in C \rangle. \xrightarrow{\pi} \mathrm{Id}_{V_\pi}$$

$$\pi(gh^{-1}) = \pi(g) \cdot \pi(h)^{-1} = \left(\frac{\chi_\pi(C)}{d_\pi}\right) \cdot \left(\frac{d_\pi}{\chi_\pi(C)}\right) \mathrm{Id}_{V_\pi} = \mathrm{Id}$$

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Lecture 15. ②

Algebra

Cor If G simple group, then there are no conjugacy classes of of size p^k $(k \geq 1, p$

Thm (Burnside) $|G| = p^a q^b \Rightarrow G$ solvable. (or complicate proof without character theory)

Proof Assume there are G which are not solvable.
Let G smallest, not solvable.
If $H \trianglelefteq G$, $H \neq G$, then $|H|, |G/H| < |G| \Rightarrow H, G/H$ solvable.
$\Rightarrow G$ solvable. Contradiction!

Hence G simple. $\Rightarrow$ cannot have conjugacy class of size p^k, q^t, $k, t \geq$

$$G \times G \xrightarrow{\text{conjugation}} G.$$
$$(g, x) \longmapsto gxg^{-1}.$$

$Z(G) \trianglelefteq G$

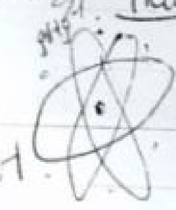
$$p^a q^b = |G| = 1 + \underbrace{\sum_{|C| \neq 1} |C|}_{\text{divisible by } pq}.$$

Contradiction!

$g \in G$, $|C| = \dfrac{|G|}{|\mathrm{stab}_G(g)|}$

Therefore. the claim is true. $\square$

(HW2)



Thm (Frobenius): Let $G \geq H$ such that $gHg^{-1} \cap H = \begin{cases} H & \text{or} \\ \{1\} \end{cases}$. $\forall g \in G$.
Then $K := \{1\} \cup (G \setminus \bigcup_{g \in G} gHg^{-1}) \trianglelefteq G$. and $G \cong K \rtimes H$

Equivalent statement. $G \circlearrowright X$ transitively (i.e. one element) and only 1 fixes 2 (or more points).
Then G is a semidirect product.

Q $X = G \cdot x_0$. $H = \mathrm{stab}_G(x_0)$, $gHg^{-1} = \mathrm{stab}(g \cdot x_0)$.
$K = \{1\} \cup \{g \mid g \text{ has no fixed points}\}$.

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Algebra 3

Lecture 15. ③

Restriction and Induction

$S \subseteq R$ subring ring

$$\begin{tikzcd} R\text{-mod} \arrow[r, shift left, "\mathrm{res}_S^R"] & S\text{-mod} \arrow[l, shift left, "\mathrm{ind}_S^R"] \\ R\otimes_S M \arrow[d, "\mathrm{id}\otimes_S f"'] & M \arrow[l, mapsto] \arrow[d, "f"] \\ R\otimes_S N & N \arrow[l, mapsto] \end{tikzcd}$$

$$R \times (R\otimes_S M) \longrightarrow R\otimes_S M$$
$$(r, a\otimes m) \longmapsto ra \otimes m.$$

Prop (Frobenius reciprocity).

robenius Reciprocity.

$$R\text{-mod} \underset{\mathrm{ind}}{\overset{\mathrm{res}}{\rightleftarrows}} S\text{-mod}$$
$$\overset{\cup\!\!\!\!|}{M} \qquad\qquad \overset{\cup\!\!\!\!|}{N}$$

$$\mathrm{Hom}_{R\text{-mod}}(\mathrm{ind}(N); M) \underset{\beta}{\overset{\alpha}{\underset{\sim}{\rightleftarrows}}} \mathrm{Hom}_{S\text{-mod}}(N, \mathrm{res}(M))$$

functorial bijection. $\|$

$$\| \qquad\qquad \mathrm{Hom}_{S\text{-mod}}(N, M)$$

$$\mathrm{Hom}_{R\text{-mod}}(R\otimes_S N; M)$$

$$f \longmapsto \alpha(f)~~(m)~~. \qquad \alpha(f)(m) = f(1\otimes_S 1$$

(S-linear)

$$\alpha(f)(sn) = f(1\otimes sn) = f(s\otimes n)$$
$$= f(s\cdot(1\otimes n)) = s f(1\otimes m) = s\,\alpha(f)n$$

$$\beta(g)(r\otimes n) = rg(n) \longleftarrow g$$

Proof. Ex. $\square$.

$$H \leq G, \quad \mathbb{C}H \leq \mathbb{C}G. \qquad \mathbb{C}G\text{-mod} \underset{\mathrm{ind}}{\overset{\mathrm{res}}{\rightleftarrows}} \mathbb{C}H\text{-mod}.$$

$$(\pi, V_\pi) \in \hat{G} \qquad\qquad (\rho, W_\rho) \in \hat{H}.$$

$$\mathrm{res}(V_\pi) = \bigoplus_{\nu\in\hat{H}} m_\nu^\pi \in \mathbb{C}H\text{-mod}.$$

$$\mathrm{ind}(W_\rho) = \bigoplus_{\tau\in\hat{G}} n_\rho^\tau V_\tau \in \mathbb{C}G\text{-mod}.$$

$\tau\neq\pi$ gives 0.

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathbb{C}G}\left(\bigoplus_{\tau\in\hat{G}} n_\rho^\tau V_\tau ; d_\pi\right)\right) = n_\rho^\pi$$

$v\in V\oplus V$

V

$\in$

$f = [f_1 \ldots f$

$f\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = f_1(v_1$

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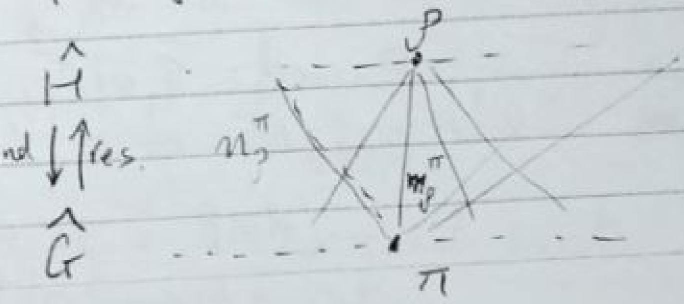
Lecture 15. ②

Thm. (Frobenius Reciprocity). $m_\rho^\pi = n_\rho^\pi$.

Bratelli diagram



Recall: $H \leq G$

$$\begin{tikzcd} \mathbb{C}G\text{-mod} \arrow[r, shift left, "res"] & \mathbb{C}H\text{-mod} \arrow[l, shift left, "ind"] \end{tikzcd}$$

(π, V_π) $\qquad$ (ρ, W_ρ)

res and ind form an adjoint pair of functors.

$$res(V_\pi) = \bigoplus_{\nu \in \hat{H}} n^\pi_\nu V_\nu$$

$$n^\pi_\nu = n^\pi_\rho$$

$$ind(W_\rho) = \bigoplus_{\pi \in \hat{G}} n^\pi_\rho V_\pi$$

$$\begin{tikzcd} R(G) \arrow[r, dash, "\sim"] \arrow[d, shift right, "res"'] & \mathcal{F}(G)_{class} \arrow[r, dash, "\sim"] \arrow[d, shift right, "res"'] & Z(\mathbb{C}G) \arrow[d, shift right, "res"'] \\ R(H) \arrow[r, dash, "\sim"] \arrow[u, shift right, "ind"'] & \mathcal{F}(H)_{class} \arrow[r, dash, "\sim"] \arrow[u, shift right, "ind"'] & Z(\mathbb{C}H) \arrow[u, shift right, "ind"'] \end{tikzcd}$$

Notation; superscript G or H to denote the group of character / class function / idempotent.

$$\begin{tikzcd} (\pi, V) \arrow[r] \arrow[d, "res"] & \chi^G_\pi = trace_V(\pi^G) \\ (res(\pi), V) & \chi^H_{res(\pi)}(h) = \chi^G_\pi(h) \end{tikzcd}$$

just restrict the domain!

Let $\chi \in \mathcal{F}(G)_{class}$, a character maps to a central idempotent by

$$\chi \longmapsto \sum_{g \in G} \overline{\chi(g)}\, g \in Z(\mathbb{C}G)$$ an idempotent up to scaling by $\frac{d_\pi}{|G|}$

This is only an $\mathbb{R}$-linear map, not $\mathbb{C}$.

Recall:

(1) Hermitian product on $\mathcal{F}(G)_{class}$: $\langle f_1, f_2 \rangle_{\mathbb{C}G} = \frac{1}{|G|} \sum f_1(x) \overline{f_2(x)}$

$$\langle f_1, f_2 \rangle_{\mathbb{C}G} = \frac{1}{|G|} \sum_{g \in G} f_1(x) \overline{f_2(x)} .$$

$$\| \chi_\pi \|^2 = 1 \quad \text{for all } \pi \in \widehat{G}$$

(2) $\mathbb{C}G \cong \mathbb{C}^{|G|}$ as a $\mathbb{C}$ vector-space

$\langle \cdot , \cdot \rangle_{\mathbb{C}G}$ is standard inner product on $\mathbb{C}^{|G|}$.

$\mathbb{C}H \cong \mathbb{C}^{|H|}$ has inner product $\langle \cdot , \cdot \rangle_{\mathbb{C}H}$ the standard inner product on $\mathbb{C}^{|H|}$, the restriction of $\langle \cdot , \cdot \rangle_{\mathbb{C}G}$

$\{ f_\pi^G \mid \pi \in \widehat{G} \} \longrightarrow$ orthogonal basis of $Z(\mathbb{C}G)$

$f_\pi^G = \sum_{g \in G} \overline{\chi_\pi(g)} g$ $\nearrow$ $\quad \| f_\pi^G \|^2 = |G|$.

$\{ f_\rho^H \mid \rho \in \widehat{H} \}$ orthogonal basis of $Z(\mathbb{C}H)$ and $\| f_\rho^H \|^2 = |H|$.

Recover χ_π from f_π^G?

$$\chi_v(g) = \langle g, f_\pi^G \rangle_{\mathbb{C}G} = \langle g, \sum_{g \in G} \overline{\chi_\pi(g)}\, g \rangle = \chi_\pi(g).$$

(3) Restriction of f_π^G?

$$Z(\mathbb{C}G) \xrightarrow{res} Z(\mathbb{C}H).$$

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "\text{res}"] & f_x^G = \sum_{g\in G} \overline{x(g)}\, g \arrow[d, "\text{projection onto } H"] \\
x\restriction_H \arrow[r] & f_x^H = \sum_{h\in H} \overline{x(h)}\, h
\end{tikzcd}$$

$$Z(\mathbb{C}G) \xrightarrow[\text{project}]{} Z(\mathbb{C}H)$$

(4) $$\left.\begin{aligned} \text{res}(f_\pi^G) &= \sum_{\nu\in\hat{H}} n_\nu^\pi f_\nu^H \\ \text{ind}(f_\rho^H) &= \sum_{\tau\in\hat{G}} p^\tau f_\tau^G \end{aligned}\right\}$$ in the basis $\{f_\pi^G\}_{\pi\in\hat{G}}$ or the basis $\{f_\rho^H\}_{\rho\in\hat{H}}$

What does induction look like in standard basis?

Guess: $$Z(\mathbb{C}H) \longrightarrow Z(\mathbb{C}G)$$
$$\mu \longmapsto \frac{1}{|G|}\sum_{g\in G} g\mu g^{-1}$$

Natural thing to do because we want ind(μ) to be invariant under conjugation by elts of g, since $\in Z(\mathbb{C}G)$

Call this guess $\widetilde{\text{ind}}(\mu)$.

Suppose μ is a basis elt of H, $g\in H$

$\langle \widetilde{\text{ind}}(\mu), g\rangle_{\mathbb{C}G} =$

$$= \frac{1}{|G|}\sum_{x\in G} \langle x\mu x^{-1}, g\rangle_{\mathbb{C}G} = \frac{1}{|G|}\sum_{x\in G}\langle \mu, xgx^{-1}\rangle_{\mathbb{C}G} = \langle \mu, \widetilde{\text{ind}}(g)\rangle_{\mathbb{C}G}$$

By linearity, this extends to when μ is not a basis elt.

$$\langle \widetilde{\text{ind}}(\mu), g\rangle_{\mathbb{C}G} = \langle \mu, \widetilde{\text{ind}}(g)\rangle_{\mathbb{C}G}.$$

Remark: $\langle \widetilde{\text{ind}}(f_\rho^H), f_\pi^G \rangle_{\mathbb{C}G}$ will be the f_π^G coefficient of $\widetilde{\text{ind}}(f_\rho^H)$ in $\{f_\pi^G\}$-basis.

$$\langle \widetilde{\text{ind}}(f_\rho^H), f_H^G \rangle_{CG} = \langle f_\rho^H, \widetilde{\text{ind}}(f_\pi^G) \rangle_{CG}$$

$$= \langle f_\rho^H, f_\pi^G \rangle_{\mathbb{C}G} = \langle f_\rho^H, \text{res}(f_\pi^G) \rangle_{\mathbb{C}H} = n_\rho^\pi |H|.$$

$\uparrow$ b/c f_H^G is already in center

$\uparrow$ since the components of f_π^G that are not in span of H will vanish anyway.

Similarly,

$$\langle \text{ind}(f_\rho^H); f_\pi^G \rangle_{\mathbb{C}G} = n_\rho^\pi |G|$$

So $$\text{ind}(f_\rho^H) = \frac{|G|}{|H|} \widetilde{\text{ind}}(f_\rho^H).$$

Thus, $$\text{ind}: Z(\mathbb{C}H) \longrightarrow Z(\mathbb{C}G)$$

$$\mu \longmapsto \frac{1}{|H|} \sum_{g \in G} g \mu g^{-1}.$$

Theorem (Frobenius): Let (ρ, W) be a representation of $H \leq G$. Then $$\chi_{\text{ind}(\rho)}^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_\rho(x^{-1}gx).$$

Proof of Theorem: $\chi^G_{ind(\rho)}(g) = \langle g, f^G_{ind(\rho)} \rangle_{\mathbb{C}G} = \langle g, ind(f_\rho^H) \rangle_{\mathbb{C}G}$

$$= \frac{1}{|H|} \langle g, \sum_{x \in G} x f_\rho^H x^{-1} \rangle_{\mathbb{C}G}$$

$$= \frac{1}{|H|} \langle \sum_{x \in G} x^{-1} g x, f_\rho^H \rangle_{\mathbb{C}G}$$

one of these is equal to its projection onto H, so the next equality follows.

$$= \frac{1}{|H|} \langle \sum_{\substack{x \in G \\ x^{-1}gx \in H}} x^{-1} g x, f_\rho^H \rangle_{\mathbb{C}H} = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_\rho^H (x^{-1} g x). \quad \blacksquare$$

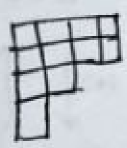
Representations of S_n

Conjugacy classes of S_n are in bijection w/ portions of n.

$$S_n // S_n \xleftrightarrow{\sim} \{ \lambda \vdash n \}$$

Def: if $(\lambda_1, \lambda_2, \ldots)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + \ldots = n$, then $(\lambda_1, \lambda_2, \ldots)$ is a partition of n, $\lambda \vdash n$.

Also write it as $\lambda =$



$= 4+4+2+1$ is a partition of 11.

Call λ a shape or [illegible].

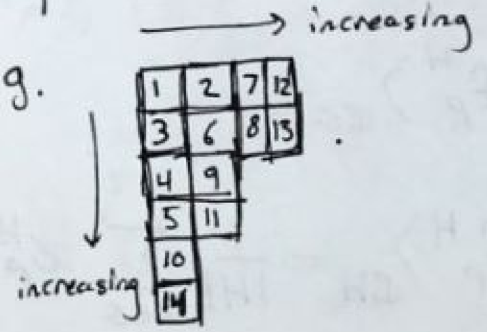
For $\lambda \vdash n$, a tableaux of shape λ is

$$T : \{1, 2, \ldots, n\} \xrightarrow{\sim} \text{boxes in } \lambda.$$

A standard (Young) tableaux is increasing left to right, and top to bottom.

$SYT(\lambda)$ is the set of standard young tableaux of shape λ.

e.g.



$$S^{\lambda} = \text{Span}_{\mathbb{C}} \{ V_T \mid T \in SYT(\lambda) \}$$

These become precisely the irreps of S_n.

10/20/14

If $H \leq G$, V a $\mathbb{C}G$-module, naturally also a $\mathbb{C}H$-module

If (ρ, W) is a $\mathbb{C}H$-module, make a $\mathbb{C}G$-module by $\mathbb{C}G \otimes_{\mathbb{C}H} W$. This is a representation of G.

$$\mathbb{C}G \otimes_{\mathbb{C}H} W \cong \{ f : G \to W \mid \pi_{\ell}(h) \cdot f = \rho(h) f \ \forall h \in G \}$$

$$\begin{tikzcd}
G \arrow[d, phantom, "\rotatebox{90}{$\leq$}"] & R(G) \arrow[r, phantom, "\simeq"] \arrow[d, shift right, "res"'] & \mathcal{F}(G)_{class} \arrow[r, phantom, "\simeq"] \arrow[d, shift right, "res"'] & Z(\mathbb{C}G) \arrow[d, shift right, "res"'] \\
H & R(H) \arrow[r, phantom, "\simeq"] \arrow[u, shift right, "ind"'] & \mathcal{F}(H)_{class} \arrow[r, phantom, "\simeq"] \arrow[u, shift right, "ind"'] & Z(\mathbb{C}H) \arrow[u, shift right, "ind"']
\end{tikzcd}$$

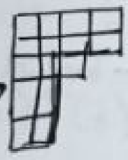
↑ restriction is restriction of domain

↑ res is projection
ind is some averaging

Can use this to find character tables, such as the character table for D_{2n}, which is $\cong C_n \rtimes C_2$

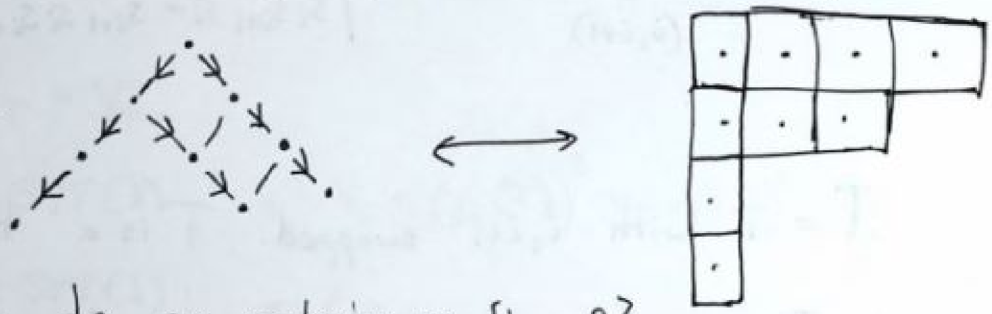
Representations of S_n

Recall: partitions of n, $\lambda \vdash n$.

- $T : \{1, \dots, n\} \longrightarrow$  is a tableaux of shape λ.

- $T \in SYT(\lambda) \iff$ increasing top to bottom, left to right

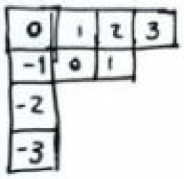
Draw as a poset



Corresponds to an ordering on $\{1, \dots, 9\}$.
An increasing function $\{1, \dots, 9\}$ to the shape λ.

Def: Pick a box in λ, $\square \in \lambda$.

contents in λ



"content" · $c(\square) = \text{column}(\square) - \text{row}(\square)$

· ~~$h(\square) = \#\{b \in \lambda \mid \square \leq b\}$~~
↑
~~poset order~~

"arm" · $a(\square)$ = # of boxes to the right, in same row

"leg" · $\ell(\square)$ = # of boxes below in same column

"hook" · $h(\square) = a(\square) + \ell(\square) + 1$

If $T \in Syt(\lambda)$, $T(i)$ is the box that contains i.

Warning: $c(T(i)) = col(T(i)) - row(T(i))$.

Let $\lambda \vdash n$. Then

$$|SYT(\lambda)| = \frac{n!}{\prod_{\square \in \lambda} h(\square)}$$

"hook length formula"

Def: $\lambda \vdash n$, $S^{\lambda} = \mathbb{C}\langle V_T \mid V_T \in SYT(\lambda)\rangle$.

Will construct an action of S_n on S^λ, making it into an S_n-module.

braid group is same w/out $s_i^2=1$.

Remark: $$S_n = \left\langle \underset{\substack{\downarrow \\ (i,i+1)}}{s_i},\ 1\le i\le n-1 \;\middle|\; \begin{array}{l} s_i^2=1 \\ s_is_j = s_js_i \text{ if } |i-j|\ge 2 \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \ \ |i-j|=1 \end{array} \right\rangle$$

Def: • $s_iT = T$ with $i, i+1$ swapped. T is a tableaux

• Let T be a standard Tableaux of shape λ.

convention: $V_{s_iT} = 0$ if $s_iT \notin SYT(\lambda)$

Example: $\lambda = \ydiagram{3,1}$ $$S^{\ydiagram{3,1}} = \mathbb{C}V_{\substack{123\\4}} \oplus \mathbb{C}V_{\substack{124\\3}} \oplus \mathbb{C}V_{\substack{134\\2}} \cong \mathbb{C}^3$$

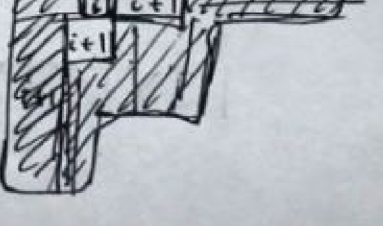
$\gamma_i(T) = \dfrac{1}{c(T(i+1)) - c(T(i))} \in \mathbb{Q}$. Denominator never vanishes b/c content constant on diagonal, yet $i, i+1$ never on same diagonal.

Define: $s_i \cdot V_T = \gamma_i(T)V_T + (1+\gamma_i(T))V_{s_iT}$.

Theorem: $\{S^\lambda \mid \lambda \vdash n\} = \widehat{S}_n$.

Proof: • S^λ is S_n-mod ← need to check relations are satisfied.

s_iT is not $SYT(\lambda)$ if $i+1$ ~~is in~~ is to the right directly, directly below, ~~or in the region below and left~~.



Theorem: $\{S^\lambda \mid \lambda \vdash n\} = \hat{S}_n$.

Need to check S^λ is an S_n-mod.

$$s_i T \notin SYT(\lambda) \iff T(i), T(i+1) \text{ in adjacent diagonals}$$
$$\iff c(T(i)) - c(T(i+1)) = \pm 1.$$

Check: $s_i^2 V_T = V_T$.

if $s_i T \notin SYT(\lambda)$, $s_i^2 \cdot V_T = (\gamma_i(T))^2 V_T = (\pm 1)^2 V_T = V_T$

if $s_i T \in SYT(\lambda)$: $s_i(\mathbb{C}\langle V_T, V_{s_iT}\rangle) = \mathbb{C}\langle V_T, V_{s_iT}\rangle$

in the basis for the subspace, s_i has matrix

[illegible] = [illegible]

$$s_i \longleftrightarrow \begin{bmatrix} \gamma_i(T) & 1+\gamma_i(s_iT) \\ \gamma_i(T)+1 & \gamma_i(s_iT) \end{bmatrix} = \begin{bmatrix} \gamma_i(T) & 1-\gamma_i(T) \\ \gamma_i(T)+1 & -\gamma_i(T) \end{bmatrix}$$

The square of this matrix is the identity. ✓

Check: $s_i s_j V_T = s_j s_i V_T \quad |i-j| \geq 2$.

This is true because exchanging i and $i+1$ and $j, j+1$ is the same no matter which order.

⟶

Check: $s_i s_{i+1} s_i \cdot V_T = s_{i+1} s_i s_{i+1} \cdot V_T$

$s_i, s_{i+1}, s_{i+1}s_i$ correspond to 6×6 matrices. in the basis

$$\langle V_T, V_{s_iT}, V_{s_{i+1}T}, V_{s_is_{i+1}T}, V_{s_{i+1}s_iT}, V_{s_is_{i+1}s_iT} \rangle.$$

Matrix of s_i is A_i

Matrix of s_{i+1} is A_{i+1}

$A_i =$ ~~$\gamma_i(T)$~~ ~~$1+\gamma_i(T)$~~ ~~$\gamma_i(s_{i+1}T)$~~ ~~$1+\gamma_i(s_{i+1}T)$~~

$$\left[\begin{array}{cc|cc|cc} \gamma_i(T) & 1-\gamma_i(T) & 0 & 0 & 0 & 0 \\ 1+\gamma_i(T) & -\gamma_i(T) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \gamma_i(s_{i+1}T) & 1-\gamma_i(s_{i+1}T) & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & 1+\gamma_i(s_{i+1}T) & -\gamma_i(s_{i+1}T) & & \\ \hline 0 & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \multicolumn{2}{c}{\multirow{2}{*}{\boxed{\cancel{A}}}} \\ 0 & 0 & & & & \end{array}\right]$$

$$\begin{array}{|cc} \gamma_i(s_{i+1}s_iT) & 1-\gamma_i(s_{i+1}s_iT) \\ 1+\gamma_i(s_{i+1}s_iT) & -\gamma_i(s_{i+1}s_iT) \end{array}$$

A_{i+1} is similar, looks like

$$\begin{matrix} x & o & x & o & o & o \\ o & x & o & o & x & o \\ x & o & x & x & o & x \\ o & o & o & o & o & o \\ o & x & o & o & x & o \\ o & o & o & x & o & x \end{matrix}$$

Can show that ~~it's the inverse of~~

$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1}.$$

Now we know that S^λ is an S_n module for every $\lambda \vdash n$. Need to check that S^λ and S^μ are non-isomorphic.

Then need to show that S^λ is irreducible.

Finally, show $|\{\lambda \vdash n\}| = |S_n//S_n| \Rightarrow \{S^\lambda\}_{\lambda\vdash n} = \widehat{S_n}$.

10/22/14

Recall: $\lambda \vdash n$, $SYT(\lambda)$, $S^\lambda = \mathbb{C}\langle v_T \mid T \in SYT(\lambda)\rangle$

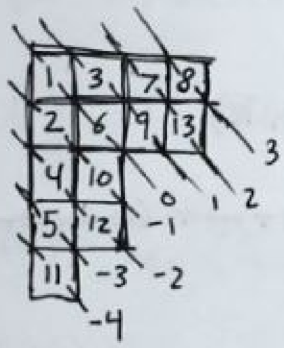
Define action $s_i \cdot v_T = \gamma_i(T)\, v_T + (1+\gamma_i(T))\, v_{s_i T}$ for~~ea~~ simple transpositions $s_i = (i \ i{+}1)$

$$\gamma_i(T) = \frac{1}{c(T(i+1)) - c(T(i))}$$

$$c(\square) = \text{col}\,\# - \text{row}\,\#.$$

Last time: S^λ is an S_n-module

Remark:



$\longmapsto$ $(0,-1,1,-2,-3,0,2,3,1,-1,-4,-2,2)$

List $c(T(i))$ for $i = 1 \ldots 13$

$$T \longmapsto \{c(T(i))\}_{i=1}^{n}$$

Sequence Completely determines the tableaux, so this map is injective.

Theorem $\hat{S}_n = \{S^\lambda \mid \lambda \vdash n\}$

Proof: (continued)

Claim: $S^\lambda \neq S^\mu$ if $\lambda \neq \mu$.

Define $m_k = (1,k) + (2,k) + \cdots + (k-1,k) \in \mathbb{C}S_n$ for $k \in \{2,\ldots,n\}$. (Tueys-Murphy elements.)

Proposition: $m_k \cdot v_T = c(T(k))\, v_T$

$\longrightarrow$

Proof of prop: By induction on k.

$k=2, \quad m_2 = (1,2) = s_1 \qquad s_1 \cdot V_T = \gamma_1(T) V_T + (1+\gamma_1(T)) V_{s_i T}$

But $s_i T$ is not a valid SYT, so $V_{s_i T} = 0$

$$s_1 \cdot V_T = \gamma_1(T) V_T = \frac{1}{c(T(2)) - c(T(1))} V_T$$

$$c(T(1)) = 0, \quad c(T(2)) = \pm 1, \text{ so } \gamma_1(T) = c(T(2)).$$

if $k>2$,

$$m_k = (1,k) + (2,k) + \dots + (k-1,k)$$

~~$= s_{k-1} \cdot ((1,k-1) + (2,k-1) + \dots + (k-2,k-1)) + s_{k-1}$~~

~~$= s_{k-1} m_{k+1} + s_k$~~

~~$m_k \cdot V_T = (s_{k-1} m_{k-1} + s_{k-1}) \cdot V_T$~~

~~$= s_{k-1} \cdot c(T(k-1)) V_T + s_{k-1} V_T$~~

~~$= \frac{c(T(k-1))}{c(T(k)) - c(T(k-1))} V_T + c(T(k-1)) \cdot \left(1 + \frac{1}{c(T(k)) - c(T(k-1))}\right) V_{s_{k-1}T}$~~

~~$+ s_{k-1} V_T$~~

~~$= \left(\frac{c(T(k))}{c(T(k)) - c(T(k-1))} - 1\right) V_T$~~

~~$(c(T(k-1)) + 1)(s_{k-1} V_T) = (c(T(k-1)) + 1)\left(\frac{1}{c(T(k)) - c(T(k-1))} V_T\right.$~~

~~$\left.+ \frac{c(T(k)) - c(T(k-1)) + 1}{c(T(k)) - c(T(k-1))} V_{s_k T}\right)$~~

Proof of prop:

for $k>2$

$$m_k = (1,k)+(2,k)+\dots+(k-1,k)$$

$$= s_{k-1}\big((1,k-1)+(2,k-1)+\dots+(k-2,k-1)\big)s_{k-1} + s_{k-1}$$

$$= s_{k-1}m_{k-1}s_{k-1} + s_{k-1}$$

$$m_k V_T = \left(s_{k-1}m_{k-1}s_{k-1} + s_{k-1}\right)\cdot V_T$$

$$= s_{k-1}m_{k-1}\left(\gamma_{k-1}(T)V_T + (1+\gamma_{k-1}(T))V_{s_{k-1}T}\right) + s_{k-1}V_T$$

$$= s_{k-1}\left(\gamma_{k-1}(T)\,c(T(k-1))\,V_T + (1+\gamma_{k-1}(T))\,c(T(k))\,V_{s_{k-1}T}\right) + s_{k-1}V_T$$

$$= s_{k-1}\left(V_T\,(1+c(T(k))\cdot\gamma_{k-1}(T)-1) + V_{s_{k-1}T}\cdot(1+\gamma_{k-1}(T))\,c(T(k))\right)$$

$$= c(T(k))\cdot s_{k-1}(s_{k-1}V_T)$$

$$= c(T(k))\,V_T. \qquad \blacksquare$$

Remark: $\{V_T\}_{T\in SYT(\lambda)}$ are common eigenvectors for $\{m_2,\dots,m_n\}$ with eigenvalues $c(T(i))\,V_T = m_i V_T$.

This sequence of eigen values determines T! (and also λ).

Therefore $S^\lambda \cong S^\mu \Rightarrow \lambda=\mu$.

(look at eigenvalues of $m_2,\dots,m_n$ on S^λ and S^μ, which must be the same $\rightsquigarrow$ recover shape.

Claim: S^λ is irreducible.

Show it has no proper submodules.

Remark: Fix $T \in SYT(\lambda)$

$$P_T = \prod_{i=2}^{n} \prod_{\substack{S \in SYT(\lambda) \\ c(T(i)) \neq c(S(i))}} \frac{m_i - c(S(i))}{c(T(i)) - c(S(i))}$$

$P_T V_T = 1$ $\qquad$ P_T is projection onto space spanned by V_T.

$T \neq R$ $\qquad P_T V_R = 0$

$P_T V_S = 0$

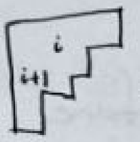
Let $0 \subsetneq V \subseteq S^\lambda$ a submodule, simple.

$$0 \neq \sum_{T \in SYT(\lambda)} a_T V_T \in V \quad , ~~a_T \neq 0~~$$

Fix S such that $a_S \neq 0$.

$$P_S\left(\sum a_T V_T\right) = a_S V_S \in V.$$

Def: (lowering).

Let $T \in SYT(\lambda)$, $i \in \{1, \ldots, n\}$ such that 

$c(T(i+1)) < c(T(i)) - 1$

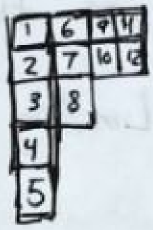
call $s_i T$ lowering of T.

similarly define s_i a raising of T. if $c(T(i+1)) > c(T(i)) + 1$.

Lowering algorithm: Given T, find smallest i that can be lowered, and then replace T by $s_i T$, repeat.

This algorithm terminates and produces the "column reading tableaux of shape λ.

Column-reading tableaux



<u>Def</u>: $\tau_i \cdot V_T = s_i V_T - \gamma_i(T) V_T = (1+\gamma_i) V_{s_i T}$.

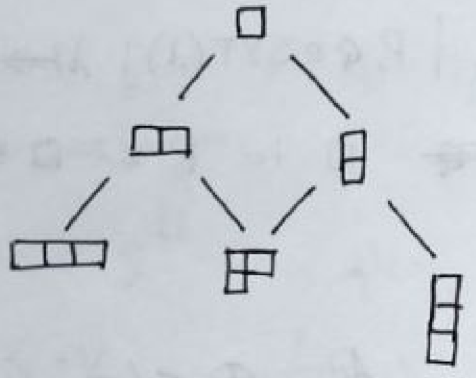
<u>Remark</u>: if $V_T \in V$, then $V_{s_i T} \in V \leftarrow$ b/c V is simple.

Therefore, $V_{\substack{\text{col reading} \\ \text{tableaux}}} \in V$, and then by raising, can get any tableaux we want. So $V_S \in V \ \forall S \in SYT(\lambda)$.

Hence $V = S^\lambda$, so S^λ is **simple**. ▨

<u>Theorem</u>: $\hat{S}_n = \{ S^\lambda \mid \lambda \vdash n \}$. (Young's Seminormal Construction)

<u>Young Lattice</u>: set of all partitions of n with partial order given by "inclusion"



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<u>Note</u>: Matrices in Young's seminormal construction may not be orthogonal. There's also

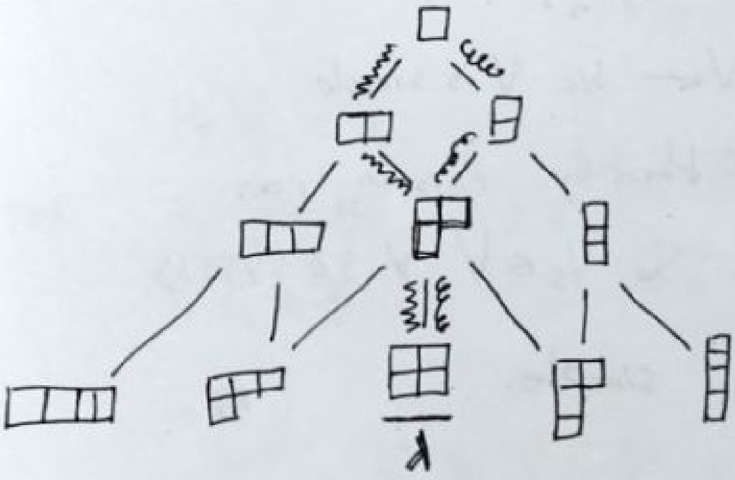
- Young's Orthogonal Construction (matrices orthogonal)
- Young Symmetrizers (Etingof's Book)
- Springer Representations (a geometric version of the same thing).

Other ways to see the representations of S_n.

- Kazhdan–Lusztig construction (on graphs)

Young Lattices:

Put an order on the partitions by "containment"



Consider paths in the lattice. Corresponds to putting a number in the next box that appears.

Results in Standard Tableaux.

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ ← squiggly path 〰

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ ← curly path ℓℓℓ

Path Algebra: $\mathcal{A}_n = \mathbb{C}\langle E_{P,Q} \mid P, Q \in SYT(\lambda);\ \lambda \vdash n \rangle$

elements are paths from ~~□ to Q~~ □ to P or ▣ to Q

$$E_{P,Q}\, E_{S,T} = \delta_{QS}\, E_{P,T}$$

$\dim \mathcal{A}_n = n!$

$$\mathcal{A}_n = \bigoplus_{\lambda \vdash n} \mathbb{C}\langle E_{P,Q} \mid P, Q \in SYT(\lambda)\rangle$$

↑ invariant subspace under multiplication.

$$\simeq \bigoplus_{\lambda \vdash n} M_{|SYT(\lambda)|}(\mathbb{C})$$

↑ looks like a matrix ring!

$$\mathcal{A}_n = \bigoplus_{\lambda \vdash n} \mathbb{C}\langle E_{P,Q} \mid P, Q \in SYT(\lambda)\rangle$$

$$\simeq \bigoplus_{\lambda \vdash n} M_{|SYT(\lambda)|}(\mathbb{C})$$ ← exactly the irreps of S_n!

$$\simeq \mathbb{C} S_n$$

$\mathcal{A}_n$ acts on S^λ by

$$\left.\begin{array}{l} \mathcal{A}_n \times S^\lambda \longrightarrow S^\lambda \\ E_{P,Q} \cdot V_T \longmapsto \delta_{Q,T} V_P \end{array}\right\} \text{An irreducible representation of } S_n.$$

<u>Corollary:</u> $$\underbrace{\sum_{\lambda \vdash n} |SYT(\lambda)|^2}_{\substack{\uparrow \\ \text{sum of} \\ \chi_\lambda(1)^2\text{, first col} \\ \text{of character} \\ \text{table}}} = \underbrace{n!}_{\leftarrow \text{ size of group } S_n}$$

We should have a bijection between $\{(P,Q) \mid P, Q \in SYT(\lambda)\} \lambda \vdash n\}$ and S_n. Called the RSK correspondence. (Robinson, Schensted, Knuth).

<u>Corollary:</u> $\chi_\lambda([g]) \in \mathbb{Z}$

<u>Example:</u> $S^{\begin{array}{|c|c|c|c|}\hline \ & \ & - & \ \\ \hline\end{array}}$. Is a one-dimensional vector space, with action given by $S_i \cdot V_{\begin{array}{|c|c|c|}\hline 1 & 2 & \cdots n \\ \hline\end{array}} = \frac{1}{i-(i-1)} V_{\begin{array}{|c|c|c|}\hline 1 & 2 & \cdots n \\ \hline\end{array}}$ is the trivial representation.

$S^{\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}$ is also 1 dimensional with action

$$s_i \cdot V_{\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}} = \frac{1}{-i-(1-i)} V_{\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array}} = -V_{\begin{array}{c} 1 \\ 2 \\ | \\ n \end{array}}$$ is the sign representation. (since it's -1 on the simple transpositions).

<u>Def:</u> Let $\lambda \vdash n$. Then $A_{\lambda+\rho} = \det \begin{bmatrix} x_1^{\lambda_1+n-1} . x_2^{\lambda_1+n-1} \cdots\cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \\ x_1^{\lambda_n} \cdots\cdots & x_n^{\lambda_n} \end{bmatrix}$

Example: $\lambda = 0$,

$$A_\rho = \det \begin{bmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ \vdots & & \vdots \\ x_1^1 & \cdots & x_n^1 \end{bmatrix} = \prod_{i<j} (x_i - x_j)$$ "Vandermonde Determinant"

$$S_\lambda = \frac{A_{\lambda+\rho}}{A_\rho} \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}.$$

Schur function/polynomial ↑ symmetric polynomials in $x_1, \dots, x_n$

Def: $P_k = x_1^k + \cdots + x_n^k \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$; $P_\lambda = \prod P_{\lambda_i}$

$\{P_\lambda\}$ forms a basis for symmetric polynomials.

Def: Frobenius Characteristic

$$FC: R(S_n) \longrightarrow \mathbb{Q}[x_1, \dots, x_n]^{S_n}.$$

$$[V] \longmapsto \frac{1}{n!} \sum_{\mu \vdash n} \chi_V(C_\mu)\, p_\mu\, |C_\mu|.$$

↑ conjugacy class corresponding to the partition μ

Note:

$$|C_\mu| = \frac{\mu!}{1^{\mu_1} \cdot \mu_1!\, 2^{\mu_2} \mu_2! \cdots n^{\mu_n} \mu_n!}$$

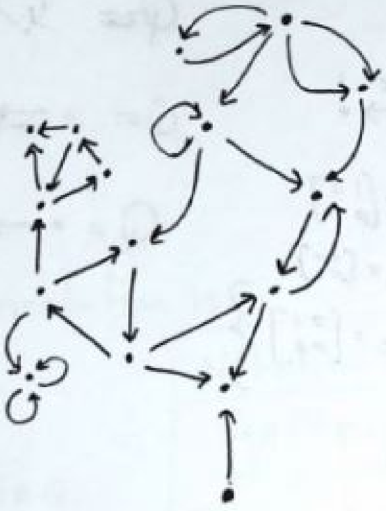
Fact: (1) $FC(S^\lambda) = S_\lambda$ ← Schur Function

(2) $$S_\lambda = \sum_{\mu \vdash n} \chi_\lambda(C_\mu) \frac{P_\mu}{1^{\mu_1}\mu_1!\, 2^{\mu_2}\mu_2! \cdots n^{\mu_n} \mu_n!}$$

(3) $$P_\lambda = \sum_{\mu \vdash n} \chi_\lambda(C_\mu)\, S_\mu$$

Quiver Algebras

A Quiver is an oriented graph on a finite vertex set, finitely many edges, possibly multiple edges and loops.



Def: $Q = (V, E, t, h)$

vtxs edges tail head

$t, h : Q \longrightarrow V$

$t(\underset{a}{\bullet} \longrightarrow \underset{b}{\bullet}) = b$

$h(\underset{a}{\bullet} \longrightarrow \underset{b}{\bullet}) = a$

For $i \in V$, e_i = trivial edge starting and ending at i.

Kronecker Quiver

Jordan Quiver

Def: (Path Algebra) Q is a quiver, $P_Q = \mathbb{C}\langle a_{\overleftarrow{p}} \mid \overleftarrow{p} \text{ is oriented path in } Q \rangle$

A path is a sequence of edges, read right to left.

The algebra structure is given by

$$a_{\overleftarrow{p}} \cdot a_{\overleftarrow{q}} = \begin{cases} a_{\overleftarrow{pq}} & \text{(concatenate paths) if } h(\overleftarrow{q}) = t(\overleftarrow{p}) \\ 0 & \text{otherwise.} \end{cases}$$

The identity path is $\sum p_i$, the path that doesn't go anywhere.

Notation: $p_i = a_{e_i}$ is the length zero path.

Remark: generators: p_i for $i \in V$, a_e for $e \in E$.

relations: $a_e p_i = \delta_{i,t(e)} a_e$ $\quad p_i p_j = \delta_{ij} p_i$

$p_i a_e = \delta_{ih(e)} a_e$

Examples: $Q = \bullet \;\; \bullet \;\; \bullet \qquad P_Q = \mathbb{C}^3$

$Q =$  $\qquad P_Q = \mathbb{C}[x]$

$Q =$  $\qquad \mathbb{C}\langle x, y\rangle$

noncommutative polynomials in two variables.

$$\cong \begin{bmatrix} a & 0 & x & 0 \\ 0 & a & 0 & y \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix}, \quad a, b, x, y \in \mathbb{C}.$$

$Q = \bullet \rightrightarrows \bullet \qquad P_Q = \begin{bmatrix} \mathbb{C} & \mathbb{C}\oplus\mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$

$Q = \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad P_Q = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{bmatrix}$

$Q = \overset{2}{\bullet} \xrightarrow{a_e} \overset{1}{\bullet}$

$$P_Q = \mathbb{C}\left\{ p_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; p_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; a_e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$$

$$\begin{pmatrix} a & \begin{pmatrix} x \\ y \end{pmatrix} \\ 0 & b \end{pmatrix} \begin{pmatrix} c & \begin{pmatrix} w \\ z \end{pmatrix} \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & \begin{pmatrix} aw+xd \\ az+yd \end{pmatrix} \\ 0 & \end{pmatrix}$$

$$\begin{pmatrix} a & 0 & x & 0 \\ 0 & a & 0 & y \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} c & 0 & w & 0 \\ 0 & c & 0 & z \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 & aw+xd & d \\ 0 & ac & [illegible] & \\ 0 & 0 & bd & 0 \\ 0 & 0 & 0 & bd \end{pmatrix}$$

Remark: Let M be a P_Q-module.

$$1 = \sum_{i \in V} p_i \implies M = \bigoplus_{i \in V} p_i M$$

$$\begin{tikzcd} \bigoplus_{i \in V} p_i M \arrow[d, "p_i"] \arrow[dr, "a_e"] & \\ p_i M & a_e\, p_{t(e)} M \subseteq p_{h(e)}\, a_e M \subseteq p_{h(e)} M \end{tikzcd}$$

$$a_e\, p_{t(e)} M = a_e M$$

Action of p_i is projection on to i^{th} component.

Action of a_e is moving between components.

Conversely, let $\begin{cases} M = \bigoplus_{i \in Q_0} V_i \\ (f_e)_{e \in Q_1}\,; \; f_e : V_{t(e)} \longrightarrow V_{h(e)} \end{cases}$

Then M is a P_Q-module with the structure

- p_i is projection onto i^{th} component
- a_e is $f_e \circ p_{t(e)}$

$$\begin{tikzcd} M \arrow[r, "a_e"] \arrow[d, "p_{t(e)}"'] & M \\ V_{t(e)} \arrow[r, "f_e"] & V_{h(e)} \arrow[u, hook] \end{tikzcd}$$

Satisfies the relations for a P_Q-mod

$$\begin{cases} a_e p_i = f_e \circ p_{t(e)} \circ p_i = \delta_{i,t(e)}\, a_e \\ p_i a_e = \delta_{i,h(e)}\, a_e \\ p_i p_j = \delta_{ij}\, p_j \end{cases}$$

Def: A Quiver representation is $((V_i)_{i\in Q_0}; (f_e)_{e\in Q_1})$ such that $f_e : V_{t(e)} \longrightarrow V_{h(e)}$ is a vector space morphism.

$$\overset{V_1}{\bullet} \xrightarrow{f_e} \overset{V_2}{\bullet}$$

Def: The dimension of a quiver representation is

$$\dim\left((V_i)_{i\in Q_0}; (f_e)_{e\in Q_1}\right) = (\dim V_i)_{i\in Q_0}$$

Notation: $\mathrm{Rep}(Q,\alpha)$ = Q-modules of dimension α

Def: A morphism of Quiver representations is $\phi: ((V_i)_{i\in Q_0}; (f_e)_{e\in Q_1}) \longrightarrow ((W_i)_{i\in Q_0}; (g_e)_{e\in Q_1})$, a collection of maps $V_i \xrightarrow{\phi_i} W_i$ such that the diagram

$$\begin{tikzcd}
V_{t(e)} \arrow[r, "\phi_{t(e)}"] \arrow[d, "f_e"'] & W_{t(e)} \arrow[d, "g_e"] \\
V_{h(e)} \arrow[r] & W_{h(e)}
\end{tikzcd}$$

commutes.

Def: a sub-(quiver rep) is $(W_i) \leq (V_i)$, W_i subspace of V_i. Maps are the restrictions of the maps of bigger module.

Def: ϕ is an ~~[illegible]~~ automorphism of quiver representations if $\phi_i \in GL(V_i)$ for all i and it's a quiver-rep morphism.

Identify V_i with $\mathbb{C}^{\dim V_i}$.

So we can say an isomorphism is $\phi: ((\mathbb{C}^{n_i})_{i\in Q_0}; (f_e)_{e\in Q_1}) \rightarrow ((\mathbb{C}^{n_i})_{i\in Q_0}; (g_e)_{e\in Q_1})$

$\Phi_i \in GL(n_i)$ and

$$g_e \phi_{t(e)} = \phi_{h(e)} f_i .$$

Question: classify quiver representations of fixed dimension up to isomorphism.

Isomorphism Classes correspond to orbits of $\prod_{i\in Q_0} GL(n_i)$ action.

Representations of dimension $(n_i)_{i\in Q_0}$: $\circlearrowright \prod_{e\in Q_1} M_{n_{h(e)};n_{t(e)}}(\mathbb{C})$

Say $(A_e)_{e\in Q_1} \in \prod_{e\in Q_1} M_{n_{h(e)};n_{t(e)}}(\mathbb{C})$

$(S_i)_{i\in Q_0} \in \prod_{i\in Q_0} GL(n_i)$

Then

$$(S_i)_{i\in Q_0} \cdot (A_e)_{e\in Q_0} = (S_{h(e)} A_e S_{t(e)}^{-1})_{e\in Q_1} .$$

Examples:

$$Q = \begin{tikzcd} n \arrow[loop above] \end{tikzcd}$$

$P_Q = \mathbb{C}[x]$

$M_n(\mathbb{C}) \leftarrow$ reps of dimension n

Isomorphic if matrices are similar.

$S \times A \mapsto SAS^{-1}$.

So the (modules over) representations of $\mathbb{C}[x]$ are basically the Jordan Canonical Forms (for fixed dimension n).

Exercise: representations of $\bullet \longrightarrow \bullet$?

$$\begin{tikzcd} \underset{\mathbb{C}^n}{\bullet} \arrow[r] & \underset{\mathbb{C}^m}{\bullet} \end{tikzcd}$$

$GL_n \times GL_m \circlearrowright M_{m\times n}(\mathbb{C})$

$(S, T) \cdot A \longmapsto TAS^{-1}$.

So TAS^{-1} can be written as $\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}$; each IM class. basically corresponds to rank(A).

11/03/14

Let Q be a quiver, $\underline{M} = ((M_i)_{i\in Q_0} ; (f_e)_{e\in Q_1})$ a Q-module.

by choosing bases, realize $\underline{M}$ as $((\mathbb{C}^{\dim M_i})_{i\in Q_0} ; (M(f_e))_{e\in Q_1})$

$\uparrow$ matrix representation

$\operatorname{Rep}(Q,\underline{\alpha}) = \bigoplus_{e\in Q_1} M_{h(e)\times t(e)}(\mathbb{C})$ has a natural action of $\prod_{i\in Q_0} GL_{\alpha_i}(\mathbb{C})$.

$$(S_i)_{i\in Q_0} \cdot (A_e)_{e\in Q_1} = \cancel{(S_{h(e)} | A_e | S_{t(e)}} \left(S_{h(e)} A_e S_{t(e)}^{-1}\right)_{e\in Q_1}$$

$$\operatorname{Rep}(Q,\alpha)/_{Iso.} \longleftrightarrow \prod_{i\in Q_0} GL_{\alpha_i}(\mathbb{C}) \text{ orbits on } \bigoplus M_{\alpha_{h(e)}\times\alpha_{t(e)}}(\mathbb{C})$$

Given a matrix $\mathbb{C}^n \xrightarrow{A} \mathbb{C}^m$, can you ~~break it up into~~ find S, T such that TAS^{-1} looks like $\left[\begin{array}{c|c} 1 \ddots 1 & 0 \\ \hline 0 & 0 \end{array}\right]$?

$n=2$
$m=3$

$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} & \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} & \end{bmatrix} =$

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$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, shift left, "A"] \arrow[d, phantom, "\parallel"] & \mathbb{C}^M \arrow[l, shift left, "A^T"] \arrow[d, phantom, "\parallel"] \\
\operatorname{Ker}(A) \arrow[d, phantom, "\oplus"] & \operatorname{Ker}(A^T) \arrow[d, phantom, "\oplus"] \\
\operatorname{im}(A^T) \arrow[r, "\sim", "A"'] & \operatorname{im}(A)
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Say $(A_e)_{e\in Q_1} \in \prod_{e\in Q_1} M_{\cancel{m}_{h(e)};n_{t(e)}}(\mathbb{C})$

$(S_i)_{i\in Q_0} \in \prod_{i\in Q_0} GL(n_i)$

Then

$$(S_i)_{i\in Q_0} \cdot (A_e)_{e\in Q_0} = (S_{h(e)} A_e S_{t(e)}^{-1})_{e\in Q_1}.$$

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11/03/14

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$$\operatorname{Rep}(Q,\alpha)/_{Iso.} \xleftrightarrow{\sim} \prod_{i\in Q_0} GL_{\alpha_i}(\mathbb{C}) \text{ orbits on } \bigoplus M_{\alpha_{h(e)}\times\alpha_{t(e)}}(\mathbb{C})$$

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\operatorname{Ker}(A) & \operatorname{Ker}(A^T) \\
\oplus & \oplus \\
\operatorname{im}(A^T) \arrow[r, "\sim", "A"'] & \operatorname{im}(A)
\end{tikzcd}$$

Example: classify the orbits of $GL_n \times GL_n \circlearrowright M_{m\times n}(\mathbb{C}) \oplus M_{m\times n}(\mathbb{C})$

$$\begin{tikzcd} \underset{\mathbb{C}^m}{\bullet} \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] & \underset{\mathbb{C}^n}{\bullet} \end{tikzcd}$$

This is doable, but very hard.

Perspective:

$$\begin{tikzcd} A \text{ artinian} \arrow[d] \\ A/_{W(A)} = S \text{ semisimple} \end{tikzcd}$$

$\hat{A} = \hat{S}$, but A has indecomposable yet not simple modules.

Take $e \in A$ idempotent, Ae idecomposable. (principal indecomposable ideal)

$$\Updownarrow$$

$Ae/_{W(A)}$ simple

$$\Updownarrow$$

e primitive idempotent.

$$\Updownarrow$$

$e \in Ae$ is unique idempotent of Ae.

Def: $\{e_1, \dots, e_n\} \in A$ is said to be a complete set of primitive idempotents if $\begin{cases} 1 = e_1 + \dots + e_n \\ e_i \text{ primitive } \forall i \end{cases}$

Can lift idempotents in $A/_{W(A)}$; complete set for $A/_{W(A)}$ gives complete set for A.

$A = \bigoplus_{i=1}^{n} Ae_i$ is decomposition of A into indecomposables.

(Krull - Schmidt Theorem).

Recall: Fix An R-module P, consider the functor $\mathrm{Hom}_R(P, -)$.

$$\underline{R\text{-mod}} \xrightarrow{\mathrm{Hom}_R(P,-)} \underline{Ab}$$

Given a short exact sequence, mapping it under this functor may not be exact on the last position.

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0 \quad \text{exact}$$

$$0 \longrightarrow \underset{\overset{*}{\text{exact}}}{\mathrm{Hom}_R(P; M)} \longrightarrow \underset{\overset{*}{\text{exact.}}}{\mathrm{Hom}_R(P; N)} \longrightarrow \underset{\overset{\uparrow}{\text{may not be exact here!}}}{\mathrm{Hom}_R(P; K)} \longrightarrow 0$$

<u>Def</u>: P is projective iff $\mathrm{Hom}_R(P, -)$ is exact functor.

$\iff \exists\, n, Q$ such that $R^n = P \oplus Q$.

In our situation, $A = \bigoplus_{i=1}^{n} Ae_i$, each Ae_i is projective.

Any f.g. A-module, projective, decomposes as a direct sum of the Ae_i with mutiplicity

Let e, f be principle idempotents.

$$\mathrm{Hom}_A(Ae, Af) = eAf$$

$$\phi \longmapsto \phi(e) = \phi(ee) = e\,\overset{\in Af}{\phi(e)} \in eAf.$$

In fact, $eAf \neq 0 \iff eW(A)f \neq 0$.

To A, associate a Quiver, $Q = (Q_0, Q_1, h, t)$

$Q_0 = \{\text{primitive idempotents for } A\} = \{e_1, \dots, e_n\}$

$Q_1 = \{(e, f) \mid fAe \neq 0\}$

$P_Q(A)$ is the path algebra of this quiver

<u>Theorem</u> (Gabriel): <u>A-mod</u> is equivalent to $\frac{P_Q(A)}{I}$-<u>mod</u> (as a category) for some admissible I.

An ideal I is called <u>admissible</u> if it is generated by relations $\sum_{i=1}^{N} c_i a_{p_i} = 0$ where $t(p_1) = t(p_2) = \dots = t(p_N)$

$h(p_1) = h(p_2) = \dots = h(p_N)$

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Notation: If A is Artinian / a fin. dim. $\mathbb{C}$-algebra indecomposable modules have well-defined length.

$n(A)$ = number of isomorphism types of ~~irreducible~~ indecomposable A-modules

$n_k(A)$ = the number of length k.

Def: A is said to be of finite representation type if $n(A) < \infty$ otherwise, infinite representation type, if $n(A) = \infty$

bounded if $n_k(A) = 0$ for $k >> 0$.

Braver-Thrall 1: A of bounded type $\Longrightarrow$ A finite type

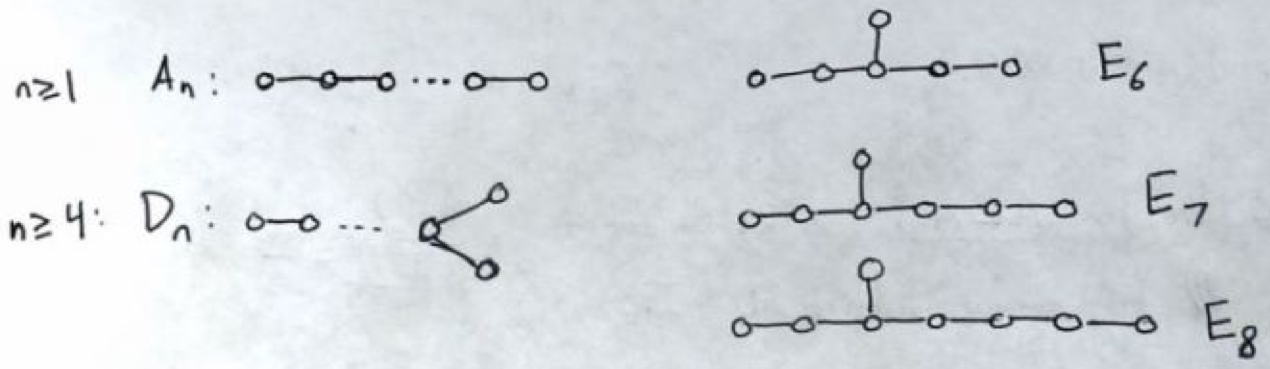
Braver-Thrall 2: If $\left\{\begin{matrix} n(A) = \infty \\ |Z(A)| = \infty \end{matrix}\right.$, then $n_k(A) = \infty$ for infinitely many k.

(proved by Roiter-Auslander-Ringel)

Question: Which Quivers have finite representation type.

Answer: (Gabriel, 1970's).

Q has finite representation type $\Longleftrightarrow$ it's connected components (modulo edge directions) are on the following list:



Dynkin Diagrams!

Examples: A_1 $\quad \bullet \quad$ $P_{A_1} = \mathbb{C}$

Representation is a vector space / $\mathbb{C}$.

One indecomposable, $\mathbb{C}$.

A_2 $\quad \bullet \longrightarrow \bullet$

Up to isomorphism, looks like $\mathbb{C}^n \xrightarrow{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \; 0 \end{bmatrix}} \mathbb{C}^m$

indecomposables are:

$$\begin{tikzcd} \underset{\mathbb{C}}{\bullet} \arrow[r] & \underset{0}{\bullet} \end{tikzcd} \qquad \begin{tikzcd} \underset{0}{\bullet} \arrow[r] & \underset{\mathbb{C}}{\bullet} \end{tikzcd} \qquad \begin{tikzcd} \underset{\mathbb{C}}{\bullet} \arrow[r] & \underset{\mathbb{C}}{\bullet} \end{tikzcd}$$

Any module decomposes as

$$\mathbb{C}^n \xrightarrow{\overset{\text{rank } k}{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{bmatrix}}} \mathbb{C}^m = \bigoplus k \left(\underset{\mathbb{C}}{\bullet} \xrightarrow[\text{id}]{} \underset{\mathbb{C}}{\bullet} \right) \oplus (n-k) \left(\underset{\mathbb{C}}{\bullet} \longrightarrow \underset{0}{\bullet} \right) \oplus (m-k) \left(\underset{0}{\bullet} \longrightarrow \underset{\mathbb{C}}{\bullet} \right)$$

$$A_3 \quad \begin{tikzcd} \bullet \arrow[r, shift left, "A"] & \bullet \arrow[l, shift left, "A^T"] \arrow[r, shift left, "B"] & \bullet \arrow[l, shift left, "B^T"] \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, shift left, "A"] & \bullet \arrow[l, shift left, "A^T"] \arrow[r, shift left, "B"] & \bullet \arrow[l, shift left, "B^T"] \end{tikzcd}$$

$$\begin{tikzcd} \ker A \arrow[r, "A"] & 0 \\ \oplus & \\ \operatorname{im} A^T \arrow[r, "\sim", "A"'] & \operatorname{im} A \end{tikzcd}$$

$$\begin{tikzcd} \ker(A^T) \cap \operatorname{im} B^T \arrow[r, dash] & \ker(BA)^T \cap \operatorname{im}(B) \\ \oplus & \\ \ker(A^T) \cap \ker B \arrow[r, "B"'] & 0 \\ \oplus & \\ \operatorname{im} A & \\ \parallel & \\ \operatorname{im} A \cap \ker B \arrow[r, "B"] & 0 \\ \oplus & \\ \operatorname{im} A \cap \operatorname{im}(B^T) \arrow[r, dash, "\sim"] & \operatorname{im}(BA) \\ & \parallel \\ & \operatorname{im}(B|_{\operatorname{im} A}) \end{tikzcd}$$

$v \in \ker(A^T) \cap \operatorname{im}(B^T)$

$A^T B^T w = 0$

$(BA)^T w = 0$

$w \in \ker(BA)^T$

~~A~~ $\qquad B^T w$

Indecomposables for A_3

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r, "id"] & \mathbb{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "id"] & \mathbb{C} \arrow[r, "id"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{C} \end{tikzcd}$$

Note that

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 \arrow[r] & \mathbb{C} \end{tikzcd} = \begin{tikzcd} \mathbb{C} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \oplus \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{C} \end{tikzcd}$$

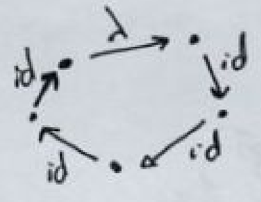
Def: If Q is a quiver, let Q^u be the un-oriented graph obtained from Q.

Remark: Let Q be a Quiver such that Q^u is a cycle.

For $\lambda \in \mathbb{C}$, construct the following Q-module: (call it $\underline{V_\lambda}$).
pick one edge $e_0 \in Q_1$.

$V_i = \mathbb{C}$

$$\begin{cases} f_e = id & e \neq e_0 \\ f_{e_0} = \lambda\, id \end{cases}$$



Claim these are not isomorphic.

If $F: \underline{V_\lambda} \cong \underline{V_\mu}$, then

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F_i"] \arrow[d, "f_e"'] & \mathbb{C} \arrow[d, "g_e"] \\ \mathbb{C} \arrow[r, "F_j"'] & \mathbb{C} \end{tikzcd}$$

commutes. In particular, $F_i = F_j$ for all i, j

$\Longrightarrow \lambda = \mu$.

Remark: If $n(Q)$ is finite, then $\mathrm{Rep}(Q,\alpha)/\!/_{Iso}$ is finite for any fixed dimension vector α.

Corollary: If Q^u is a cycle, then Q is not of finite type.

Def: If $R \subseteq Q$ is a subquiver, have $P_R \leq P_Q$ subalgebra inclusion.

$$\begin{tikzcd} \underline{R\text{-mod}} \arrow[r, shift left, "ext"] & \underline{Q\text{-mod}} \arrow[l, shift left, "res"] \end{tikzcd}$$

res = restriction functor
ext = trivial extension functor

$\mathrm{res} \circ \mathrm{ext} = \mathrm{id}$

Corollary:
If Q is of finite type, then R is also finite type.

Corollary:
- If Q^u contains a cycle, then Q is NOT of finite type.
- If Q is finite type, then all connected components are of finite type, and Q^u has no cycles.

Enough to consider connected components of Q^u.

11/10/14

Recall:
- if Q is of finite representation type, $\mathrm{Rep}(Q,\underline{\alpha})/\!/_{Iso.}$ is finite.
- if Q is of finite type, then subquivers are of finite representation type
- if Q^u is a cycle $\longrightarrow$ Q is of infinite representation type
- Q of finite representation type $\Longrightarrow$ Q^u has no cycles.

Convention: from now on, Q^u has no cycles.

Therefore, the path algebra P_Q is finite dimensional.

Exercise: Find the principal indecomposable Q-modules and the simple Q-modules. (as many as $|Q_0|$ up to IM).

Eg. Fix $i \in Q_0$

$$V_j = \begin{cases} 0 & j \neq i \\ \mathbb{C} & j = i \end{cases}$$

$f_e = 0$ for all $e \in Q_1$

Call this representation $\underline{\mathbb{C}}_i$

Here are $|Q_0|$ nonisomorphic Q-representations

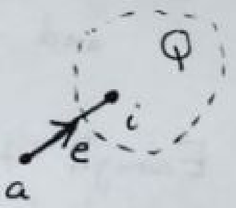
Def: Fix $i \in Q_0$. We say that $\underline{V}$ (a Q-mod.) is rigid at i if for any $F \in Aut(\underline{V})$ we have $F_i = \lambda\, id_{V_i}$

Example: If $\underline{V_i}$ is 1-dimensional then $\underline{V}$ is rigid at i

Remark: Let $\begin{cases} i \in Q \\ \underline{V} \in Q\text{-mod}, \text{ rigid at } i, \ \dim V_i \geq 2 \end{cases}$

Let $R = Q$ with the node a and edge $e = (a, i)$

Then R is of infinite type.



For $\phi : \mathbb{C} \longrightarrow V_i$ we construct $\underline{V}_\phi \in R\text{-mod}$ such that $\underline{V}_\phi|_Q = \underline{V}$, $(\underline{V}_\phi)_a = \mathbb{C}$, $f_e = \phi$.

When are two such maps isomorphic?

Suppose $\underline{V}_\phi \xrightarrow[\sim]{F} \underline{V}_\psi$. Restricted to Q, $F|_Q$ is isomorphism (and Automorphism!)

$$\begin{tikzcd} \underline{V}_\phi|_Q \arrow[r, "F", "\sim"'] \arrow[d, phantom, "\shortparallel"] & \underline{V}_\psi|_Q \arrow[d, phantom, "\shortparallel"] \\ \underline{V} & \underline{V} \end{tikzcd}$$

$F|_Q \in Aut_Q(\underline{V}) \Longrightarrow F_i = \lambda\, id_V.$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\phi"] \arrow[d, "\wr", "\mu Id_{\mathbb{C}} = F_a"'] & V_i \arrow[d, "F_i = \lambda id_{V_i}", "\wr"'] \\
\mathbb{C} \arrow[r, "\psi"'] & V_i
\end{tikzcd}$$

$\lambda \cdot \phi = \mu \cdot \psi \Rightarrow \mathbb{C}\phi = \mathbb{C}\psi.$

$\{ \underline{V}_\phi \mid \phi \in \mathbb{P}^1 V_i \}$ is a family of non isomorphic reps of R of same dimension, infinite.

So R is not of finite type.

Example: (of rigid Q-modules)

I

$$\begin{tikzcd}
& & & & & \mathbb{C} \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"'] \\
\mathbb{C} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"'] & \overset{\mathbb{C}^2}{*} \arrow[r, "id"'] & \mathbb{C}^2 \arrow[r, "id"'] & \mathbb{C}^2 \cdots \mathbb{C}^2 \arrow[r, "id"'] & \mathbb{C}^2 & \\
& & & & & \mathbb{C} \arrow[ul, "\begin{bmatrix}0\\1\end{bmatrix}"]
\end{tikzcd}$$

rigid at $*$

Let $F \in Aut_Q(\underline{V})$

$F_2 = F_3 = \cdots = F_{n-2}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"] \arrow[d, "\lambda id_{\mathbb{C}} = F_1"'] & \mathbb{C}^2 \arrow[d, "F_2"] \\
\mathbb{C} \arrow[r, "\begin{bmatrix}1\\1\end{bmatrix}"'] & \mathbb{C}^2
\end{tikzcd}$$

$F_2(\begin{bmatrix}1\\1\end{bmatrix}) = \lambda \begin{bmatrix}1\\1\end{bmatrix}$

F_2 preserves $\mathbb{C}\begin{bmatrix}1\\1\end{bmatrix}$.

II

$$\begin{tikzcd}
& & \overset{4}{\bullet}\ \mathbb{C}^2 \arrow[d, "\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}"] & \text{rigid at this node.} \arrow[l] \arrow[dl, bend left] & \\
\overset{1}{\bullet} \arrow[r, "\begin{bmatrix}0\\1\end{bmatrix}"'] & \overset{2}{\bullet} \arrow[r, "\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"'] & \overset{3}{\bullet} & \overset{5}{\bullet} \arrow[l, "\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}"] & \overset{6}{\bullet} \arrow[l, "\begin{bmatrix}1\\0\end{bmatrix}"] \\
\mathbb{C} & \mathbb{C}^2 & \mathbb{C}^3 & \mathbb{C}^2 & \mathbb{C}
\end{tikzcd}$$

Let F be an automorphism.

$$\begin{tikzcd}
\mathbb{C}^1 \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[d, "\wr", "F_1"'] & \mathbb{C}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"] \arrow[d, "\wr"', "F_2"] & \mathbb{C}^3 \arrow[d, "\wr", "F_3"'] \\
\mathbb{C}^1 \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"'] & \mathbb{C}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"'] & \mathbb{C}^3
\end{tikzcd}$$

$$F_3 \Big(\underbrace{\mathbb{C}\begin{bmatrix}1\\0\\0\end{bmatrix} + \mathbb{C}\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{column space of } \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} \Big) = \mathbb{C}\begin{bmatrix}1\\0\\0\end{bmatrix} + \mathbb{C}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$F_2\left(\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}\right) = \mathbb{C}\begin{bmatrix}1\\0\\0\end{bmatrix}$

$F_3\left(\mathbb{C}\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \mathbb{C}\begin{bmatrix}1\\0\\0\end{bmatrix}$ ← (and by looking at the rest of the diagram as well)

Combining all of this, we know that F_3 must be a scalar multiple of the identity by observing the spaces it must preserve: ~~namely~~ its the column spaces of these matrices and $\mathbb{C}\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$.

Also rigid at node 4.

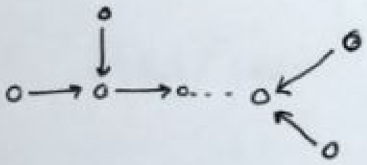
III

$$\begin{tikzcd}
 & & & \mathbb{C}^3 \arrow[d, "\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}"] & & \\
\mathbb{C} \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"'] & \mathbb{C}^2 \arrow[r, "\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"'] & \mathbb{C}^3 \arrow[r, "\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}"'] & \mathbb{C}^4 & \mathbb{C}^3 \arrow[l, "\begin{bmatrix}1&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}"] & \overset{\ast}{\mathbb{C}^2} \arrow[l, "\begin{bmatrix}1&0\\-1&0\\0&1\end{bmatrix}"]
\end{tikzcd}$$

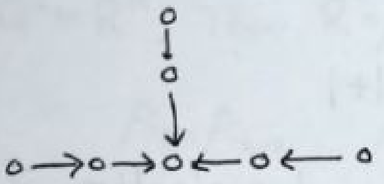
IV

$$\begin{tikzcd}
 & & & & \mathbb{C}^3 \arrow[d, "\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}"] & & \\
\overset{\ast}{\mathbb{C}^2} \arrow[r, "\begin{bmatrix}1&1\\0&1\\-1&-1\end{bmatrix}"] & \mathbb{C}^3 \arrow[r, "\begin{bmatrix}1&0&1\\1&-1&0\\0&1&0\\0&0&1\end{bmatrix}"] & \mathbb{C}^4 \arrow[r, "\begin{bmatrix}0&1&1&0\\1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}"] & \mathbb{C}^5 \arrow[r, "\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\1&1&1&1&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}"'] & \mathbb{C}^6 & \mathbb{C}^4 \arrow[l, "\begin{bmatrix}I_4\\0\end{bmatrix}"'] & \mathbb{C}^2 \arrow[l, "\begin{bmatrix}I_2\\0_2\end{bmatrix}"']
\end{tikzcd}$$

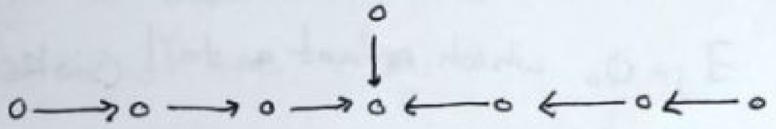
Corollary:



is of infinite type



is of infinite type



∞ type

∞ type

So any quiver containing these is of infinite type.

Later: if $Q^u = R^u$, then, Q and R have the same representation type.

Prop: If Q is finite type, then the connected components of Q^u are A_n, D_n, E_6, E_7, E_8.

Proof: No cycles, assume Q^u is connected $\longrightarrow$ Q^u is a tree

No two branches, nothing larger than E_6, E_7, E_8 by above Corollary, max degree 3.

To prove Gabriel's Theorem, must show the "later" statement above and also show A_n, D_n, E_6, E_7, E_8 have finite type.

11/12/14

Reflection Functors:

Recall: Q^u is a tree.

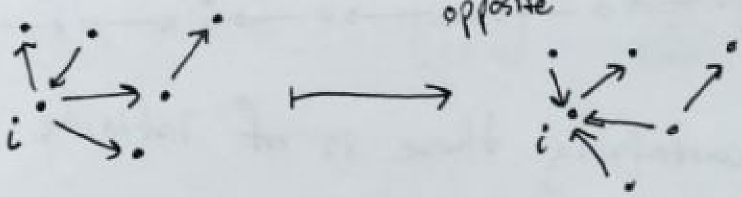
In particular, $|Q_0| = |Q_1|+1$

so $h: Q_1 \longrightarrow Q_0$ is not surjective. $\exists\, i \in Q_0$ which is not a head. (source).

similarly, $\exists\, j \in Q_0$ which is not a tail (sinks)

pick $i \in Q_0$

Def: $\rho_i Q$ = same as Q, except for the edges connecting at i, which have the ~~same~~ opposite orientation.



Remark: Label Q_0 by $\{1, 2, \dots, |Q_0|\}$ such that for any edge $e \in Q_1$, $t(e) < h(e)$ (standard labelling)

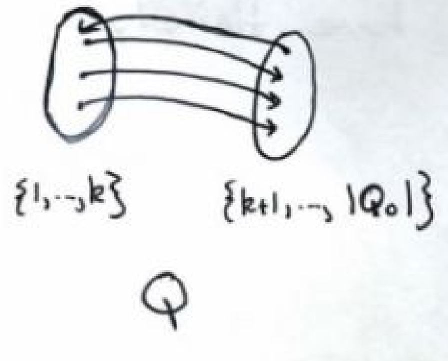
Proof: by induction on $|Q_0|$

remove a sink from Q, do to the new subgraphs (subtrees).

Label the sink by $|Q_0|$. ■

Remark: Let Q be a quiver with a standard labelling.

$\rho_k \cdots \rho_2 \rho_1 Q \longleftarrow$ changes all orientations of edges ~~between~~ from $\{1, \dots k\}$ ~~and~~ to $\{k+1, \dots, |Q_0|\}$ and k becomes a sink.



Precisely:

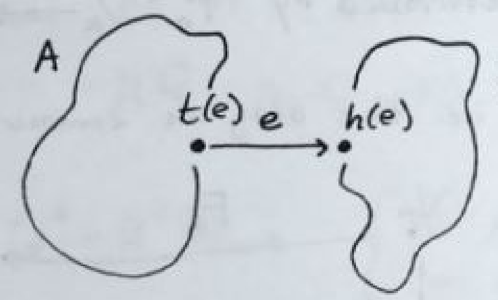
- edges (i,j) with $i,j \le k$ or $i,j > k$ are fixed
- edges such that $t(e) \le k$, $h(e) > k$ are flipped
- k becomes a sink, $k+1$ a source.

In particular $\rho_{|Q_0|}\rho_{|Q_0|-1}\cdots\rho_1 Q = Q$.

<u>Proposition:</u> Let $Q^u = R^u$. Then $R = \rho_{i_s}\cdots\rho_{i_1}Q$, such that i_k is a source in $\rho_{i_{k-1}}\rho_{i_{k-2}}\cdots\rho_{i_1}Q$ for some $i_j \in Q_0$

<u>Proof:</u> Enough to assume that Q and R differ only in the orientation of one edge, e.

Removing e, create two connected components.



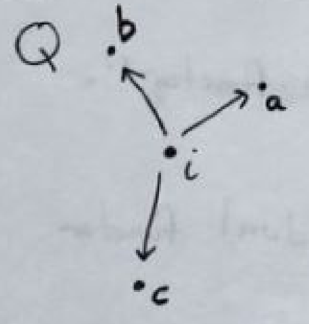
Put standard labelling on A, using $\{1, \dots, |A|\}$.

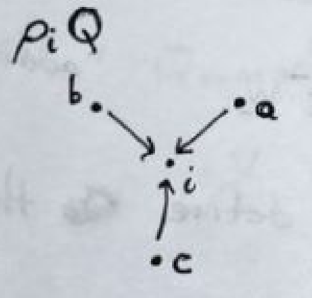
Put standard labelling on B using $\{|A|+1, \dots, |A|+|B|\}$. ■

Then $\rho_{|A|}\cdots\rho_1 Q = R$.

<u>Def:</u> Let $i \in Q_0$ be a source, and let $\underline{V} = ((V_j)_{j\in Q_0}, (f_e)_{e\in Q_1})$ be a Q-module.

Let $S_i^- \underline{V} \in \rho_i Q\text{-mod}$ by $$(S_i^- V)_j = \begin{cases} V_j & \text{if } j \neq i \\ \bigoplus_{\substack{t(e)=i \\ \text{in } Q}} V_{h(e)}/\mathrm{Im} f_e & \text{if } j = i \end{cases}$$





$$\begin{cases} g_e = f_e \text{ if } e \neq i \leftarrow a \\ g_{i\leftarrow a} \div V_a \longrightarrow \dfrac{V_a}{\mathrm{Im} f_{i\to a}} \hookrightarrow \boxed{\bigoplus_{i\to b} V_b/\mathrm{Im} f} \end{cases}$$

$(S_i^- V)_i$

$V_i = V_i$

$(S_i^- V)_i = V_a/\mathrm{Im} f_{i\to a} \oplus V_b/\mathrm{Im} f_{i\to b} \oplus V_c/\mathrm{Im} f_{i\to c}$

Wanted p_i to be a functor, so lets define how it behaves on morphisms. Let $\underline{V}, \underline{W} \in Q\text{-mod}$, $\underline{F}: \underline{V} \to \underline{W}$ Q-linear.

$$S_i^- \underline{F} : S_i^- \underline{V} \longrightarrow S_i^- \underline{W}$$

$$(S_i^- F)_j = F_j \quad \text{if } j \neq i$$

$$(S_i^- F)_i : \bigoplus_{i \to a} V_a / \mathrm{Im} f_{i \to a} \longrightarrow \bigoplus_{i \to a} W_a / \mathrm{Im}\, g_{i \to a}$$

the canonical map determined by $F_a : V_a \longrightarrow W_a$.

Check that $S_i^- \underline{F}$ is $p_i Q$-linear, i.e. this diagram commutes.

$$\begin{tikzcd}
V_a \arrow[r, "F_a"] \arrow[d] & W_a \arrow[d] \\
(S_i^- V)_i \arrow[r, "\bigoplus_{i \to b} F_b"'] & (S_i^- W)_i
\end{tikzcd}
\qquad
\begin{tikzcd}
V_a \arrow[r, "F_a"] \arrow[d] & W_a \arrow[d] \\
V_a / \mathrm{Im} f_{i \to a} \arrow[r, "F_a"'] & W_a / \mathrm{Im}\, g_{i \to a}
\end{tikzcd}$$

This commutes by definition, so the ~~map~~ diagram on the left does as well.

So if $i \in Q_0$ is a source

$$Q_{mod} \xrightarrow{S_i^-} p_i Q\text{-mod}$$

is a genuine functor.

Exercise: Check that it behaves nicely under compositions, and that it sends direct sums to direct sums.

$$S_i^- (\underline{V} \oplus \underline{W}) = S_i^- \underline{V} \oplus S_i^- \underline{W} \qquad \text{"additive functor"}$$

Similarly, if $i \in Q_0$ is a sink, can define the dual functor

$$Q\text{-mod} \xrightarrow{S_i^+} p_i Q\text{-mod}.$$

Defining S_i^+

$$\underline{V} = \left((V_i)_{i\in Q_0}, (f_e)_{e\in Q_1}\right)$$

$$(S_i^+ V)_j = V_j \text{ if } j \neq i \qquad g_e = f_e \text{ if } e \neq i \to a$$

$$(S_i^+ V)_i = \bigoplus_{a\to i} \ker f_{a\to i}$$

$$\begin{tikzcd} g_{i\to a} : \bigoplus_{b\to i} \ker f_{b\to i} \arrow[r, two heads] & \ker f_{a\to i} \arrow[d, hook] \\ & V_a. \end{tikzcd}$$

WRONG

11/17/14

Recall: $\cdot$ Q^u is a tree

- $\rho_i Q$ is Q with edges at i reversed.

- $Q^u = R^u \Rightarrow \begin{cases} R = \rho_{i_s} \cdots \rho_{i_1} Q \\ \text{can arrange } i_k \text{ is a source in } \rho_{i_{k-1}} \cdots \rho_{i_1} Q \end{cases}$

- i a source $\quad Q\text{-mod} \xrightarrow{S_i^-} \rho_i Q\text{-mod}$

$$\underline{V} \longmapsto S_i^- \underline{V}$$

$$(S_i^- \underline{V})_i = \frac{\bigoplus_{i\to a} V_a}{\operatorname{Im}\left(\bigoplus_{i\to a} f_{i\to a}\right)} \leftarrow \text{Correction!!}$$

- i a sink $\quad Q\text{-mod} \xrightarrow{S_i^+} \rho_i Q\text{-mod}$

$$\underline{V} \longmapsto S_i^+ \underline{V}$$

$$(S_i^+ \underline{V})_i = \ker\left(\bigoplus_{a\to i} f_{a\to i}\right) \longleftarrow \text{CORRECTION!!}$$

Exercise: If i is a source, [illegible]

$$\underline{V} \simeq S_i^+ S_i^- \underline{V} \oplus \dim(\ker \oplus f_{i\to a})\,\mathbb{C}_i$$

If i is a sink,

$$\underline{V} \simeq S_i^- S_i^+ \underline{V} \oplus \dim(\text{coker} \oplus f_{i\to a})\,\mathbb{C}_i$$

Proposition: Let i be a source.

(1) $S_i^- \underline{\mathbb{C}}_i = 0$

(2) If $\underline{V}$ is indecomposable, $\underline{V} \neq \underline{\mathbb{C}}_i$ then $S_i^- \underline{V}$ is indecomposable $\rho_i Q$-mod, and

$$\underline{V} \simeq S_i^+ S_i^- \underline{V}$$

In this case, $\dim (S_i^- \underline{V})_i = \left(\sum_{i\to a} \dim V_a\right) - \dim V_i$

There is a similar statement for i a sink.

Proof: (1) Pretty clear

(2) Let $\underline{V}$ be indecomposable. If $S_i^- \underline{V} = \underline{A} \oplus \underline{B}$

Then $S_i^+ S_i^- \underline{V} = S_i^+ \underline{A} \oplus S_i^+ \underline{B}$

$$\Longrightarrow \underline{V} \cong S_i^+ \underline{A} \oplus S_i^+ \underline{B} \oplus \dim(\ker \bigoplus_{i\to a} f_{i\to a})\,\underline{\mathbb{C}}_i$$

If both $S_i^+ \underline{A}, S_i^+ \underline{B}$ nonzero, then $V \cong \dim(\ker \oplus f_{i\to a})\mathbb{C}_i$, but we excluded the case that $\underline{V} \cong \mathbb{C}_i$.

So one of $S_i^+ \underline{A}, S_i^+ \underline{B}$ is nonzero, wlog $S_i^+ \underline{A} \neq 0$.

Then b/c $\underline{V}$ is indecomposable, $\underline{V} \simeq S_i^+ \underline{A}$. Then

$\underline{A} \oplus \underline{B} = S_i^- V = S_i^- S_i^+ \underline{A}$ and also, $S_i^- S_i^+ \underline{A} \oplus \underline{B} \oplus \underset{}{\overset{\text{for some } n}{\swarrow}} n\,\mathbb{C} = \underline{A} \oplus \underline{B}$

$$\Longrightarrow B = 0.$$

Then, $\dim(S_i^- \underline{V})_i = \left(\sum_{i\to a} \dim V_a\right) - \dim \mathrm{Im}\left(\bigoplus_{i\to a} f_{i\to a}\right)$

because $$(S_i^- V)_i = \frac{\bigoplus_{i\to a} V_a}{\mathrm{Im}(\oplus f_{i\to a})}$$

But $V_i \xrightarrow{\oplus f_{i\to a}} \bigoplus_{i\to o} V_a$, use rank nullity to conclude

$$\dim \mathrm{Im}\left(\bigoplus_{i\to a} f_{i\to a}\right) = \dim V_i \quad ■$$

<u>Theorem:</u> If $Q^u = R^u$ then Q and R have the same representation type.

<u>Proof:</u> Enough to show that if i is a source in Q then Q and $p_i Q$ have the same representation type.

$$\begin{tikzcd}
Q\text{-mod} \arrow[r, "S_i^-"] & p_i Q\text{-mod} \\
\underline{\mathbb{C}_i} \arrow[r, mapsto] & \underline{0} \\
\underset{\neq \mathbb{C}_i}{\text{idecomposable}} \arrow[r, mapsto] & \text{indecomposable.}
\end{tikzcd}$$

} injective on indecomposables

If $p_i Q$ has finite representation type, then so does Q.

Similarly for S_i^+, also have the proposition, so conclude if Q has finite type, then so does $p_i Q$. ■

<u>Corollary:</u> If Q has finite representation type, then $Q^u = A_n, D_n, E_{6,7,8}$

The B_n, C_n, G_2, F_4 have double edges and therefore cycles. So they are bad.

<u>Question:</u> is converse also true?

The Tits form:

A symmetric bilinear form on ~~Rep~~ dimension vectors due to Jacques Tits.

Remark: "dim" $\text{Rep}(Q,\underline{\alpha})/_{\text{Iso}} \longrightarrow$ size of the GL_n orbits on these vector spaces

$$\text{"dim" } \text{Rep}(Q,\underline{\alpha})/\text{Iso} = \dim \text{Rep}(Q,\underline{\alpha}) - \sum_i \dim(GL_{\alpha_i}(V_{i})) \swarrow \text{ the product of } GL_{\alpha_i}(V_{i})\text{.}$$

$$= \sum_{e\in Q_1} \alpha_{h(e)}\alpha_{t(e)} - \sum_{i\in Q_0} \alpha_i^2 \qquad \prod_{i\in Q_0} GL_{\alpha_i}(V_i)$$

Def; $\beta_Q : \mathbb{R}^{|Q_0|} \times \mathbb{R}^{|Q_0|} \longrightarrow \mathbb{R}$ (The Tits form)

$$\beta_Q(\underline{v},\underline{w}) = 2\sum_{i\in Q_0} v_i w_i - \sum_{e\in Q_1}\left(v_{h(e)}w_{t(e)} + w_{h(e)}v_{t(e)}\right)$$

Proposition: If $Q^u = A_n, D_n$ or $E_{6,7,8}$, then β is positive definite.

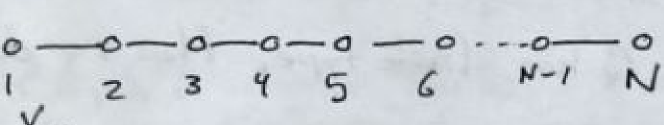
Proof:

$$\sum_{k=1}^{N-1}\binom{k+1}{2}\left(\frac{1}{k}x_k - \frac{1}{k+1}x_{k+1}\right)^2 + \frac{N+1}{2N}x_N^2$$

$$= x_1^2 + x_2^2 + \dots + x_N^2 - x_1x_2 - \dots - x_{N-1}x$$

call it $\phi(x_1,\dots,x_N)$

For A_n,

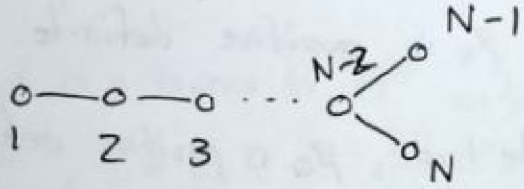


for dimension vector $\underline{v}$,

$$\beta(\underline{v},\underline{v}) = \phi(v_1,\dots,v_N) \geq 0 \quad \text{and} \quad \phi(v_1,\dots,v_N) = 0 \Longleftrightarrow \underline{v} = 0.$$

$\longrightarrow$

For D_n



$$\beta(\underline{v},\underline{v}) = \phi(v_1,\dots,v_{N-2}) + \phi(N-1,N-2) + \phi(N,N-2)$$

$$= (\quad)^2 + (\quad)^2 + (\quad)^2 + \left(\frac{N-1}{2(N-2)} + \frac{3}{4} + \frac{3}{4} - 2\right) x_N^2$$

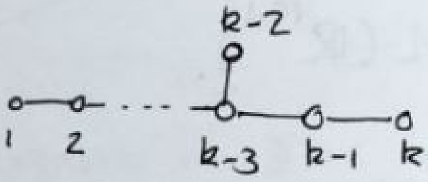
So $\beta(\underline{v},\underline{v})$ is positive definite.

11/19/17

Recall: $$\sum_{k=1}^{N-1} \frac{k(k+1)}{2}\left(\frac{1}{k}x_k - \frac{1}{k+1}x_{k+1}\right)^2 + \frac{N+1}{2N}x_N^2$$

$$= x_1^2 + x_2^2 + \dots + x_N^2 - x_1x_2 - x_2x_3 - \dots - x_{N-1}x_N$$

E_k:



$$\frac{1}{2}\beta(\underline{v},\underline{v}) = v_1^2 + v_2^2 + \dots + v_k^2 - v_1v_2 - v_2v_3 - \dots - v_{k-4}v_{k-3} - v_{k-3}v_{k-2}$$
$$- v_{k-3}v_{k-1}$$
$$- v_{k-1}v_k$$

$$= \phi(v_1,\dots,v_{k-3}) + \phi(v_{k-2},v_{k-3}) + \phi(v_k,v_{k-1},v_{k-3})$$
$$-2v_{k-3}^2$$

$$= (\text{sum of squares}) + v_{k-3}^2\left(\frac{k-2}{2(k-3)} + \frac{3}{4} + \frac{4}{6} - 2\right)$$

$$= (\text{sum of squares}) + v_{k-3}^2 \underbrace{\left(\frac{9-k}{12(k-3)}\right)}_{\geq 0 \text{ if } k=6,7,8.}$$

So β is positive definite on $E_{6,7,8}$. ▨

So if $Q^u = A, D, E$ then β_Q is positive definite.

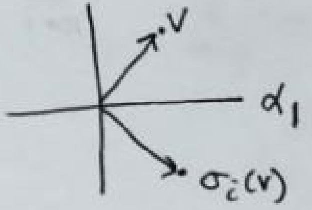
<u>Corollary</u>: if Q is of finite type, β_Q is positive definite.

$$\beta_Q : \mathbb{R}^{|Q_0|} \times \mathbb{R}^{|Q_0|} \longrightarrow \mathbb{R}$$

<u>Notation</u>: $\underline{\alpha_i} = \dim(\underline{\mathbb{C}_i}) = (0, \dots, \underset{\substack{\uparrow \\ \text{1 in the } i^{th} \text{ position.}}}{1}, \dots 0)$

<u>Def</u>: $\sigma_i : \mathbb{R}^{|Q_0|} \longrightarrow \mathbb{R}^{|Q_0|}$ orthogonal reflection with respect to α_i

$$\sigma_i(\underline{v}) = \underline{v} - \beta(\underline{v}, \alpha_i)\alpha_i$$

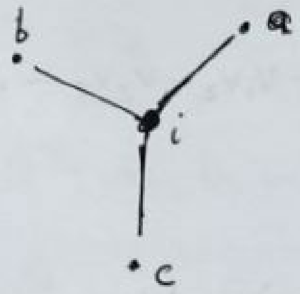


~~sets the i^{th} component to its negative.~~

<u>Def</u>: If β is non-degenerate,

$$W_Q = \langle \sigma_i \mid i \in Q_0 \rangle \leq GL(\mathbb{R}^{|Q_0|}).$$

<u>Remark</u>: Q



$$\sigma_i(\underline{v}) = \underline{v} - (0, \dots, 0, \beta(\underline{v}, \alpha_i), 0, \dots, 0)$$

$\sigma_i(\underline{v})$ and $\underline{v}$ are equal except for ith entry.

$$(\sigma_i(\underline{v}))_i = v_i - \beta(\underline{v}, \alpha_i) = v_i - \Big(2v_i - \sum_{(i,a) \in Q^u} v_a\Big) = \sum_{(i,a) \in Q^u} v_a - v_i.$$

Corollary: i is a source in Q^u and $\underline{V}$ is indecomposable as a Q-module, $\underline{V} \neq \underline{\mathbb{C}_i}$.

$$\Rightarrow \underline{\dim}(S_i^-(\underline{V})) = \sigma_i(\underline{\dim}(\underline{V})).$$

if instead i is a sink,

$$\underline{\dim}(S_i^+ \underline{V}) = \sigma_i(\underline{\dim}(\underline{V})).$$

Def: Let Q be such that β is positive definite.

root system

$$\Phi := \Phi_Q = \left\{ \underline{v} \in \mathbb{Z}^{|Q_0|} \;\middle|\; \beta(\underline{v},\underline{v}) = 2 \right\}.$$

Note that this set is finite.

$$\Phi^+ := \Phi \cap \mathbb{Z}_{\geq 0}^{|Q_0|} \qquad \Phi^- := \Phi \cap \mathbb{Z}_{\leq 0}^{|Q_0|}$$

positive roots

negative roots

Terminology: elements of Φ are called roots, Φ^+ positive roots, Φ^- negative roots.

Prop: $\Phi = \Phi^+ \sqcup \Phi^-$.

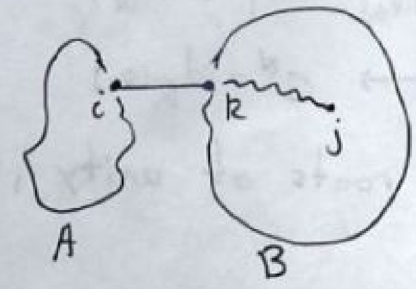
Pf: ~~let $\phi \in \Phi^+ \cap \Phi^-$~~ Let $\underline{v} \in \Phi$ such that $v_i > 0$, $v_j < 0$. May assume i, j are close as possible, i.e. $v_{k=0}$ for all $i \leq k \leq j$. Let (i,k) be an edge on this path.

unique path in Q

$i \rightsquigarrow j$

Convention: Q is such that Q^u is a tree and β_{Q^u} is positive definite.

proof continued.



$\underline{v} = \underline{v}_A + \underline{v}_B$

$\longrightarrow$

$$2 = \beta(\underline{v},\underline{v}) = \beta(v_A,v_A) + \beta(v_B,v_B) + 2\beta(v_A,v_B)$$

$$= \beta(v_A,v_A) + \beta(v_B,v_B) + 2(-v_i v_k)$$ ← use definition of Tits form.

$$\underset{\substack{\uparrow \\ \text{zero only possible for zero vector,} \\ \text{2 min value otherwise if } v_A \in \mathbb{Z}^{|Q_0|}}}{\geq 2} + 2 + 0 \geq 4 \quad \divideontimes.$$

So ~~th~~ components of v are either all positive or all negative.

■

<u>Remark</u>: W_Q acts on Φ, b/c W_Q is orthogonal transformations.

(1) Action is faithful.

$\alpha_i \in \Phi$ for any $i \in Q_0$

σ_i is never the identity on Φ, because none of the α_i are preserved under every σ_i (reflection).

$$W \hookrightarrow \underset{\substack{\uparrow \\ \text{permutations} \\ \text{of } \Phi}}{S_\Phi} \Rightarrow W_Q \text{ is a finite group.}$$

(2) $\underset{\Phi^+}{\underset{\cap}{\sigma_i(\alpha_i)}} = \underset{\Phi^-}{\underset{\cap}{-\alpha_i}}$ if $v \in \Phi^+$ $\sigma_i(v) \in \Phi^+$ unless $v = \alpha_i$

(if one entry of a vector $v \in \Phi$ is positive, then so are all of the entries)

(3) Label nodes in Q by $1, \dots, |Q_0|$ and let

$$c = \sigma_{|Q_0|} \cdots \sigma_1 \in W_Q.$$

Let $N = \mathrm{ord}(c) \longrightarrow c^N = \mathrm{id}_{\mathbb{R}^{|Q_0|}}$

Eigenvalues are roots of unity, but 1 is <u>not</u> an eigenvalue

Why is 1 not an eigenvalue?

Suppose $\sigma_{|Q_0|} \cdots \sigma_1 \underline{v} = \underline{v} \implies \sigma_{|Q_0|-1} \cdots \sigma_1 \underline{v} = \sigma_{|Q_0|} \underline{v}$

But $\sigma_{|Q_0|}$ only changes the $|Q_0|$th element of $\underline{v}$,
and σ_i only changes the ith element of $\underline{v}$, so
Now we must have $(\sigma_{|Q_0|} v)_{|Q_0|} = (\sigma_{|Q_0|-1} \cdots \sigma_1 v)_{|Q_0|} = v_{|Q_0|}$

So v is fixed by $\sigma_{|Q_0|}$. Repeat, get $\underline{v} \perp \alpha_i$ for all i, so $\underline{v} = 0$.

$$C^N - 1 = \underbrace{(C-1)}_{\text{invertible.}}(C^{N-1} + C^{N-2} + \ldots + C + 1) = 0 \implies C^{N-1} + \ldots + C + 1 = 0.$$

(4) Let $v \in \Phi^+$. ~~Then there is some $k \geq 0$ such that~~ [illegible]. ~~such that~~

~~[illegible]~~

By applying $\sigma_1, \sigma_2, \ldots, \sigma_{|Q_0|}, \sigma_1, \sigma_2, \ldots$ to v in this order, eventually obtain something in Φ^-.

(Eventually get to α_i)

In the sequence $\underline{v}, C\underline{v}, C^2\underline{v}, \ldots, C^{N-1}\underline{v}$, there must be some element of Φ^- because

$$C^{N-1} + \cdots + C + 1 = 0.$$

But $C' = \sigma_{|Q_0|} \cdots \sigma_1$, so in the sequence ~~the~~

$\sigma_1 \underline{v}, \sigma_1 \sigma_2 \underline{v}, \ldots$ we pass from Φ^+ to Φ^- at some point.

<u>Theorem</u> (Gabriel): Q is of finite representation type if and only if $Q^u = A_n, D_n$ or $E_{6,7,8}$([redacted]).

In this case, $\underline{V} \longmapsto \underline{\dim}(\underline{V})$ is a bijection between indecomposable Q-modules and Φ^+.

up to isomorphism

<u>Proof</u>: ($\Longrightarrow$) Done.

($\Longleftarrow$) Put a standard labelling on Q.

Let $\underline{V}$ be an indecomposable Q-mod, and consider

$$\underline{V},\ S_1^-\underline{V},\ S_2^-S_1^-\underline{V}, \dots, \underline{C}^-\underline{V},\ S_1^-\underline{C}^-\underline{V}, \dots$$

where $C^- = S_{|Q_0|}^- \cdots S_1^-$. In this sequence, we can find $\underline{\mathbb{C}}_i$; look at corresponding sequence of dimensions, which is

$$\dim(\underline{V}),\ \sigma_1 \dim(\underline{V}),\ \sigma_1\sigma_2 \dim(\underline{V}), \dots$$

only as long as $S_{i-1}^- \cdots (\underline{V})$ is an indecomposable $P_{i-1}, \cdots P_1 Q$-mod, $\neq \underline{\mathbb{C}}_i$

Eventually, one of these is $\alpha_i = \dim(\underline{\mathbb{C}}_i)$. So the corresponding element in the original sequence is $\underline{\mathbb{C}}_i$

Remains to show $\dim(\text{indecomposable}) \in \Phi^+$.

Now $S_1^+(S_1^-\underline{V}) \cong \underline{V}$, because $\underline{V}$ indecomposable.

$$\underline{V} \cong S_1^+(S_1^-\underline{V}) \cong S_1^+S_2^+(S_2^-S_1^-\underline{V}) \cong S_1^+S_2^+ \cdots S_{i-1}^+(\underline{\mathbb{C}}_i)$$

Thus, $\underline{V}$ is completely determined by its dimension b/c each of the functors is determined by σ_i on dimension vectors. Conversely, if $\underline{v} \in \Phi^+$, then $\underline{v} \xrightarrow{\sigma_2\sigma_1\cdots} \alpha_i$ and applying corresponding functors gives indecomposable Q-mod.

We have shown that

$$\underline{V} \longmapsto \dim \underline{V}$$ is a bijection between indecomposables and Φ^+, and so the indecomposable Q-modules is a finite set.

Remains to show that the dimension of indecomposables are in Φ^+.

11/24/14

Clarification:

Idea in Gabriel's theorem is that indecomposables should be characterized by dimension.

On the level of ~~functors~~, dimension, each functor is a reflection.

- Q^u is a tree, β_Q positive definite
- $\alpha_i = (0, \dots, 1, \dots, 0)$ standard basis for $\mathbb{R}^{|Q_0|}$
- σ_i orthogonal reflection
- $c = \sigma_{|Q_0|} \cdots \sigma_1$ $\underline{\Phi} = \{\underline{v} \in \mathbb{Z}^{|Q_0|} \mid \beta(\underline{v}, \underline{v}) = 2\}$
- $\Phi = \underset{\substack{\uparrow \\ \text{positive} \\ \text{coeffs}}}{\Phi^+} \sqcup \underset{\substack{\nwarrow \\ \text{negative} \\ \text{coeffs}}}{\Phi^-}$ $\sigma_i(\Phi^+ \setminus \{\alpha_i\}) = \Phi^+ \setminus \{\alpha_i\}$

  $\sigma_i(\alpha_i) = -\alpha_i$

(2) (a). if $\underline{v} \in \mathbb{Z}^{|Q_0|}_{\geq 0}$, then there is $k > 0$ such that $c^k \underline{v}$ has a negative coordinate.

(b). if $\underline{v} \in \Phi^+$, then $\exists\, k > 0$ such that $c^k \underline{v} \in \Phi^-$

(c). if $v \in \Phi^+$, then $\exists\, k > 0$, $1 < s < |Q_0|$ such that $\sigma_{s-1} \cdots \sigma_1 \underline{c}^k \cdot \underline{v} = \alpha_s$

Proof of Gabriel's Theorem (revisited)

Already ~~[illegible]~~ know that if Q is of finite type, then it is of type A, D or E.

do by counting.

Need to show that if Q^u is A, D, E, then it has finite type.

Know β_Q is positive definite. Let $1, 2, \ldots, |Q_0|$ be a standard labelling of Q.

Let $\underline{V}$ be an indecomposable Q-mod.

$$\begin{tikzcd}
Q\text{-mod} & \rho_i Q\text{-mod} & \\
\underline{V} \arrow[r, bend left, "S_1^-"] & S_1^-(\underline{V}) \arrow[r, bend left, "S_2^-"] & S_2^- S_1^-(\underline{V})
\end{tikzcd}$$

indecomposable Q-mod

if indecomposable, continue
else it ~~[illegible]~~ is 0, in which
case $\underline{V} = \mathbb{C}_1$

indecomposable $\rho_2\rho_1$-mod
~~or $S_1^- \underline{V} = \mathbb{C}_2$~~.
or $S_1^- \underline{V} = \mathbb{C}_2$.

As long as these modules are nonzero, continue.

$$\begin{tikzcd}
\dim(\underline{V}) \arrow[r] & \sigma_1(\dim \underline{V}) \arrow[r] & \sigma_2\sigma_1(\dim \underline{V}) \\
\rotatebox{90}{$\in$} & \text{OR} & \text{OR} \\
\mathbb{Z}_{\geq 0}^{|Q_0|} & \vec{0} & \vec{0}
\end{tikzcd}$$

By (2a), $(C^-)^k(\dim \underline{V})$ has a strictly negative entry for some k. But this cannot be the dimension of a module, so at some point applying $\sigma_1, \sigma_2\sigma_1, \ldots$ will give the zero vector.

$\Longrightarrow \exists\, k \geq 0,\ 1 < s \leq |Q_0|$ such that $S_{s-1}^- \cdots S_1^- (C^-)^k \underline{V} = \underline{\mathbb{C}}_s$.
(by 2b, looking at dimensions). Has dimension α_s.

$$\sigma_{s-1} \cdots \sigma_1 c^k(\dim \underline{V}) = \alpha_s.$$

Hence $\beta(\dim \underline{V}, \dim \underline{V}) = \beta(\alpha_s, \alpha_s) = 2$.

Therefore, $\dim \underline{V} \in \Phi^+$

Hence, the map is well-defined. $\dim : \left\{ \begin{matrix} \text{indecomposable} \\ Q\text{-mod} \\ \text{up to ISO} \end{matrix} \right\} \longrightarrow \Phi^+$

$\underline{V} \longmapsto \dim \underline{V}$

This also tells us that

$$\underline{V} \cong (C^+)^k S_1^+ S_2^+ \cdots S_{s-1}^+ (\underline{\mathbb{C}}_s)$$

s and k can be read off from the dimension of $\underline{V}$, which means that $\underline{V}$ can be recovered from $\dim \underline{V}$. Hence dim is injective. It is surjective because any element of Φ^+ may be turned into a Q-mod in this way: let $\underline{v} \in \Phi^+$

Step 1: Produce s, k s.t. $\sigma_{s-1} \cdots \sigma_1 C^k \cdot v = \alpha_s$

Step 2: $\underline{V} := (C^+)^k S_1^+ S_2^+ \cdots S_{s-1}^+ (\underline{\mathbb{C}}_s)$ is indecomposable, has $\dim \underline{V} = \underline{v}$.

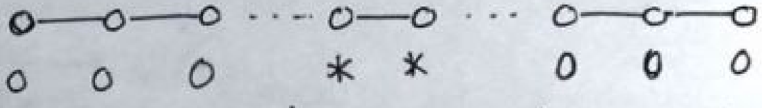
Hence, $\left| \left\{ \begin{matrix} \text{indecomposable} \\ Q\text{-mod up to} \\ \text{isomorphism} \end{matrix} \right\} \right| = |\Phi^+| = \text{finite}.$ ■

Example: $Q^u = A_n$

$$\beta(\underline{v}, \underline{v}) = 2\left(v_1^2 + v_2^2 + \dots + v_n^2 - v_1 v_2 - v_2 v_3 - \dots - v_{n-1} v_n\right).$$

Want $\underline{v} \in \Phi^+$, that is, $v \in \mathbb{Z}^n_{\geq 0}$ and $\sum_{i=1}^{n} v_i^2 - \sum_{i=1}^{n-1} v_i v_{i+1} = 1$.

Because if there is an interrupted string of zeros, $\underline{V}$ is a sum of other dimensions for other, smaller A-type reps, then it is not indecomposable. Hence, can only have uninterrupted strings of nonzero nodes, like this: Eg.



So reduce it to the case when $v_i \geq 1$ for all i.

$\underline{v} \in \Phi^+$ and $v_i \geq 1$ for all i.

$$\beta(\underline{1}, \underline{1}) = 2(n-(n-1)) = 2 \qquad \text{if } \underline{v} = \underline{1}$$

$$2 = \beta(\underline{v}, \underline{v}) = \beta((\underline{v}-\underline{1})+\underline{1}\,;\ (\underline{v}-\underline{1})+\underline{1}) \qquad \text{if } \underline{v} \neq \underline{1}$$

$$= \beta(\underline{v}-\underline{1}, \underline{v}-\underline{1}) + 2 + 2\beta(\underline{v}-1, 1)$$

let $\underline{w} = \underline{v} - 1$

$$\beta(\underline{1}, \underline{w}) = 2\sum_{i \in Q_0} w_i - \sum_{Q_1}(w_i + w_j) = 2\sum_{Q_0} w_i - 2\sum_{Q_1}(w_i) + w_1 + w_n$$

$$= w_1 + w_n$$

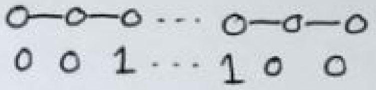
So

$$2 = \beta(\underline{v}, \underline{v}) = \beta(\underline{v}-\underline{1}, \underline{v}-\underline{1}) + 2 + 2\beta(\underline{v}-\underline{1}, \underline{1})$$

$$= (\geq 2) + 2 + 2(\geq 0) > 2.$$

So $\underline{v} = 1$ is the only option.

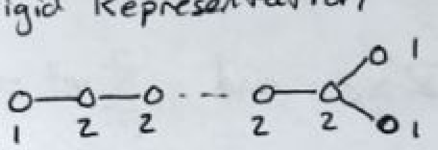
Indecomposable modules are



There are exactly $\binom{n}{2}$ of these.

<u>Example:</u> Rigid Representation

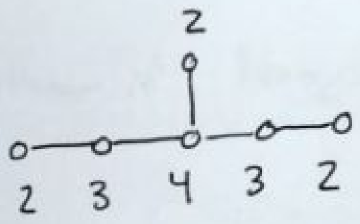
Is it indecomposable?

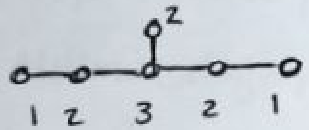


$$\beta(\dim \underline{V}, \dim \underline{V}) = 2((3 + 4(n-3)) - (6 + 4(n-4)))$$

$$= 6 + 8(n-3) - 12 - 12(n-4)$$

$$= 8 + 6 - 12 = 2.$$

Example: E_6  is not irreducible!

But  is indecomposable.

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CATEGORIFICATION

Categories

$\mathcal{C}$: objects $A, B, \ldots$ $\qquad \mathrm{Hom}_{\mathcal{C}}(A, B) = \{A \to B\}$

arrows/morphisms $A \to B$ $\qquad id_A \in \mathrm{Hom}(A, A)$ is identity w.r.t. composition.

respect composition

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \xrightarrow{\circ} \mathrm{Hom}(A, C)$$

$$(f, g) \longmapsto g \circ f.$$

Functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "f"'] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, mapsto] & F(B) \end{tikzcd}$$

$F(id_A) = id_{F(A)}$

F is compatible with composition.

$$F(g \circ f) = F(g) \circ F(f)$$

Natural Transformation / Functorial Morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D} \end{tikzcd}$$

$F \xrightarrow{\phi} G$ consists of $\phi_A : F(A) \to G(A)$ for each $A \in \mathcal{C}$, each compatible with composition.

$$\begin{tikzcd} F(A) \arrow[r, "\phi_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\phi_B"'] & G(B) \end{tikzcd}$$

Isomorphism of Categories:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad \begin{matrix} F \circ G = id_{\mathcal{D}} \\ G \circ F = id_{\mathcal{C}} \end{matrix}$$

Equivalence of Categories:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad \left.\begin{matrix} F \circ G \simeq id_{\mathcal{D}} \\ G \circ F \simeq id_{\mathcal{C}} \end{matrix}\right\}$$

Functorial isomorphism: i.e. natural transformation between $F \circ G$ and $id_{\mathcal{D}}$.

Example: $\left\{\begin{matrix}\text{finite dim.} \\ \mathbb{R}\text{-vector spaces}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z} \text{ as objects} \\ Hom(m,n) = M_{m \times n}(\mathbb{R})\end{matrix}\right\}$

$$V \longmapsto \dim V$$

$$\mathbb{R}^n \longleftarrow\!\shortmid\; n$$

Equivalent but <u>not</u> isomorphic: $V \neq \mathbb{R}^{\dim V}$.

Fact: $\mathcal{C} \xrightarrow{F} \mathcal{D}$

$$F \text{ is an equivalence} \iff \begin{cases} Hom_{\mathcal{C}}(A,B) \xrightarrow{F} Hom_{\mathcal{D}}(F(A),F(B)) \text{ bijection} \\ \forall X \in Obj(\mathcal{D}) \; \exists A \in Obj(\mathcal{C}) \text{ s.t. } F(A) \cong X \end{cases}$$

"Faithful" and "Full"

"Dense"

Abelian Categories:

Each $Hom_{\mathcal{C}}(A,B)$ is an abelian group, composition is bilinear.
The category has both direct sums and direct products (finite)
kernels, cokernels make sense, kernels are monomorphisms, epimorphisms are cokernels

Example:

Groups are not an abelian category because the "kernel" of a homomorphism need not be normal

A category is k-linear if $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a k-vector space.

<u>Theorem:</u> (Freyd-Mitchell)

Any Abelian category is equivalent to a full subcategory of A-mod for some ring A.

<u>Recall:</u> G a finite group, $V, W \in G$-mod.

$V \otimes_{\mathbb{C}} W$ is another G-module, with action $g \cdot (v \otimes w) = gv \otimes gw$.

<u>Monoidal</u> <u>Category:</u> Consists of $(\mathcal{C}, \otimes, a, \mathbb{1}, i)$

- $\mathcal{C}$ is a category
- $\otimes$ is a bifunctor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- a is a functorial isomorphism (means that $\otimes$ is associative) "associativity constraint"

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[rr, bend left, "\otimes\circ(\otimes\times id)", ""{name=U, below}] \arrow[rr, bend right, "\otimes\circ(id\times\otimes)"', ""{name=D, above}] & & \mathcal{C} \arrow[from=U, to=D, "\wr\, a"]
\end{tikzcd}$$

- $\mathbb{1}$ is an object in $\mathcal{C}$
- $i : \mathbb{1} \otimes \mathbb{1} \xrightarrow{\sim} \mathbb{1}$ is an isomorphism

$(\mathbb{1}, i)$ is called the "unit" of $\mathcal{C}$.

such that

- (pentagon axiom)

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes W \arrow[rr, "a_{X,Y,Z}\otimes id_W", "\sim"'] \arrow[d, "a_{X\otimes Y,Z,W}"', "\wr"] & & (X\otimes(Y\otimes Z))\otimes W \arrow[d, "\wr"', "a_{X,Y\otimes Z,W}"] \\
(X\otimes Y)\otimes(Z\otimes W) \arrow[dr, "\sim", "a_{X,Y,Z\otimes W}"'] & & X\otimes((Y\otimes Z)\otimes W) \arrow[dl, "\sim"', "id_X\otimes a_{Y,Z,W}"] \\
& X\otimes(Y\otimes(Z\otimes W)) &
\end{tikzcd}$$

- (unit axiom)

$$\left.\begin{array}{l} L_{\mathbb{1}} = \mathbb{1} \otimes - : \mathcal{C} \longrightarrow \mathcal{C} \\ R_{\mathbb{1}} = - \otimes \mathbb{1} : \mathcal{C} \longrightarrow \mathcal{C} \end{array}\right\}$$ equivalence of categories from $\mathcal{C}$ to $\mathcal{C}$.

Remarks:

$$\begin{tikzcd}
\mathbb{1} \otimes (\mathbb{1} \otimes X) \arrow[rr, bend right, "\mathrm{id}_{\mathbb{1}} \otimes \ell_X"'] & (\mathbb{1} \otimes \mathbb{1}) \otimes X \arrow[l, "\sim", "a_{1,1,X}"'] \arrow[r, "\sim", "i \otimes \mathrm{id}_X"'] & \mathbb{1} \otimes X
\end{tikzcd}$$

$\exists\, \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$ such that the diagram commutes.

$L_{\mathbb{1}} \xrightarrow[\sim]{\ell} \mathrm{id}_{\mathcal{C}}$ is a functorial isomorphism. Similarly, get a right functorial isomorphism.

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "a_{X,1,1}"] \arrow[rr, bend right, "r_X \otimes \mathrm{id}_{\mathbb{1}}"'] & X \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[r, "\mathrm{id}_X \otimes i"] & X \otimes \mathbb{1}
\end{tikzcd}$$

$R_{\mathbb{1}} \xrightarrow[\sim]{r} \mathrm{id}_{\mathcal{C}}$ functorial isomorphism.

Proposition (triangle diagrams)

I.

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "\sim", "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \ell_Y"] \\
& X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{1} \otimes X) \otimes Y \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathbb{1} \otimes (X \otimes Y) \arrow[dl, "\sim"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes Y) \otimes \mathbb{1} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & X \otimes (Y \otimes \mathbb{1}) \arrow[dl, "\sim"] \\
& X \otimes Y &
\end{tikzcd}$$

II. $\ell_{\mathbb{1}} = r_{\mathbb{1}} = i$

III $\ell_{\mathbb{1} \otimes X} = \mathrm{id}_{\mathbb{1}} \otimes \ell_X$

$r_{X \otimes \mathbb{1}} = r_X \otimes \mathrm{id}_{\mathbb{1}}$

Proof: use the pentagon axiom

$$\begin{tikzcd}
((X\otimes\mathbb{1})\otimes\mathbb{1})\otimes Y \arrow[rr, "\sim"] \arrow[dd, "\wr"'] \arrow[dr, "r_X"] \arrow[rr, phantom, "///", shift right=4ex] & & (X\otimes(\mathbb{1}\otimes\mathbb{1}))\otimes Y \arrow[dl, "i"'] \arrow[dd, "\wr"] \\
\text{|||} & (X\otimes\{\mathbb{1}\}\otimes Y\} \arrow[d, "\wr"] & \text{|||} \\
(X\otimes\mathbb{1})\otimes(\mathbb{1}\otimes Y) \arrow[r, "r_X"] \arrow[dr, "\wr"'] & X\otimes(\mathbb{1}\otimes Y) & X\otimes((\mathbb{1}\otimes\mathbb{1})\otimes Y) \arrow[l, "i"'] \arrow[dl, "\sim"] \\
\bigstar & X\otimes(\mathbb{1}\otimes(\mathbb{1}\otimes Y)) \arrow[u, "l_Y"'] & \text{|||}
\end{tikzcd}$$

Commutes because Associativity is natural, defn of r_X, l_Y $\blacksquare$

the pair $(\mathbb{1}, i)$

Replace in $\bigstar$ $\mathbb{1}\otimes Y$ with Y, using equivalence of categories.

Proposition: $\mathbb{1}$ is unique up to unique isomorphism

Proposition: $(\mathrm{End}_C(\mathbb{1}), \circ)$ is commutative ~~[illegible] category~~

if C is abelian, $\mathrm{End}(\mathbb{1})$ is a commutative ring.

Proof: $\mathbb{1}\otimes\mathbb{1} \xrightarrow{\sim} \mathbb{1}$

$$\mathrm{End}(\mathbb{1}\otimes\mathbb{1}) \xrightarrow{\sim} \mathrm{End}(\mathbb{1})$$

$$\begin{tikzcd}
\mathbb{1} \arrow[d, "f"'] & \mathbb{1}\otimes\mathbb{1} \arrow[l, "i"', "\sim"] \arrow[d, "\mathbb{1}\otimes f = f\otimes\mathbb{1}"] \\
\mathbb{1} & \mathbb{1}\otimes\mathbb{1} \arrow[l, "i"', "\sim"]
\end{tikzcd}$$

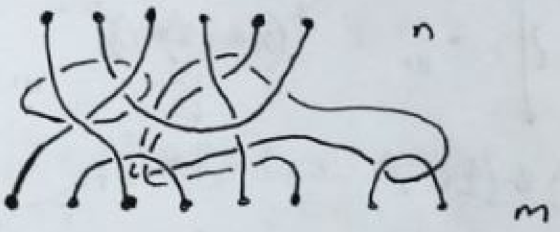
$$f = \chi(\mathbb{1}\otimes f) = \chi(f\otimes\mathbb{1})$$

$$fg = \chi(f\otimes\mathbb{1})\chi(\mathbb{1}\otimes g) = \chi(f\otimes g) = \chi(\mathbb{1}\otimes g)\,\chi(f\otimes\mathbb{1})$$
$$= gf. \quad \blacksquare$$

Example: Tangles

objects: $\mathbb{Z}$.

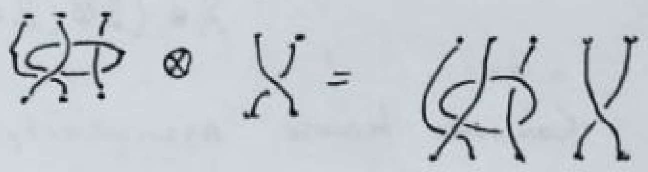
morphisms:



Composition is concatenation (when it makes sense)

Tensor product structure

$m \otimes n = m + n$



Monoidal Functors

Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories. A monoidal functor between $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, J) such that

(1) $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor

(2) $J : \otimes \circ (F \times F) \xrightarrow{\sim} F \circ \otimes$ natural transformation

$$F(X) \otimes_D F(Y) \xrightarrow[\sim]{J_{X,Y}} F(X \otimes_C Y).$$

(3) $F(\mathbb{1}_c) \simeq F(\mathbb{1}_D)$ and $F(i_c) \simeq F(i_D)$

$$F((\mathbb{1}_c, i_c)) \simeq F((\mathbb{1}_D, i_D)).$$

(4) ~~Another~~ Hexagon ~~Pentagon~~ condition (compatibility with associativity)

$$\begin{tikzcd}
(F(x) \otimes F(y)) \otimes F(z) \arrow[r, "a", "\sim"'] \arrow[d, "J", "\wr"'] & F(x) \otimes (F(y) \otimes F(z)) \arrow[r, "J", "\sim"'] & F(x) \otimes F(y \otimes z) \arrow[d, "J", "\wr"'] \\
F(x \otimes y) \otimes F(z) \arrow[r, "J", "\sim"'] & F((x \otimes y) \otimes z) \arrow[r, "F(a)", "\sim"'] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

Remark:

$$\begin{tikzcd} \mathbb{1}_D & \mathbb{1}_D \otimes F(\mathbb{1}_c) \arrow[l, "r_{F(\mathbb{1})}"', "\sim"] \arrow[r, "l_{\mathbb{1}_D}", "\sim"'] & F(\mathbb{1}_c) \arrow[ll, bend right, "\phi"'] \end{tikzcd}$$

ϕ is unique

$$\begin{tikzcd} F(\mathbb{1}) \otimes F(\mathbb{1}) \arrow[r, "J", "\sim"'] \arrow[dr, "i_F"'] & F(\mathbb{1} \otimes \mathbb{1}) \arrow[d, "\wr\ F(i)"] \\ & F(\mathbb{1}) \end{tikzcd}$$

$F((\mathbb{1}, i)) = (F(\mathbb{1}), i_F)$

$$\begin{tikzcd} F(\mathbb{1} \otimes X) \arrow[r, "\sim"] \arrow[d, "\phi \otimes id_{F(x)}"', "\wr"] & F(X) \arrow[d, "\wr"] \\ F(\mathbb{1}) \otimes F(X) \arrow[r, "\sim"] & \mathbb{1}_D \otimes F(X) \end{tikzcd}$$

This shows that $F(\mathbb{1}_c)$ is isomorphic to $\mathbb{1}_D$ by unique isomorphism.

Therefore, we may assume that $F(\mathbb{1}_c) = \mathbb{1}_D$.

Examples: Forgetful Functors

$\mathbb{C}G\text{-mod} \longrightarrow \mathbb{C}$-Vector Spaces.

Let R be a ring, $M \in R\text{-mod-}R$ can form

$$M \otimes_R - : R\text{-mod} \longrightarrow R\text{-mod}.$$

$$\begin{aligned} (R\text{-mod-}R, \otimes) &\longrightarrow (\text{End}(R\text{-mod}), \circ) \\ M &\longmapsto M \otimes_R - . \end{aligned}$$

$\nearrow$ End(R-mod) is a monoidal category w/ composition.

R-mod-R is monoidal with usual tensor product

Remark: let $\mathcal{C}$ be a monoidal category. MonEnd($\mathcal{C}$), the category of monoidal functors from $\mathcal{C}$ to $\mathcal{C}$ is a monoidal category under composition.

There is also a notion of monoidal natural transformations.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

objects $X \longrightarrow F(X)$

arrows are diagrams
$$\begin{tikzcd} X \arrow[r] \arrow[d] & F(X) \arrow[d] \\ Y \arrow[r] & F(Y) \end{tikzcd}$$

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

Def: A monoidal category $\mathcal{C}$ is said to be strict if $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ (equal, not isomorphic) $\forall X, Y, Z$.

Example: $\overline{\text{SET}}$ objects are finite sets up to bijection i.e. pos. integers

morphisms are $f: \{0, \dots, n-1\} \longrightarrow \{0, \dots, m-1\}$

$$\begin{tikzcd} m \otimes n = mn \arrow[d, "f"'] & \arrow[d, "g"] & \arrow[d, "f \otimes g"] \\ a \otimes b = ab & & \end{tikzcd}$$

$(f \otimes g)(mx+y) := af(y)+g(x)$.

This is a strict monoidal category.

$\underline{\text{Set}} \longrightarrow \overline{\text{set}}$ is a monoidal functor (and equivalence of categories).

$X \longmapsto |X|$

$\underline{\text{Vect}} \longrightarrow \overline{\text{Vect}}$ is another such example.

$V \longmapsto \dim V$

Theorem: (MacLane Strictness Theorem)
Let $\mathcal{C}$ moni~~a~~idal category. Then it is monoidially equival~~ant~~ent to a strict monoidal category.

Remark: cannot in general replace isomorphism with equivalence

Corollary: (MacLane Coherence Thm)
Let $\mathcal{C}$ be a monoidal category, $X_1, \dots, X_n \in \mathcal{C}$. Any ~~parenthe~~ two ways to parenthesize are same.

Corollary: (MacLane Coherence)

Let $\mathcal{C}$ be a monoidal category, $X_1, \dots, X_n \in \mathcal{C}$.

"Any two ways to parenthesize are the same, as well as with any ~~ways~~ number of identities inserted."

Let P_1, P_2 be two ways to parenthesize $X_1 \otimes \dots \otimes X_n$ in this order, with possible insertions of $\mathbb{1}$.

Let $f, g : P_1 \longrightarrow P_2$ isomorphisms from compositions, unit constraints.

Then $f = g$.

Def: Let $\mathcal{C}$ be a monoidal category, $X \in \mathcal{C}$.

A right dual for X is X^* together with

$$ev_X : X^* \otimes X \longrightarrow \mathbb{1} \qquad coev : \mathbb{1} \longrightarrow X \otimes X^*.$$

~~$coev_X \; X \otimes X^* \to \mathbb{1}$~~

such that

$$\begin{tikzcd} X \arrow[r] \arrow[rrr, bend right, "id_X"] & (X \otimes X^*) \otimes X \arrow[r] & X \otimes (X^* \otimes X) \arrow[r] & X \end{tikzcd}$$

and the same for X^*.

Also a similar notion of left dual, ${}^*X$

$${}_X ev : X \otimes {}^*X \longrightarrow \mathbb{1}$$

$${}_X coev : \mathbb{1} \longrightarrow X \otimes {}^*X$$

Remark: If a right/left dual exists, it is unique up to unique isomorphism.

Remark: Let $X \xrightarrow{f} Y$. Then there is

$$\begin{tikzcd} Y^* \arrow[r] \arrow[dd, bend right, "f^*"'] & Y^* \otimes (X \otimes X^*) \arrow[d] \\ & (Y^* \otimes Y) \otimes X^* \arrow[d] \\ & X^* \end{tikzcd}$$

Also ${}^*Y \xrightarrow{{}^*f} {}^*X$

Def: A monoidal category in which right/left duals exist is called rigid.

Example finite dimensional $\mathbb{C}$-vector spaces

fin. dim. $\mathbb{C}G$-mod

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Lecture 30 ①

Algebra 3

Recall monoidal
- categories
- functors
- natural transformation

rigid categories.

Def $\mathcal{C}$ is said to be locally finite /k if
- Obj / Iso is a set.
- $\text{Hom}_k(X, Y)$ are finite dimensional /k.
- $\forall X \in \text{Obj}(\mathcal{C})$ has finite length.

Rmk In locally finite categories, the following hold:
1). Shur's Lemma.
2). Jordan-Holder
3). Krull-Schmidt.
4). Grothendieck group : $\text{Gr}_{\mathbb{Z}}(\mathcal{C})$. ($\mathbb{Z}$-mod, operation $\oplus$.), $\text{Gr}_k(\mathcal{C})$

Def. $\mathcal{C}$ is said to be ~~finata of~~ finite if
- $\mathcal{C}$ is locally finite.
- (Simple Obj)/Iso is finite.
- $\mathcal{C}$ has enough projective object ($\forall X, \exists P \to X \to 0$ projective cover)

Def.

tensor {
- $\text{End}_{\mathcal{C}}(\mathbb{1}) \cong k$.
- multi-tensor {
  - monoidal, rigid
  - locally finite/k
  - $\otimes$ is bilinear on morphisms

monoidal, rigid; locally finite/k; $\otimes$ is bilinear on morphisms; finite, semisimple } multi-fusion

$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong k$; multi-fusion } fusion

Exp. finite-dimensional k-vector space, finite-dimensional kG-mod, are fusion.

Def Let $\mathcal{C}$ be monoidal & k-linear. A monoidal functor $\mathcal{C} \xrightarrow{F}$ k-Vect is called fiber functor.

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Lecture 30. ②

Thm (Reconstruction Theorems.) Equivalence of category between

{(monoidal finite category; fiber functors)} $\cong$ {finite dim./k bialgebras}.

finiteness is not neccessary

{(finite tenser category; fiber functors)} $\cong$ {finite dim/k Hopf algebras}.

{(fusion category; fiber functor)} $\cong$ {finite dim/k semisimple Hopf algebras}

$$(\mathcal{C}, F) \longmapsto \mathrm{End}(F).$$

$$(H\text{-mod}, \text{Forget}) \longleftarrow\!\shortmid \quad H.$$

forgetful functors ?

Def 1). $(A, A\otimes A \xrightarrow{m} A)$ $\quad k \xrightarrow{u} A$ $\qquad$ algebra

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\cdot id"] \arrow[d, "id\otimes m"'] \arrow[dr, phantom, "\wr\wr"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

m, multiplication

2). $(C, C\otimes C \xleftarrow{\Delta} C)$ $\qquad$ coalgebra

$k \xleftarrow{\varepsilon} C$,

$$\begin{tikzcd}
C\otimes C\otimes C \arrow[dr, phantom, "\wr\wr"] & C\otimes C \arrow[l, "\Delta\otimes id"'] \\
C\otimes C \arrow[u, "id\otimes\Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

Δ, comultiplication ?

3). $\left(H, \begin{matrix} H\otimes H \xrightarrow{m} H \\ H\otimes H \xleftarrow{\Delta} H \end{matrix}\right)$ such that $\begin{cases} (H, m) \text{ algebra} \\ (H, \Delta) \text{ coalgebra} \\ \Delta \text{ is an algebra map} \end{cases}$ is called bialgebra

4). Hopf algebra: (H, m, Δ, S) where $\begin{cases} (H, m, \Delta) \text{ bialgebra} \\ S: H\to H \text{ (antipode) is an inverse} \\ \text{of } id: H\to H \text{ in } (\mathrm{Hom}_k(H,H), *). \end{cases}$

[$f, g \in \mathrm{Hom}_k(H,H)$.

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[rrr, bend right, "f*g"'] & H\otimes H \arrow[r, "f\otimes g"] & H\otimes H \arrow[r, "m"] & H
\end{tikzcd} ; \quad \begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[rr, bend right, "u\circ\varepsilon"'] & k \arrow[r, "u"] & H
\end{tikzcd}$$

u unit

$(\mathrm{Hom}(H,H), *, u\circ\varepsilon)$ associative ring with unit.

$S * id_H = id_H * S = u\circ\varepsilon$.]

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Algebra 3

Lecture 30. ③

E.g. 1) $\mathbb{C}G$; $|G| < \infty$. $\Delta : \mathbb{C}G \longrightarrow \mathbb{C}G \otimes \mathbb{C}G$

$g \longmapsto g \otimes g$

$S : \mathbb{C}G \longrightarrow \mathbb{C}G$

$g \longmapsto g^{-1}$.

2). G Lie group $\rightsquigarrow D(G)$ = left-invariant differential operator on G. is a Hopf algebra.

Egs [Eg. $G = (\mathbb{R}^n, +)$; $D(\mathbb{R}^n) = \mathbb{C}[\partial_1, \dots, \partial_n] \longrightarrow \mathbb{C}[\partial_1, \dots \partial_n] \otimes \mathbb{C}[\partial_1, \dots, \partial_n]$.

$\partial_i \longmapsto \partial_i \otimes 1 + 1 \otimes \partial_i$]

3). X (= H-space) then ~~the~~ $H_*(X)$, $H^*(X)$ are Hopf algebra. (graded algebra) in super vector spaces.

Physics. 4). FRT $\rightsquigarrow$ Hopf algebras (quantum groups).

Dimension.

Rmk Let $\mathcal{C}$ be rigid category, $a : V \to V^{**}$ not associativity

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\mathrm{coev}_V"'] \arrow[rrr, bend right, "tr(a)"'] & V \otimes V^* \arrow[r, "a \otimes id"] & V^{**} \otimes V^* \arrow[r, "ev_{V^*}"] & \mathbb{1} \end{tikzcd}$$

Similarly for ${}^{**}V$.

tr is compatible with $\oplus$, $\otimes$. exact sequences.

Def ($\mathcal{C}$ (rigid category); $\Phi : Id_{\mathcal{C}} \xrightarrow{\sim} **$ (functorial ~~morphism~~ isomorphism)) is called ~~pivotal~~ pivotal category.

Def. Let $(\mathcal{C}, \varphi)$ pivotal. ~~dim X =~~

$$\dim_{(\mathcal{C},\varphi)} X = tr(\Phi_X).$$

Rmk. In general $\dim X \neq \dim X^*$.

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Lecture 30. ④

Algebra 3.

Def. $(\mathcal{C}, \phi)$ pivotal is said to be spherical if $\dim(X) = \dim(X^*)$, $\forall X \in \mathrm{Obj}(\mathcal{C}$,

Rmk. $\mathcal{C}$ finite pivotal.

- * $Gr_0(\mathcal{C}) \xrightarrow{\dim} \mathbb{C}$ is a ring morphism.
- * $\dim(X)$ algebraic integers.

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Recall: · Monoidal Categories

$\cup|$

· rigid monoidal categories

$\cup|$

are they equal? unknown {

· pivotal categories ← isomorphism between V and its dual

$\cup|$

· spherical categories ← V and its dual have the same dimension

}

- abelian categories

  $\cup|$

  k-linear

  $\cup|$

  locally finite ← finite length, Hom(A,B) fin.dim

  ↑

  Kroll Schmidt Theorem holds

  notion of Grothendieck group

finite ⊆ locally finite

↑

finitely many simple objects

projective covers

"enough projectives"

- tensor categories = locally finite, rigid monoidal

  $\cup|$

  fusion categories = finite tensor category, semisimple

- {(fusion category, fiber functor)} ≃ {fin. dim semisimple Hopf Algebras}

  ↑

  ~~homomorphism~~ to k-vector spaces

Example of the "Reconstruction Theorems"

Idea: want to recover algebra from the category, because the category is equal to the category of modules over a ring.

Some Linear Algebra

Frobenius-Perron Eigenvalues

- $A \in M_{n\times n}(\mathbb{R})$, $A_{ij} \geq 0$ for all i,j
- A is not conjugate to a block diagonal by a permutation matrix

Then

(1) A has at least 1 positive eigenvalue

(2) The largest eigenvalue λ has algebraic multiplicity 1.

(3) $\lambda = |\lambda| \geq |\mu|$ if μ is an eigenvalue (spectral radius $= |\lambda|$)

(4) Normalize the eigenvector of λ to have strictly positive entries, and sum to 1. Call it $\vec{v}_\lambda$

(5) If $\vec{v}$ is an eigenvector $\neq \vec{0}$ for A with entries in $\mathbb{R}_{\geq 0}$, then $\vec{v}$ is a scaling of $\vec{v}_\lambda$.

Def: Let $\mathcal{C}$ be a finite tensor category over $\mathbb{C}$

Grothendieck Group of $\mathcal{C}$ over $\mathbb{Z}$ $\rightarrow$

$$Gr_{\mathbb{Z}}(\mathcal{C}) \xrightarrow{[x]\cdot -} Gr_{\mathbb{Z}}(\mathcal{C})$$

The map is determined by its action on the generators x_i

$$[x_i] \longmapsto [x \otimes x_i] = \sum \underset{\underset{\mathbb{Z}_{\geq 0}}{\cap}}{m_{ij}(x)} [x_j]$$

The Frobenius-Perron dimension of X is the Frobenius Perron eigenvalue of m_{ij}, denoted FPdim(X).

Facts: (1) $\mathrm{FPdim}(X) \geqslant 1$

(2) $\mathrm{FPdim}(X) = 1 \iff X$ invertible $\quad (X \otimes X^* \simeq \mathbb{1} \simeq X^* \otimes X)$

(3) $\mathrm{FPdim}(X) \in \overline{\mathbb{Z}}$

(4) $\mathcal{C} \xrightarrow{F} \mathcal{D}$, monoidal, exact, faithful $\Longrightarrow$

$$\mathrm{FPdim}_{\mathcal{C}}(X) = \mathrm{FPdim}_{\mathcal{D}}(F(X))$$

(5) Notation: X_i simple

P_i projective covers

$$[R_{\mathcal{C}}] = \sum \mathrm{FPdim}(X_i)[P_i] \in Gr_{\mathbb{Z}}(\mathcal{C})$$

is called the regular (virtual) object of $\mathcal{C}$.

$R_{\mathcal{C}}$ corresponds in the $\mathbb{C}G$-mod case to $\mathbb{C}G$ itself — the regular rep.

$$\dim \mathcal{C} := \mathrm{FPdim}\, R_{\mathcal{C}}$$

(6) $Gr(\mathcal{C}) \xrightarrow{\mathrm{FPdim}} \mathbb{R}$ is the unique algebra map on simple objects taking values in $\mathbb{Z}_{>0}$

(7) $\mathcal{C} \xrightarrow{F} \mathcal{D} : [R_{\mathcal{D}}] = \dfrac{\mathrm{FPdim}(\mathcal{D})}{\mathrm{FPdim}(\mathcal{C})}[F(R_{\mathcal{C}})]$

F monoidal
exact
faithful

Analogue of La Grange's theorem.

If $\mathcal{C} = G$-mod, $\mathcal{D} = H$-mod, $R_{\mathcal{D}} = \mathbb{C}H$, $R_{\mathcal{C}} = \mathbb{C}G$, then this reduces to Lagrange's Theorem.

Remark: If H is a finite dimensional Hopf Algebra,

$$\mathrm{FPdim}(H\text{-mod}) = \dim(H)$$

In this case, $Gr_{\mathbb{Z}}(\mathcal{C}) \xrightarrow{\mathrm{FPdim}} \mathbb{Z}$

Defn: If $\mathrm{FPdim}(Gr_{\mathbb{Z}}(\mathcal{C})) \subseteq \mathbb{Z}$ we say that $\mathcal{C}$ is integral.

Theorem:

$$\left\{\begin{matrix}\text{integral}\\ \text{fusion}\\ \text{categories}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{finite dimensional}\\ \text{semisimple}\\ \text{Hopf Algebras}\end{matrix}\right\}$$

But there are fusion categories that are not integral.