Category Theory

webpage: tr.im/c2267

History of Category Theory

1945 - Eilenberg, MacLane
"General Theory of natural equivalences"
really wanted a definition of a "natural transformation"
↳ introduced functors, which led to categories

1957 - Grothendieck's Tohuku Paper
1958 - Daniel Kan, introduces adjunctions
, Lawvere - categorical foundations of math
1960's - Lambek - category theory ↔ proof theory
1970's - topos theory

applications: CS, Cog sci, linguistics, philosophy
↓
Universal properties
lcm ↔ direct product

1.2 in the book
Sets & fns

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

association

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

$h\circ(g\circ f) = (h\circ g)\circ f$

unit laws

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "f = f\circ 1_A"'] & A \arrow[d, "f"] \arrow[dr, "1_B\circ f = f"] & \\ & B \arrow[r, "1_B"'] & B \end{tikzcd}$$

1.3 in the book
Defn: A category consists of
- a collection of objects: $A, B, C, \dots$
- a collection of arrows: $f, g, h, \dots$
- For each arrow f, must be given two objects $\mathrm{dom}(f)$, $\mathrm{cod}(f)$
- Composition: given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, can form an arrow $A \xrightarrow{g\circ f} C$
- For each object A there is an arrow $A \xrightarrow{1_A} A$

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St - Composition is associative
- for $A \xrightarrow{f} B$, $f \circ \mathbb{1}_A = f = \mathbb{1}_B \circ f$

1.4 in the book

1) Sets & fns (Sets)
finite sets & fns (Sets_{fin})
Sets & bijections
etc.

2) Structured sets, homomorphisms
groups, vector spaces, graphs, rings, topological spaces, smooth manifolds, pointed sets

we will frequently look @ posets w/ monotone maps (Pos)

Let Rel be the category where the objects are sets, arrows: A, B sets, then $f \subseteq A \times B$ are arrows
- identity: $\mathbb{1}_A = \{(a,a) \mid a \in A\} \subseteq A \times A$
- Composition: if $R \subseteq A \times B$, $S \subseteq B \times C$, $S \circ R = \{(a,c) \in A \times C \mid \exists b \in B \text{ st } (a,b) \in R, (b,c) \in S\}$

- now take objects are elt's of $\mathbb{Z}^+$ and arrows $n \to m$ are $n \times m$ matrices
Composition is matrix multiplication

- Consider a single object $\mathbb{R}$
arrows: rational fns $p(x)/q(x)$

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<u>Finite Categories</u>

- 1 object, 1 arrow : $\star \xrightarrow{\mathbb{1}} \star$
- 2 objects, 3 arrows (2 id, 1 b/t, i.e. $\star \to \ast$)
- 3 objects $\begin{tikzcd} \star \arrow[r] \arrow[dr] & \ast \arrow[d] \\ & \circ \end{tikzcd}$

- discrete category (not necessarily finite) on set A has objects which are the elt's of A, and only identity arrows

now let C, D be categories.

<u>Definition</u> : a <u>functor</u> $F : C \to D$ is a mapping of - objects in $C \to$ objects in D
- arrows in $C \to$ arrows in D

s.t.

a) $F(f : A \to B) = F(f) : F(A) \to F(B)$

b) $F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$

c) $F(g \circ f) = F(g) \circ F(f)$

Cat is the category of categories where the arrows are functors

there are subtleties w/ this we'll see later

Take any preordered set, A

then we can define a category w/ objects the elts of A

& arrows $a \to b$ iff $a \leq_A b$

any category with at most one arrow b/t any pair of obj gives a preorder

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take as objects formulas $\phi, \psi, \ldots$

arrows from ϕ to ψ are deductions $\frac{\phi}{\psi}$

Category of proofs

Take any functional programming language, L

objects : types

arrows : programs of types $A \to B$

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Another important example is monoids & their homomorphisms

__Definition__: Category, an arrow $A \xrightarrow{f} B$ is an __isomorphism__ (iso) if
$\exists\, B \xrightarrow{g} A$ st $f \circ g = \mathbb{1}_B$ & $g \circ f = \mathbb{1}_A$

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Category Theory

Plan: Cayley's Thm
Constructions
Free Category

Def: $f : A \longrightarrow B$ is Isomorphism if it has an inverse, ie $\exists g : B \longrightarrow A$ $g \circ f = id_A$, $f \circ g = id_B$.

Notation: $g = f^{-1}$

NB: a group homomorphism preserves inverses.

Theorem (Cayley) Every group is isomorphic to a subgroup of $Aut(X)$ for X a set.

Proof: Define $\overline{G} \subseteq Aut(|G|)$ by $\overline{G} = \{\overline{g} \mid g \in G\}$, where

$|G|$ ↑ underlying set

$\overline{g} : |G| \longrightarrow |G|$
$\overline{g}(h) = g \cdot h$

$\overline{g}$ is a permutation w/ inverse $\overline{g^{-1}}$

Clearly isomorphic to G. ■

Theorem: $\mathbb{C}$ is a category with a set of objects and arrows (small category) is isomorphic to a category of sets and functions.

Proof: Define $\overline{C} = \{f \in \mathbb{C} \mid cod(f) = C\}$
$\overline{g} : \overline{C} \longrightarrow \overline{D}$, $\overline{g}(f) = g \circ f$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "f \circ g"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

New category $\overline{\mathbb{C}}$, isomorphic via $i : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$ which adds a bar to things. $j : \overline{\mathbb{C}} \longrightarrow \mathbb{C}$ is inverse defined by $j(\overline{g}) = \overline{g}(id_{dom(g)}) = g$. ■

Def: Such a category is called concrete.

Theorem: (Freyd) Ho(Top) is <u>not</u> concrete.

$\uparrow$
topological spaces, cts maps up to homotopy.

Constructions:

- Product of categories $\mathbb{C}$, $\mathbb{D}$

$\mathbb{C} \times \mathbb{D}$ has $\begin{cases} \text{objects} & (C, D) \text{ where } C \in \mathbb{C},\ D \in \mathbb{D} \\ \text{arrows} & (f, g) : (C, D) \longrightarrow (C', D') \text{ for } f : C \longrightarrow C',\ g : D \longrightarrow D' \end{cases}$

$$\begin{tikzcd} & \mathbb{C} \times \mathbb{D} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C} & & \mathbb{D} \end{tikzcd}$$

Projection functors

- Opposite Category $\mathbb{C}^{op}$

$\mathbb{C}^{op}$ has $\begin{cases} \text{objects as in } \mathbb{C} \\ \text{arrows } C \longrightarrow D \text{ in } \mathbb{C}^{op} \text{ is an arrow } D \longrightarrow C \text{ in } \mathbb{C} \end{cases}$

<u>Notation:</u> C^* is object in $\mathbb{C}^{op}$ corresponding to $C \in \mathbb{C}$

f^* is arrow in $\mathbb{C}^{op}$ corresponding to $f \in \mathbb{C}$

in $\mathbb{C}$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

in $\mathbb{C}^*$
$$\begin{tikzcd} A^* & B^* \arrow[l, "f^*"'] \\ & C^* \arrow[u, "g^*"'] \arrow[ul, "f^* \circ g^* = (g \circ f)^*"] \end{tikzcd}$$

<u>Example:</u> Affine Schemes are opposite Commutative Rings.

- arrow category $\mathbb{C}^{\to}$

$\mathbb{C}^{\to}$ has $\begin{cases} \text{objects} & f: A \to B \text{ an arrow in } \mathbb{C} \\ \text{arrows} & \text{commutative squares} \end{cases}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g_1"'] & B \arrow[d, "g_2"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd} \qquad f' \circ g_1 = g_2 \circ f$$

Projection functors

$$\begin{tikzcd} & \mathbb{C}^{\to} \arrow[dl, "dom"'] \arrow[dr, "cod"] & \\ \mathbb{C} & & \mathbb{C} \end{tikzcd}$$

- slice category $\mathbb{C}/C$

given a category $\mathbb{C}$, an object $C \in \mathbb{C}$,

$\mathbb{C}/C$ has $\begin{cases} \text{objects} & X \xrightarrow{f} C \\ \text{arrows} & \end{cases}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & C \\ X' \arrow[ur, "f'"'] & \end{tikzcd} \qquad f' \circ a = f$$

Functor $U: \mathbb{C}/C \longrightarrow \mathbb{C}$ given by taking the domain of an object

if $g: C \to D$, $g_*: \mathbb{C}/C \longrightarrow \mathbb{C}/D$ is given by left compose

$$g_* \left(\begin{matrix} X \\ \downarrow f \\ C \end{matrix} \right) = \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

If $\mathbb{C}$ is a small category,

$$\overline{(-)} : \mathbb{C} \xrightarrow{\overline{(-)}} \underline{Cat} \longrightarrow \underline{Set}$$

small categories = sets (by Cayley's Thm) if P is a poset, $p \in P$

$$P/p \cong \downarrow(p) = \{q \in P \mid q \leq p\}$$

- Coslice Category $C/\mathbb{C}$

$C/\mathbb{C}$ has objects $C \xrightarrow{f} X$

arrows

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "f'"'] & X \\ & X' \arrow[u, "a"'] \end{tikzcd}$$

$a \circ f' = f$

<u>Fact:</u> $\left(\mathbb{C}^{op}/c^*\right)^{op} \cong C/\mathbb{C}$

$$(-)/\mathbb{C} : \mathbb{C}^{op} \longrightarrow Cat$$

if $g : C \to D$, arrow $g^*(f) = f \circ g$ is functor $g^* : D/\mathbb{C} \longrightarrow C/\mathbb{C}$

$\underline{Set_*} \cong 1/\underline{set}$

pointed set

coslice with any one element object

Free Categories

<u>Def:</u> Free Monoid on set A (the alphabet)

$$A^* = \{\text{words over } A\} \quad \text{(Kleene Construction)}$$

$$= \{a_1, \ldots a_k \mid a_i \in A, k \geq 0\}$$

$$- = \text{empty word}$$

Monoid Multiplication is just concatenate words.

$|-| : \mathrm{Mon} \longrightarrow \mathrm{Set}$ forgetful functor

$i : A \longrightarrow |A^*| \qquad i(a) = a$

The free monoid satisfies a universal property:
if N is any monoid, $f : A \to |N|$ any set map, then there is a unique monoid HM $h : A^* \to N$ such that $|h| \circ i = f$

$$\begin{tikzcd}
\underline{\mathrm{Mon}} \arrow[d, bend right, "|-|"'] & A^* \arrow[rr, dashed, "\exists! h"] & & N \\
\underline{\mathrm{Set}} & {|A^*|} \arrow[rr, "|h|"] & & {|N|} \\
& & A \arrow[ul, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

<u>Proof</u>: existence

$h(-) = u_N$

$h(w * a) = h(w) \cdot_N h(a)$

check that it's a monoid HM

uniqueness

only one way to define it. ■

<u>Def</u>: Free Category

Think of small categories as diagrams in <u>Set</u>

$$\begin{tikzcd}
C_2 \arrow[r, "\circ"] & C_1 \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] & C_0 \arrow[l, "i"']
\end{tikzcd}$$

$C_2 = \{ (f,g) \in C_1 \times C_1 \mid \mathrm{cod}(f) = \mathrm{dom}(g) \}$

has a forgetful functor to graphs, forget i and $\circ$ and C_2

<u>Def</u>: Directed graph is $E \overset{s}{\underset{t}{\rightrightarrows}} V$

graph morphism is g

$$\begin{tikzcd}
E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "g_E"'] & V \arrow[d, "g_V"] \\
E' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & V'
\end{tikzcd}$$

$t' \circ g_E = g_V \circ t$

$s' \circ g_E = g_V \circ s$

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Recall: A set, A^* monoid, $i : A \longrightarrow |A^*|$ inclusion of generators.

Universal Mapping Property: given $f : A \to |N|$ for any monoid N,

$$\begin{tikzcd} \underline{\text{Mon}} & A^* \arrow[r, "\exists! \bar{f}"] & N \\ \text{Set} & {|A^*|} \arrow[r, "|\bar{f}|"] & {|N|} \\ & A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Free Categories

<u>Graphs</u> $\quad$ $\begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$

<u>Categories</u> $\quad$ $\begin{tikzcd} C_2 \arrow[r, "\circ"] & C_1 \arrow[r, bend left, "dom"] \arrow[r, bend right, "cod"'] & C_0 \arrow[l, "id"'] \end{tikzcd}$ $\qquad C_2 = \{(f,g) \in C_1 \times C_1 \mid cod(f) = dom(g)\}$

<u>Def</u>: the <u>free</u> <u>~~categoy~~</u> on graph $G = \begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd}$ is $\underline{C}(G)$ with objects V

arrows are paths $(v_0, v_n, e_1, \dots, e_n)$

start $\quad$ end $\quad$ edges

$s(e_1) = v_0$

$t(e_i) = s(e_{i+1})$ for $0 < i < n$

$t(e_n) = v_n$

$$\overset{\bullet}{v_0} \xrightarrow{e_1} \overset{\bullet}{v_1} \xrightarrow{e_2} \overset{\bullet}{v_2} \cdots \overset{\bullet}{v_{n-1}} \xrightarrow{e_n} \overset{\bullet}{v_n}$$

Example: recover $\mathbb{N}$ as $\{0\}^*$.

the category structure on $\underline{C}(G)$ is with

$id_v = (v,v,-)$

composition of $f=(v_0, v_n, e_1, \dots, e_n)$ and $g=(v_n, v_m, e_{n+1}, \dots, e_m)$

$g \circ f = (v_0, v_m, e_1, \dots, e_m)$.

NB: if $V = \{*\}$ is a singleton, then $\underline{C}(G) \cong E^*$ as a monoid category (E^* is a monoid/also category)

Universal Mapping Property: given any category $\mathbb{D}$ and $f: G \to |\mathbb{D}|$ (where $|\mathbb{D}|$ is $\mathbb{D}$ as a graph), $\exists!\ \bar{f}$ such that $|\bar{f}| \circ i = f$.

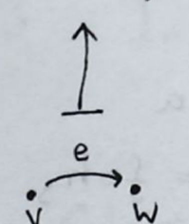
Cat

$$\underline{C}(G) \xrightarrow{\exists!\ \bar{f}} \mathbb{D}$$

Graphs

$$\begin{tikzcd} {|\underline{C}(G)|} \arrow[r, "|\bar{f}|"] & {|\mathbb{D}|} \\ G \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

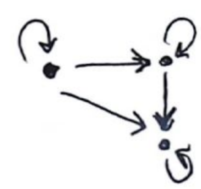
(v,w,e)

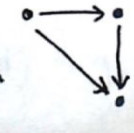
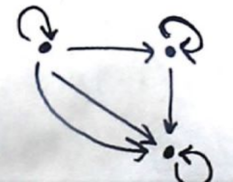


Examples: $V = \{a,b\}$, $E = \emptyset$

$\underline{C}(G)$ = discrete category on V

$\underline{2} = \underline{C}(* \longrightarrow \bullet) =$ 

$\underline{3} = \underline{C}(\cdot \longrightarrow \cdot \longrightarrow \cdot) =$ 

$\underline{C}($  $) =$ 

Foundations

<u>Def</u> a category $\mathbb{C}$ is <u>small</u> if the objects are a set and the arrows are a set.

<u>Examples</u>: finite categories, free categories on a set, discrete categories on a set, a monoids, preorder as category

<u>Non-examples</u>: <u>Sets</u>, <u>Pre</u>, <u>Groups</u>, <u>Graphs</u>, <u>Cat</u>

<u>Q</u>: Category of finite sets – is it small? No, otherwise <u>Sets</u> is small: put each set in a singleton $\Longrightarrow$ set of all sets.

But the category where objects, arrows are <u>hereditarily finite</u> sets is small. Every element of each set also finite.

<u>Def</u> $\mathbb{C}$ is <u>locally small</u> if $\mathrm{Hom}_{\mathbb{C}}(X,Y)$ is a set for all objects X, Y of $\mathbb{C}$.

Types of Arrows

<u>Def</u> In a category $\mathbb{C}$, an arrow $f : A \longrightarrow B$ is a <u>monomorphism</u> if for any $g, h : C \longrightarrow A$, $f \circ g = f \circ h$, then $g = h$

$$C \overset{g}{\underset{h}{\rightrightarrows}} A \xrightarrow{f} B$$

<u>Def</u>: $f : A \to B$ is an <u>epimorphism</u> if for any $i, j : B \longrightarrow D$

$i \circ f = j \circ f \Longrightarrow i = j$ $\qquad A \xrightarrow{f} B \overset{i}{\underset{j}{\rightrightarrows}} D$

Notation: $A \rightarrowtail^{f} B$ monomorphism

$A \overset{f}{\twoheadrightarrow} B$ epimorphism

Prop: in Sets, f mono iff f injective

proof: Suppose $f : A \rightarrowtail B$.

Let $x, y \in A$, $f(x) = f(y)$.

$$\begin{tikzcd} 1 \arrow[r, shift left, "\overline{x}"] \arrow[r, shift right, "\overline{y}"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

$$f \circ \overline{x} = f \circ \overline{y} \implies \overline{x} = \overline{y} \implies x = y.$$

Conversely, suppose f is injective

Let
$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$
say $f \circ g = f \circ h$

Let $z \in C$. $f(g(z)) = f(h(z)) \implies g(z) = h(z)$

Hence $g = h$. $\blacksquare$

Exercise f epic $\iff$ f surjective in Sets

NB: In Mon $\mathbb{N} \overset{i}{\rightarrowtail} \mathbb{Z}$

$(\mathbb{N}, 0, +)$ $(\mathbb{Z}, 0, +)$

Claim: i is epic. Let
$$\begin{tikzcd} \mathbb{N} \arrow[r, tail, "i"] & \mathbb{Z} \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & (M, 1, *) \end{tikzcd}$$

Suppose $f \circ i = g \circ i$

enough to show $f(-1) = g(-1)$

$$\begin{aligned} f(-1) = f(-1+1-1) &= f(-1) * u = f(-1) * g(1-1) = f(-1) * g(1) * g(-1) \\ &= f(-1) * g(i(1)) * g(-1) = f(-1) * f(i(1)) * g(-1) \\ &= f(-1) * f(1) * g(-1) = f(-1+1) * g(-1) = u * g(-1) \\ &= g(-1) \end{aligned}$$ $\blacksquare$

Hence, this is an epimorphism which is <u>not</u> surjective! Epic & Monic $\not\Rightarrow$ Iso. However,

<u>NB</u> every iso is both monic and epic

<u>Claim:</u> $f : A \to B$ has left inverse $g : B \to A$, $g \circ f = id_A$.
Then f is monic, g epic

$$C \underset{y}{\overset{x}{\rightrightarrows}} A \xrightarrow{f} B \qquad f \circ x = f \circ y \Rightarrow g \circ f \circ x = g \circ f \circ y$$
$$\Rightarrow x = y$$

And similarly

$$B \xrightarrow{g} A \underset{y}{\overset{x}{\rightrightarrows}} C \qquad x \circ g = y \circ g \Rightarrow x \circ g \circ f = y \circ g \circ f$$
$$\Rightarrow x = y. \quad \blacksquare$$

<u>Def:</u> An arrow is a <u>split mono</u> if it has a left inverse, and a <u>split epi</u> if it has a right inverse.

<u>Terminology</u> / <u>Notation</u>

$$E \underset{s}{\overset{e}{\rightleftarrows}} X$$

$e \circ s = id_X$

e is a <u>retraction</u>

s is a <u>section</u>

X is ~~the~~ a <u>retract of E</u>

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Initial and Terminal Objects

Def In a category $\underline{C}$, an object is initial if for any $C \in \underline{C}$, there is a unique arrow to C. Initial object is written 0.

$$0 \longrightarrow C$$

Def An object 1 is terminal if there is a unique arrow $C \longrightarrow 1$ for any other object C.

Prop: Initial and Terminal objects are unique up to unique isomorphism.

Examples in Sets, $\emptyset$ is initial, $\{*\}$ is terminal

in Rings (commutative w/ identity) $\mathbb{Z}$ is initial, 0 terminal.

in Cat, 0 is initial, 1 is terminal

in Groups, trivial group is both

in a poset P, initial is least element, terminal is greatest

in $\underline{C}/X$, terminal is $X \xrightarrow{id_X} X$.

in $X/\underline{C}$, initial is $X \xrightarrow{id_X} X$

Products:

Def: if $\underline{C}$ is a category, the product of A and B is an object P with maps $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ such that for any other object X with maps $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there is a unique $u : X \to P$ such that $x_1 = p_1 \circ u$, $x_2 = p_2 \circ u$.

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, "\exists!"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}.$$

Examples: in Sets, cartesian product

in Groups, $G \times H$

in Cat, $\underline{C} \times \underline{D}$

in a poset P, the greatest lower bound of p and q.

Simply typed λ-calculus with products

- base types
- types from base types using $A \to B$, $A \times B$, etc.
- terms: $x, y, z : A$
- constants $a : A$, $b : B$

if $a : A$, $b : B$, then $(a, b) : A \times B$

first $(c) : A$ if $c : A \times B$

second $(c) : B$ if $c : A \times B$

if $c : A \to B$ and $a : A$, then $ca : B$

if $b : B$, $x : A$ variable, $\lambda x.\, b : A \to B$.

$$\left.\begin{array}{l} \text{first}((a,b)) = a \\ \text{second}((a,b)) = b \end{array}\right\} \begin{array}{l} \text{equations} \\ \beta \text{ rules for pair} \end{array}$$

More λ-calculus

$(\mathrm{first}(c), \mathrm{second}(c)) = c$ η-rules for product

$(\lambda x.\ b)\, a = b[a/x]$

$\lambda x.\ cx = c$ if x is not a free variable

$\lambda x.\ b = \lambda y.\ b[y/x]$ if y is not a free variable

$c \sim c'$ if $c = c'$ if $c \sim c'$ and $a \sim a'$, then $ca \sim c'a'$.

Define a category using this with

$\begin{cases} \text{objects are the types} \\ \text{arrows are closed terms of function type } A \longrightarrow B \end{cases}$

identity on A is $\lambda x.\ x$ where $x : A$

Composition of $b : A \longrightarrow B$, $c : B \to C$

$c \circ b = \lambda x.\ c(bx)$, x not a free variable in c or b

Check that it's a category.

$c : B \longrightarrow C$

$id_C \circ c = \lambda x.(\lambda y. y)(cx) = \lambda x.\ cx = c$.

$c \circ id_B = \lambda x.\ c((\lambda y. y)\, x) = \lambda x. cx = c$.

Exercise: associativity

This category has products

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B \qquad \begin{aligned} p_1 &= \lambda c.\ \mathrm{first}(c) \\ p_2 &= \lambda c.\ \mathrm{second}(c) \end{aligned}$$

given X,

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & A\times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

define $u = \lambda x.\, (x_1(x), x_2(x))$

<u>Check</u> $p_1 \circ u = x_1$

$$\lambda x.\, p_1(u\,x) = \lambda x.\, p_1\,(x_1(x), x_2(x)) = \lambda x.(\lambda y.\, \mathrm{first}(y))\,(x_1(x), x_2(x))$$
$$= \lambda x.\, \mathrm{first}\,((x_1(x), x_2(x))) = \lambda x.\, x_1(x) = x_1 \;. \checkmark$$

Curry–Howard correspondence

Types	Propositions
$a : A$, $b : B$	
$(a;b) : A \times B$	$A \wedge B$
$a \to b : A \to B$	$A \Longrightarrow B$

<u>Categories with binary products</u>:

if $\underline{C}$ has binary products, then it also has ternary products

$$\begin{tikzcd} & X \arrow[dl, bend right] \arrow[d, "\exists!"'] \arrow[dr, bend left] \arrow[dd, bend left] & \\ A & A\times B\times C \arrow[l] \arrow[r] \arrow[d] & C \\ & B & \end{tikzcd}$$

Just take
$$A \times B \times C = (A\times B)\times C$$
$\cong$ uniquely IM'ic
$$A \times (B \times C)$$

Terminal object is nullary product.

An object A is a unary product of itself.

If $\underline{C}$ has binary products, get a functor

$\times : \underline{C} \times \underline{C} \longrightarrow \underline{C}$

In any locally small category, $\mathrm{Hom}_{\underline{C}}(A,B) = \{ f \in \underline{C} \mid f: A \to B \}$

For $g: B \longrightarrow B'$, have $g_* : \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B')$

$f \longmapsto g \circ f$

$\mathrm{Hom}(A,-)$ is a functor from $\underline{C}$ to $\underline{\mathrm{Sets}}$, and preserves products.

Given a product $A \longleftarrow P \longrightarrow B$, then there is an iso

$\theta_X : \mathrm{Hom}(X,P) \longrightarrow \mathrm{Hom}(X,A) \times \mathrm{Hom}(X,B)$.

01/28/15

Duality

Language of categories has objects, domain, codomain, arrows

1_A, $g \circ f$

$\mathrm{dom}(1_A) = \mathrm{cod}(1_A) = A$

$f \circ 1_{\mathrm{dom}(f)} = f = 1_{\mathrm{cod}(f)} \circ f$

$\mathrm{dom}(g) = \mathrm{cod}(f) \Longrightarrow \begin{cases} \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \\ \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \end{cases}$

$\begin{matrix} \mathrm{cod}(f) = \mathrm{dom}(g) \\ \mathrm{cod}(g) = \mathrm{dom}(h) \end{matrix} \Longrightarrow h \circ (g \circ f) = (h \circ g) \circ f$

Any statement Σ has a dual Σ^* made by

replace $f \circ g$ with $g \circ f$

replace $\mathrm{dom}(f)$ with $\mathrm{cod}(f)$

replace $\mathrm{cod}(f)$ with $\mathrm{dom}(f)$.

$$\left(\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot\right)^* = \left(\cdot \xleftarrow{f} \cdot \xleftarrow{g} \cdot\right)$$

<u>Prop</u>: If $\underline{C}$ satisfies Σ, then $\underline{C}^{op}$ satisfies Σ^*

<u>Prop</u>: If Σ is true for ~~some~~ all category, then so is Σ^*

<u>Defn</u>: the <u>coproduct</u> of A and B is an object Q with maps $i : A \longrightarrow Q$, $j : B \longrightarrow Q$ such that for any object Z with maps $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$, there is a unique map $u : Q \longrightarrow Z$ such that the diagram below commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dr, "z_1"'] & Q \arrow[d, "u"] & B \arrow[l, "j"'] \arrow[dl, "z_2"] \\
& Z &
\end{tikzcd}$$

<u>Examples</u>: in <u>Sets</u>, disjoint union $A \sqcup B$.

in <u>Mon</u>, for free monoids $M(A), M(B)$; $A, B \in$ <u>Sets</u>

coproduct is $M(A \sqcup B)$

More generally, the coproduct of monoids M and N is $M(|M| \sqcup |N|)/\sim$, with $v u_M u \sim v u \sim v u_N u$, $v\, m_1 m_2\, u \sim v(m_1 \cdot_M m_2)$.

Examples: in Ab, direct sum $G \oplus H$

in Simply-Typed λ-Calculus, a new type $A+B$

$$\begin{cases} \text{if } a:A, \text{ then } \mathrm{left}(a): A+B \\ \text{if } b:B, \text{ then } \mathrm{right}(b): A+B \\ \mathrm{Case}\,(c: s,t): C \text{ if } c: A+B,\ s: A \to C,\ t: B \to C \end{cases}$$

$$\left.\begin{aligned} \mathrm{Case}(\mathrm{left}(a): s,t) &= s(a) \\ \mathrm{case}(\mathrm{right}(b): s,t) &= t(b) \end{aligned}\right\} \beta\text{-rules}$$

η-rule $\{ c: A+B \to C \implies c = \lambda x.\ \mathrm{case}(x, c\circ\mathrm{left}, c\circ\mathrm{right})$

Equalizers.

Def: An equalizer of a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is $E \xrightarrow{e} A$ such that $f\circ e = g\circ e$ and e is universal with respect to that property: If $Z \xrightarrow{z} A$ satisfies $f\circ z = g\circ z$, then $\exists!\ u: Z \to E$, $z = e\circ u$

Note that e must be monic by the universal property.

Def: a regular monomorphism is a mono that occurs as an equalizer.

examples: in Sets, $\left\{ \begin{smallmatrix} a\in A, \\ f(a)=g(a) \end{smallmatrix} \right\} \rightarrowtail A \underset{g}{\overset{f}{\rightrightarrows}} B$

Exercises: (1) Any split mono is regular?

(2) an epic, regular mono is iso?

(3) is $\mathbb{Z} \rightarrowtail \mathbb{Q}$ monic? split? regular? in Ab

(4) give $\underline{C}$ where there is a non-regular mono.

Answers: (1) True!

$$\begin{tikzcd} E \arrow[r, tail, "e"'] & A \arrow[l, bend right, "r"'] \arrow[r, shift left, "e\circ r"] \arrow[r, shift right, "1_A"'] & A \\ Z \arrow[u, dotted, "r\circ z"] \arrow[ur, "z"'] & & \end{tikzcd}$$

if ~~…~~: $e\circ r\circ z = z$, then take $r\circ z$ to be map $z \longrightarrow E$.

(2)

$$\begin{tikzcd} E \arrow[r, tail, two heads, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ A \arrow[u, dotted, "u"] \arrow[ur, "id_A"'] & & \end{tikzcd}$$

$f\circ e = g\circ e \Longrightarrow f = g$ because e is epic, so $\exists!\ u: A \to E$ $e\circ u = id_A \Longrightarrow e$ is iso (by Homework)

Def: A coequalizer of a parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is $B \xrightarrow{s} Q$ such that $s\circ f = s\circ g$ and B is universal with this property.

Coequalizers are **surjective**.

examples: in Sets, if $\sim$ is an equivalence relation on A, $A/\sim$ coequalizes.

$$\begin{tikzcd} R_\sim \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "q"] \arrow[dr, "f"'] & A/\sim \arrow[d, dotted] \\ & & Y \end{tikzcd}$$

02/02/15

Representable Functors

Covariant version: $\underline{C}$ a (locally) small category, X an object

$$\mathrm{Hom}_{\underline{C}}(X,-) : \underline{C} \longrightarrow \underline{\mathrm{Sets}}$$

Contravariant version: $\underline{C}$ locally small category, X an object

$$\mathrm{Hom}_{\underline{C}}(-,X) : \underline{C}^{op} \longrightarrow \underline{\mathrm{Sets}}$$

Theorem: If $\underline{C}$ has a terminal object, binary products, and equalizers, then these are preserved by $\mathrm{Hom}(X,-)$.

Proof: $1 \in \underline{C}$ terminal. So $\mathrm{Hom}(X,1)$ is a singleton, hence terminal in $\underline{\mathrm{Sets}}$.

$$\begin{tikzcd} & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & B \end{tikzcd}$$

a product diagram in $\underline{C}$.

$$\begin{tikzcd} & \mathrm{Hom}(X, A\times B) \arrow[dl, "p_1\circ(-)"'] \arrow[dr, "p_2\circ(-)"] & \\ \mathrm{Hom}(X,A) & & \mathrm{Hom}(X,B) \end{tikzcd}$$

is also a product diagram in $\underline{\mathrm{Sets}}$, because of the universal property in $\underline{C}$: each arrow $X \xrightarrow{u} A\times B$ arises exactly from maps $p_1\circ u$, $p_2 \circ u$, and likewise for any two maps, $X \longrightarrow A$, $X \longrightarrow B$, there is unique $u: \cancel{X} \longrightarrow A\times B$.

Now say $E \overset{e}{\rightarrowtail} A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an equalizer in $\underline{C}$.

$$\mathrm{Hom}(\cancel{X},E) \xrightarrow{e\circ(-)} \mathrm{Hom}(X,A) \underset{g\circ(-)}{\overset{f\circ(-)}{\rightrightarrows}} \mathrm{Hom}(X,B)$$

is also an equalizer, by the universal properties. ■

Groups and Categories

Fix a category $\underline{C}$ with finite products.

Def: A group object in $\underline{C}$ consists of an object G and arrows

$$G \times G \xrightarrow{m} G \quad \text{(multiplication)}$$
$$G \xrightarrow{i} G \quad \text{(inverse)}$$
$$1 \xrightarrow{u} G \quad \text{(identity)}$$

such that $G \longrightarrow 1 \xrightarrow{u} G$ unit is also map $G \to G$ by composing w/ canonical map $G \to 1$

(1) m is associative : this diagram commutes

$$\begin{tikzcd}
(G \times G) \times G \arrow[rr, "\sim"] \arrow[d, "m \times id_G"'] & & G \times (G \times G) \arrow[d, "id_G \times m"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

(2) u is a unit for m : this diagram commutes

$$\begin{tikzcd}
G \arrow[r, "\langle u, id_G \rangle"] \arrow[d, "\langle id_G, u \rangle"'] \arrow[dr, "id_G"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

unique map into product gives this map from the maps $G \xrightarrow{id} G$ and $G \xrightarrow{u} G$

(3) multiplication by inverses on both sides

$$\begin{tikzcd}
G \times G \arrow[d, "id_G \times i"'] & G \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u"] & G \times G \arrow[d, "i \times id_G"] \\
G \times G \arrow[r, "m"'] & G & G \times G \arrow[l, "m"]
\end{tikzcd}$$

diagonal map to product obtained by $G \xrightarrow{id} G$, $G \xrightarrow{id} G$.

Note: for any $x, y, z : Z \to G$ ~~$x,y,z \in G$~~, we have $m(m(x,y),z) = m(x, m(y,z))$

$$m(x,u) = x = m(u,x).$$

$$m(x, i(x)) = u = m(i(x), x).$$

What is a group homomorphism?

An arrow $G \xrightarrow{\phi} H$ such that all diagrams below commute.

$$\begin{tikzcd} G \times G \arrow[r, "\phi \times \phi"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_H"] \\ G \arrow[r, "\phi"] & H \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, "u_G"'] \arrow[dr, "u_H"] & \\ G \arrow[rr, "h"] & & H \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "i_G"'] & H \arrow[d, "i_H"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

What are the groups in Groups?

an object G, with $\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$

m is a group homomorphism

$$m(g_1 h_1, g_2 h_2) = m(g_1, g_2)\, m(h_1, h_2)$$

Prop (Eckmann – Hilton Argument)

G a set, $\circ, *$ binary operations on G, ~~then~~ ~~with units $1^\circ, 1^*$, then~~ with units $1^\circ, 1^*$, then $1^\circ = 1^*$ and $\circ = *$.

Also assume $(g_1 \circ h_1) * (g_2 \circ h_2) = (g_1 * g_2) \circ (h_1 * h_2)$

Proof: $1^{\circ} = 1^{\circ}\cdot 1^{\circ} = (1^{\circ} * 1^{*})\cdot(1^{*} * 1^{\circ}) = (1^{\circ}\cdot 1^{*}) * (1^{*}\cdot 1^{\circ}) = 1^{*} * 1^{*} = 1^{*}$

$$x\cdot y = (x*1)\cdot(1*y) = (x\cdot 1)*(1\cdot y) = x*y.$$ ■

Furthermore, $\cdot$ is commutative.

$$x\cdot y = (x\cdot 1)\cdot(1\cdot y) = x\cdot y = (1\cdot x)\cdot(y\cdot 1) = (1\cdot y)\cdot(x\cdot 1) = x\cdot y$$

Thus, for groups in the category of groups, we have two operations satisfying this law so it must be abelian.

Categorical Semantics

→ generalization of model theory, but less high-power.
→ example of the kind of question we ask is:

What is a group in Cat? (Also a category in Groups?)

Called Strict 2-groups.

What is a monoid in Cat?

Def: $\underline{C}$ a category. A strict ~~category is~~ monoidal category is

$$\begin{cases} \otimes : \underline{C}\times\underline{C} \longrightarrow \underline{C} & \text{a functor} \\ I \in \underline{C} & (\text{unit for } \otimes) \end{cases}$$

such that

$$\begin{cases} A\otimes(B\otimes C) = (A\otimes B)\otimes C \\ I\otimes A = A = A\otimes I. \end{cases}$$

Example: Objects $\emptyset = 0$, $\{\emptyset\} = 1$, $\{0,1\}$, ----, $\{0, \ldots, n-1\} = n$, ----

arrows: functions between these

$n\otimes m = n+m$.

Recall: $\ker(\phi) = \{g \in G \mid \phi(g) = 1\}$

$N \trianglelefteq G \iff \forall n \in N, \forall g \in G, gng^{-1} \in N.$

Kernels are equalizers:

$$\begin{tikzcd} \ker(\phi) \arrow[r, tail] & G \arrow[r, shift left, "h"] \arrow[r, shift right, "1"'] & H \end{tikzcd}$$

Theorem: If $N \trianglelefteq G$, $N \subseteq \ker(\phi)$ iff ϕ factors through G/N.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, two heads] & H \\ G/N \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

02/04/15

Homomorphism Theorem for Groups

G, H groups,
$u, \phi : G \to H$
$u(g) = 1_H$

$$\begin{tikzcd} \ker\phi \arrow[r] & G \arrow[r, shift left, "\phi"] \arrow[r, shift right, "u"'] & H \end{tikzcd}$$ equalizer

$N \trianglelefteq G$ (normal)

$$\begin{tikzcd} N \arrow[r, hook, shift left, "i"] \arrow[r, shift right, "u"'] & G \arrow[r, two heads, "\pi"] & G/N \end{tikzcd}$$ coequalizer

Theorem: $N \trianglelefteq G$

$N \subseteq \ker(h)$ iff h factors through $G \overset{\pi}{\twoheadrightarrow} G/N$

Proof: $(\Rightarrow)$ Universal Property of G/N; because $N \subseteq \ker(h)$, then $h \circ i = h \circ u$, so use coequalizer universal property.

$$\begin{tikzcd} N \arrow[r, hook, shift left, "i"] \arrow[r, shift right, "u"'] & G \arrow[r, "h"] \arrow[dr, two heads] & H \\ & & G/N \arrow[u, dashed, "\bar{h}"'] \end{tikzcd}$$

over $\to$

<u>Proof (continued)</u>: ($\Longleftarrow$) Assume $\exists \bar{h}$ s.t. $h = \bar{h} \circ \pi$.

for $n \in N$, $h(n) = \bar{h}(\pi(n)) = u$, so $nu^{-1} \in N$, thus $n \sim u$ in G/N. So $n \in \ker(h)$. ∎

<u>Corollary</u>: 1^{st} Isomorphism Theorem. Take $N = \ker(h)$.

<u>Def</u>: A congruence relation on a category $\underline{C}$ is an equivalence relation $\sim$ on arrows of $\underline{C}$ such that

(1) $f \sim g \implies \mathrm{dom}(f) = \mathrm{dom}(g)$
~~dom(~~ $\mathrm{cod}(f) = \mathrm{cod}(g)$

(2) $f \sim g$ and
$$\begin{tikzcd} \bullet \arrow[r, "a"] & \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$$

$\implies b \circ f \circ a = b \circ g \circ a$, when $\mathrm{cod}(a) = \mathrm{dom}(f) = \mathrm{dom}(g)$
$\mathrm{dom}(b) = \mathrm{cod}(f) = \mathrm{cod}(g)$.

<u>Example</u>: $(G, \cdot, 1)$ as a one-object category.

Say $\sim$ is such a congruence, and let $N = \{ g \in G \mid g \sim u \}$

$u \sim u$, $(x \sim u, y \sim u) \implies xy = uxy \sim uuy = y \sim u$

$x \sim u \implies x^{-1} = uux^{-1} \sim uxx^{-1} \sim u$

$x \sim u \implies gxg^{-1} \sim gg^{-1} \sim u$

Hence N is a normal subgroup!

Conversely, given $N \trianglelefteq G$, let $x \sim_N y$ if $xy^{-1} \in N$ defines such a congruence.

Def: Quotient by the congruence $\sim$ is coequalizer

$$\underline{C}^{\sim} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \underline{C} \twoheadrightarrow \underline{C}/\sim$$

Objects of $\underline{C}^{\sim}$, $\underline{C}$ are the same as those of $\underline{C}$

Arrows of $\underline{C}^{\sim}$ are ~~arrows~~ pairs $(f, g : A \to B)$ with $f \sim g$

Arrows of $\underline{C}/\sim$ are equivalence classes of arrows under $\sim$

Def: "kernel" of ~~the~~ a functor.

$F : \underline{C} \longrightarrow \underline{D}$ gives $\sim_F$ on $\underline{C}$ by $f \sim_F g$ iff $F(f) = F(g)$

$$\mathrm{Ker}(F) = \underline{C}^{\sim_F} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \underline{C} \xrightarrow{F} \underline{D}$$ ~~is equalizer.~~

Theorem: $F : \underline{C} \longrightarrow \underline{D}$, $\sim$ congruence on $\underline{C}$ such that $f \sim g \Rightarrow f \sim_F g$ iff F factors through $\underline{C} \overset{\pi}{\twoheadrightarrow} \underline{C}/\sim$

$$\begin{tikzcd}
\underline{C} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \underline{D} \\
& \underline{C}/\sim \arrow[ur, dotted] &
\end{tikzcd}$$

Proof: $(\Leftarrow)$ $f \sim g \Rightarrow F(f) = \bar{F}(\pi(f)) = \bar{F}(\pi(g)) = F(g)$

$(\Rightarrow)$ ~~Define $f \sim g \Leftrightarrow$~~

Let $f \sim g$. Then $\bar{F}(\pi(f)) = \bar{F}(\pi(g)) \Rightarrow F(f) = F(g)$.

∎

Finitely-Presented Categories

Finitely Presented Groups, e.g. $Q = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = ijk^2 = -1 \rangle$

To make the group $\langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle \cong Q$,

$$F(\{r_1, r_2, r_3\}) \underset{r}{\overset{l}{\rightrightarrows}} F(\{x, y\}) \twoheadrightarrow Q \ (\leftarrow \text{coequalizer})$$

$l(r_1) = x^4$ $\quad r(r_1) = 1$

$l(r_2) = x^2$ $\quad r(r_2) = y^2$

$l(r_3) = y^{-1}xy$ $\quad \cancel{r(x^{-1})} \equiv r(r_3) = x^{-1}$

Free Category on a Quiver

$\underline{C}(Q)$ the free category on Q a quiver

Σ set of equations $g_1 \circ \cdots \circ g_n = h_1 \circ \cdots \circ h_m$,
paths with same source and target.

gives relation $\sim_\Sigma$ on $\underline{C}(Q)$.

$\underline{C}(Q)/\sim_\Sigma$ is finitely presented category $\underline{C}(Q, \Sigma)$.

Obtain a coequalizer diagram

$$\underline{C}(n \times \underline{2}) \rightrightarrows \underline{C}(Q) \twoheadrightarrow \underline{C}(Q, \Sigma).$$

n is cardinality of Σ $\quad$ maps are left/right side of equations in Σ.

$n \times \underline{2}$ is

$$n \text{ times} \left\{ \begin{array}{c} \bullet \to \bullet \\ \bullet \to \bullet \\ \vdots \\ \bullet \to \bullet \end{array} \right.$$

Examples:

$$\begin{tikzcd} \bullet \arrow[loop, "f"] \end{tikzcd} \quad f^2 = 1 \text{ gives } \mathbb{Z}/(2)$$

$$\begin{tikzcd} \bullet \arrow[loop, "f"] \end{tikzcd} \quad f^2 = f, \quad \text{monoid with one idempotent}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left] \end{tikzcd} \quad f \circ g = 1_B \quad g \circ f = 1_A, \quad \text{two isomorphic objects}$$

$\underline{3}$ is presented as $\underline{C}(\bullet \to \bullet \to \bullet)$ or as

$$\begin{tikzcd} & \bullet \arrow[d, "g"] \\ \bullet \arrow[ur, "f"] \arrow[r, "h"'] & \bullet \end{tikzcd} \quad \text{with } h = g \circ f.$$

Subobjects:

Def: a subobject of $X \in \underline{C}$ is a mono $M \overset{m}{\rightarrowtail} X$.

Forms a category contained inside $\underline{C}/X$, $\mathrm{Sub}_{\underline{C}}(X)$.

$$\begin{tikzcd} M \arrow[dr, tail, "m"] \arrow[d, "f"'] & \\ M' \arrow[r, tail, "m'"'] & X \end{tikzcd}$$

$m = m' \circ f$

$\Longrightarrow f$ must be a mono.

"f is a map from m to m'"

Def: We say $m \subseteq m'$ iff $m \longrightarrow m'$ in $\mathrm{Sub}_{\underline{C}}(X)$,
and $m \equiv m'$ iff $\begin{tikzcd} m \arrow[r, bend left] & m' \arrow[l, bend left] \end{tikzcd}$ in $\mathrm{Sub}_{\underline{C}}(X)$

it follows that $M \cong M'$ in $\underline{C}$.

$\mathrm{Sub}_{\underline{C}}(X)$ is a preorder category, and

$\mathrm{Sub}_{\underline{C}}(X)/\equiv$ is a poset, also sometimes called $\mathrm{Sub}_{\underline{C}}(X)$.

Example: In $\underline{\mathrm{Sets}}$, $\mathrm{Sub}(X) \cong P(X)$

powerset

Say M is a subobject of X, M' subobject of M, get another subobject of X.

$$\begin{tikzcd} M' \arrow[r, tail, "f"] \arrow[rr, bend right, "m\circ f"'] & M \arrow[r, tail, "m"] & X \end{tikzcd}$$

Gives functor $\mathrm{Sub}_{\underline{C}}(M) \longrightarrow \mathrm{Sub}_{\underline{C}}(X)$.

Def: local membership

$$\begin{tikzcd} & M \arrow[d, tail, "m"] \\ Z \arrow[ur, dashed, "\bar{z}"] \arrow[r, "z"'] & X \end{tikzcd}$$

$z \in_X m \iff \exists \bar{z}$ s.t. $z = m\circ\bar{z}$

Pullbacks / Fibered Products

Def: The pullback of $A \xrightarrow{f} C \xleftarrow{g} B$, or the fibered product of A with B over C, is $P = A \times_C B$

such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

and P is universal among all such diagrams.

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Examples: if A, B are k-algebras,

$$\mathrm{Spec}(A \otimes_k B) = \mathrm{Spec}\,A \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,B.$$

Proposition: If $\underline{C}$ has products and equalizers, then $\underline{C}$ has equalizers. Take product $A \times B$, and from there take equalizer of maps $g \circ \pi_2$, $f \circ \pi_1$.

$$\begin{tikzcd}
E \arrow[dr, "e"] \arrow[drr, bend left, "\pi_2 \circ e = p_2"] \arrow[ddr, bend right, "\pi_1 \circ e = p_1"'] & & \\
& A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\text{DOES NOT COMMUTE}"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Claim that E is the fibered product of A, B over C.

Let Z be another object, with maps $z_2 : Z \longrightarrow B$ and $z_1 : Z \longrightarrow A$. Then there is a unique $\langle z_1, z_2 \rangle : Z \longrightarrow A \times B$ such that $z_1 = \pi_1 \circ \langle z_1, z_2 \rangle$, $z_2 = \pi_2 \circ \langle z_1, z_2 \rangle$.

Further assume $f \circ z_1 = g \circ z_2$, so then

$f \circ \pi_1 \circ \langle z_1, z_2 \rangle = g \circ \pi_2 \circ \langle z_1, z_2 \rangle$. Hence, we get a unique map $u : Z \longrightarrow E$ such that $\langle z_1, z_2 \rangle = e \circ u$.

Then, The maps z_1, z_2 factor through E, because

$$z_1 = \pi_1 \circ \langle z_1, z_2 \rangle = \pi_1 \circ e \circ u = p_1 \circ u$$
$$z_2 = \pi_2 \circ \langle z_1, z_2 \rangle = \pi_2 \circ e \circ u = p_2 \circ u.$$

Examples: In $\underline{\text{Sets}}$, $A \times_C B = \{ (a, b) \mid f(a) = g(b) \}$.

Def: In $\underline{C}$, $f : A \to B$, $m : M \rightarrowtail B$, get

$$\begin{tikzcd} A \times_B M \arrow[r, "\bar{f}"] \arrow[d, tail, "\bar{m}"'] & M \arrow[d, tail, "m"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with $\bar{m}$ monic.

Write $f^{-1}(m) := \bar{m}$.

$f^{-1} : Sub_{\underline{C}}(B) \longrightarrow Sub_{\underline{C}}(A)$

Two Pullbacks Lemma: Suppose the following commutes:

$$\begin{tikzcd} F \arrow[r, "f'"] \arrow[d, "h''"'] & E \arrow[r, "g'"] \arrow[d, "h'"] & D \arrow[d, "h"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

Then

(1) if small squares are pullbacks, then so is the whole rectangle. If $E = B \times_C D$, $F = E \times_B A$, then $F = A \times_C D$.

(2) If the rectangle and right square are pullbacks, then so is the left square. If $F = A \times_C D$ and $E = B \times_C D$, then $F = E \times_B A$.

Proof: (1)

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, dashed] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\ & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d] \\ & A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

Proof continued:

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "u"] \arrow[drr, bend left, "z_2"] \arrow[drrr, bend left, "g'\circ z_2"] \arrow[ddr, bend right, "z_1"'] & & & \\
& F \arrow[r, "f'"'] \arrow[d, "h''"] & E \arrow[r, "g'"'] \arrow[d, "h'"'] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d, "h"] \\
& A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

Assume $f\circ z_1 = h'\circ z_2$

Then $g\circ f\circ z_1 = h\circ g'\circ z_2 = \cancel{h}\, g h' z_2$

So $\exists!\ u: Z \longrightarrow F$, $z_1 = h''u$, $g'z_2 = g'f'u$

Note that $hg'z_2 = gfz_1 = hg'f'u$, so $f'u = z_2$ b/c both have universal property of pullback of $B\times_C D$. Hence, ~~$f\circ z_1 =$~~

$h''\circ u = z_1$ and $f'\circ u = z_2$, so ~~Z~~

$F = A\times_B E$. ∎

Corollary: Pullback of commutative triangle is commuting triangle.

$$
\begin{tikzcd}
C'\times_C A \arrow[r, dashed] \arrow[rrr, bend left] \arrow[dr] & C'\times_C B \arrow[d] \arrow[rrr] & & A \arrow[dr] \arrow[d] & \\
& C' \arrow[rr] & & C & B \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
C'\times_C A \arrow[r] \arrow[d] & A \arrow[d] \\
C'\times_C B \arrow[r] \arrow[d] & B \arrow[d] \\
C' \arrow[r] & C
\end{tikzcd}
$$

$C' \xrightarrow{h} C$ in $\underline{C}$ gives a functor $\underline{C}/C \xrightarrow{h^*} \underline{C}/C'$.

By the previous corollary, this is actually a functor.

$$\begin{tikzcd}
C'\times_C A \arrow[rrr] \arrow[dr, dashed, "h^*(id_A)"] \arrow[dd] & & & A \arrow[dr, "id_A"] \arrow[dd] & \\
& C'\times_C A \arrow[rrr] \arrow[dl] & & & A \arrow[dl] \\
C' \arrow[rrr, "h"'] & & & C &
\end{tikzcd}$$

In $\underline{Cat}$, there is a commutative square defined by.

$$\begin{tikzcd}
Sub_{\underline{C}}(C) \arrow[r, "h^{-1}"] \arrow[d] & Sub_{\underline{C}}(C') \arrow[d] \\
\underline{C}/C \arrow[r, "h^*"] & \underline{C}/C'
\end{tikzcd}$$

Furthermore, $h^{-1}: Sub_{\underline{C}}(C)/_{\equiv} \longrightarrow Sub_{\underline{C}}(C')/_{\equiv}$

is ~~an~~ a map of posets.

02/11/15

Inverse Images:

In **Sets**, $2^A \cong P(A)$, $2 = \{\top, \bot\}$.

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[rr, bend left, "!"] \arrow[dr, bend right, "x"'] & U \arrow[r, "!"] \arrow[d, tail] & 1 \arrow[d, tail, "\top"] \\
& A \arrow[r, "x_U"'] & 2
\end{tikzcd}$$

$$x_U(a) = \begin{cases} \top, & a \in U \\ \bot, & a \notin U \end{cases}.$$

$$2^A = \{f : A \to 2\}$$

In **Sets**, The pullback of $\phi : A \to 2$ is an object V_ϕ with maps

$V_\phi \xrightarrow{a} A$ such that

$$\begin{tikzcd}
V_\phi \arrow[r, "!"] \arrow[d, tail, "a"'] & 1 \arrow[d, tail, "\top"] \\
A \arrow[r, "\phi"'] & 2
\end{tikzcd}$$

is a pullback.

$V_\phi = \{ a \in A \mid \phi(a) = \top \}$.

Prop: pullbacks are products in the slice category.

$$\begin{tikzcd}
& P \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dotted] & \\
A \arrow[r] & C & B \arrow[l]
\end{tikzcd}$$

Limits: A category $\underline{C}$ has finite products and equalizers iff $\underline{C}$ has pullbacks and terminal objects.

Proof: ($\Longleftarrow$) $A \times B = A \times_1 B$ is recovering ~~[illegible]~~ product with pullbacks.

The equalizer can be constructed as follows:

given $A \overset{f}{\underset{g}{\rightrightarrows}} B$, the equalizer E is the pullback of the diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "e"'] & B \arrow[d, tail, "\Delta"] \\
A \arrow[r, "\langle f, g \rangle"'] & B \times B
\end{tikzcd}.$$

over $\longrightarrow$

<u>proof continued:</u> ($\Longrightarrow$) Nullary product is terminal.

Equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$

is pullback of

$$\begin{tikzcd} & B \arrow[d, "\Delta"] \\ A \arrow[r, "\langle f, g\rangle"'] & B \times B \end{tikzcd}$$

. $\blacksquare$

Def: $\underline{J}$ a category, $\underline{D}: \underline{J} \longrightarrow \underline{C}$ a functor, $\underline{D}$ is called a <u>diagram of type J</u>.

<u>Examples:</u>

$\underline{J}$	$D: \underline{J} \to \underline{C}$	Cone ~~Limit~~	Limit
$\underline{C}(\cdot\ \ \cdot)$	$A \quad B$	$A \leftarrow C \rightarrow B$	product $A \times B$
$\underline{C}(\cdot \to \cdot \leftarrow \cdot)$	$\begin{array}{ccc} & & B \\ & & \downarrow g \\ A & \xrightarrow{f} & C' \end{array}$	$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & \searrow & \downarrow g \\ A & \xrightarrow{f} & C' \end{array}$	pullback $A \times_{C'} B$.
$\underline{C}(\cdot \rightrightarrows \cdot)$	$A \underset{g}{\overset{f}{\rightrightarrows}} B$	$E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$	equalizer
$\underline{C}(\emptyset)$	!	C	~~initial~~ terminal object.

Def: A <u>cone over D</u> consists of an object $C \in \underline{C}$, arrows $c_j : C \longrightarrow D_j$ for $j \in \underline{J}$, such that for all $\alpha: i \to j$ in J,

$$\begin{tikzcd} & D_i \arrow[d, "D\alpha"] \\ C \arrow[ur, "c_i"] \arrow[r, "c_j"'] & D_j \end{tikzcd}$$

commutes.

Def: The <u>limit</u> over the diagram D is the terminal cone in the category of cones over D. Denoted $\varprojlim_j D_j$.

Why the notation $\varprojlim D$?

$\underline{J}$	Diagram	Cone
$\underline{C}\ (\cdots \to \overset{3}{\bullet} \leftarrow \overset{2}{\bullet} \leftarrow \overset{1}{\bullet} \leftarrow \overset{0}{\bullet})$ $(\mathbb{N}, \leq)$	$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \to \cdots$	$\begin{tikzcd} & C \arrow[dl, "c_0"'] \arrow[d, "c_1"] \arrow[dr, "c_2"] & \\ A_0 \arrow[r] & A_1 \arrow[r] & A_2 \end{tikzcd}$
$\underline{C}\ (\cdots \to \overset{3}{\bullet} \to \overset{2}{\bullet} \to \overset{1}{\bullet} \to \overset{0}{\bullet})$ $(\mathbb{N}, \geq)$	$A_0 \xleftarrow{\alpha_0} A_1 \xleftarrow{\alpha_1} A_2 \leftarrow \cdots$	$\begin{tikzcd} & C \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ A_0 & A_1 \arrow[l, "\alpha_0"'] & A_2 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] \end{tikzcd}$

In case 1, the limit is just A_0 ... uninteresting.

In case 2, the limit is interesting, e.g. p-adics, inverse limit, etc.

<u>Proposition</u>: $\underline{C}$ has all (finite / $\leq \kappa$) limits iff it has all (finite / $\leq \kappa$) products and equalizers, where κ is some cardinal.

<u>Proof</u>: $(\Longrightarrow)$ trivial.

$(\Longleftarrow)$ The limit of a diagram $D: \underline{J} \to \underline{C}$ is the equalizer of the diagram with ~~objects~~ arrows $e \circ \pi_j =: p_j$

$$\phi = \langle \pi_j \mid \alpha: i \to j \text{ in } \underline{J} \rangle$$

$$\psi = \langle D\alpha \circ \pi_i \mid \alpha: i \to j \text{ in } \underline{J} \rangle$$

$$\begin{tikzcd}
\varprojlim D_j \arrow[r, "e"] \arrow[dr, "p_j"'] & \prod_{j \in \underline{J}} D_j \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] \arrow[d, "\pi_j"] \arrow[dr, "\pi_i"] & \prod_{\alpha: i \to j \text{ in } \underline{J}} D_j \arrow[dl, "\pi_\alpha"] \\
& D_j & D_i \arrow[l, "D\alpha"]
\end{tikzcd}$$

$$\pi_\alpha \circ \phi = \pi_j$$

$$\pi_\alpha \circ \psi = D\alpha \circ \pi_i$$

The condition $\phi \circ e = \psi \circ e$ means that $p_j = D\alpha \circ p_i$, and the universal property of the equalizer gives the ~~equalizer of the~~ universal property of limits. ■

Corollary: $\mathrm{Hom}_{\underline{C}}(X, -)$ preserves limits, because it preserves products and equalizers.

Def: A colimit of $D : \underline{J} \longrightarrow \underline{C}$ is a limit of $D^{op} : \underline{J}^{op} \longrightarrow \underline{C}^{op}$.
Alternatively, an initial cocone, denoted $\varinjlim_{j} D_j$.

$\underline{J}$	Diagram	Cocone	Colimit $\varinjlim D_j$
$\underline{C}\ (\emptyset)$		C	initial object
$\underline{C}\ (\cdot \rightrightarrows \cdot)$	$A \rightrightarrows B$	$A \rightrightarrows B \to C$	coequalizer
$\underline{C}\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot \end{tikzcd}\right)$	$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C \end{tikzcd}$	$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$	pushout
$\underline{C}\ (\cdot \quad \cdot)$	$A \quad B$	$A \to C \leftarrow B$	coproduct

By duality, a category $\underline{C}$ has all (finite / $\leq \kappa$) colimits iff it has all (finite / $\leq \kappa$) coproducts and coequalizers.

Furthermore, $\mathrm{Hom}_{\underline{C}}(-, X)$ ~~preserves all colimits.~~ sends colimits in $\underline{C}$ to limits in $\underline{\text{Sets}}$.

Colimits:

Prop: A category $\underline{C}$ has all colimits iff it has all coproducts and coequalizers.

Proof: $\alpha : i \to j$ in $\underline{J}$.

$$\begin{tikzcd}
D_i \arrow[r, "q_i"] \arrow[d, "D_\alpha"'] \arrow[rr, bend left, "c_i"] & Q & \\
D_j \arrow[ur, "q_j"'] \arrow[rr, bend right, "c_j"'] & & C
\end{tikzcd}$$

$\phi \circ i_\alpha = i_i$

$\psi \circ i_\alpha = i_j \circ D_\alpha$

$$\begin{tikzcd}
\coprod\limits_{\alpha : i \to j} D_i \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \coprod\limits_{i \in \underline{J}} D_i \arrow[r, two heads, "q"] \arrow[dr, "c"'] & Q \arrow[d, dashed] \\
& & C \\
D_i \arrow[uu, "i_\alpha"] \arrow[uur, "i_i"] \arrow[r, "D_\alpha"'] & D_j \arrow[uu, "i_j"'] &
\end{tikzcd}$$

$c \circ \phi \circ i_\alpha = c \circ i_i$
$= c_i = c_j \circ D_\alpha$
$= c \circ i_j \circ D_\alpha$
$= c \circ \psi \circ i_\alpha$
$\Rightarrow c\psi = c\phi.$

Q is the coequalizer of ϕ and ψ, and also the colimit over the diagram $D : \underline{J} \longrightarrow \underline{C}$. ∎

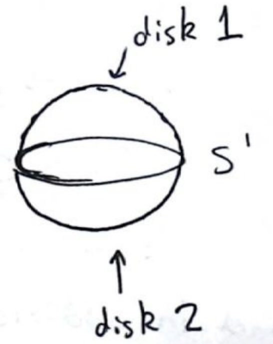
Examples:

In $\underline{\text{Top}}$, form colimits by taking the colimit of sets and topologize in the usual way. $D : \underline{J} \longrightarrow \underline{\text{Top}}$.

Q is the colimit of D with maps $D_j \xrightarrow{q_j} Q$.

Give Q the largest topology in which all q_i are continuous.

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\
D^2 \arrow[r] & S^2 = D^2 \times_{S^1} D^2 .
\end{tikzcd}$$



"Glue the disks together along their boundary S^1"

In Groups, the amalgamated sums, which is not the pushout of underlying sets.

Example: Direct limits, as colimits over $\omega = 0 \to 1 \to 2 \to \dots$

In Groups,

$$\begin{tikzcd} G_0 \arrow[r, "g_0"] \arrow[drr, "u_0"'] & G_1 \arrow[r, "g_1"] \arrow[dr, "u_1"] & G_2 \arrow[r] \arrow[d, "u_2"] & \dots \arrow[dl, "u_3"] \\ & & G_\infty & \end{tikzcd}$$

$G_\infty = \varinjlim G_i$ is initial cocone over the sequence $(G_i)_{i \in \mathbb{N}}$.

G_∞ is the colimit of underlying sets with maps $G_n \longrightarrow G_\infty$ taking an element to its equivalence class, with group structure

Let $g_{ij} = g_{j-1} \circ \dots \circ g_i$

$$\cancel{[a_0]} = \quad 1_\infty = [1_0]$$

$$\cancel{[x][y] = [xy]}$$

$$[x][y] = [g_{nk}(x)\, g_{mk}(y)]$$

$x \in G_n$, $y \in G_m$, for some $k \geq n, m$.

$$[x]^{-1} = [x^{-1}]$$

Prop: U: Groups $\to$ Sets forgetful functor creates all ω-colimits (direct limits)

Def: $F: \underline{C} \to \underline{D}$ creates $\underline{J}$-limits iff for each diagram $C: \underline{J} \to \underline{C}$ and limit (L, p_j) in $\underline{D}$, $p_j: L \to FC_j$, there is a unique cone $(\bar{L}, \bar{p_j})$ with $L = F\bar{L}$, $p_j = F(\bar{p_j})$ and in addition $(\bar{L}, \bar{p})$ is the limit of C in $\underline{C}$.

Def: $F : \underline{C} \to \underline{D}$ preserves $\underline{J}$-limits if and only if for each diagram $C : \underline{J} \longrightarrow \underline{C}$ and limit (L, p_j) over C, then (FL, Fp_j) is a limit for FC in $\underline{D}$.

$$F(\varprojlim_j C) = \varprojlim_j FC_j$$

F is also called continuous if it preserves all limits.

Prop: $Hom_{\underline{C}}(X, -) : \underline{C} \longrightarrow \underline{Sets}$ preserves all limits.

ω CPOs

Def: an ω CPO is an ω-cocomplete poset (has all ω-colimits). Meaning all increasing sequences have a least upper bound.

A monotone map of ω CPOs is an arrow in the category of ω CPOs if it is ω-cocontinuous: preserves ω-colimits.

Prop: If D is an ω CPO with initial element 0 and $h : D \to D$ is monotone and ω-cocontinuous, then h has a least fixed point. That is, an element x such that $h(x) = x$ and if $y = h(y)$, $x \leq y$.

Proof: Define a sequence $(a_n)_{n \in \mathbb{N}}$ by $a_0 = 0$, $a_{n+1} = h(a_n)$.

Want $a_n \leq a_{n+1}$ for all n. Proof by induction on n:

$a_0 = 0 \leq a_1$ ✓

if $a_n \leq a_{n+1}$, then $h(a_n) \leq h(a_{n+1}) \implies a_{n+1} \leq a_{n+2}$.

over $\longrightarrow$

Proof continued: let $a_\omega = \varinjlim_\omega a_n$.

$$h(a_\omega) = h(\varinjlim_\omega a_n) = \varinjlim_\omega h(a_n) = \varinjlim_\omega a_{n+1} = a_\omega$$

If $h(x) = x$, then we want for all n, $a_n \leq x$.

By induction, $a_0 = 0 \leq x$ ✓

if $a_n \leq x$, then $a_{n+1} = h(a_n) \leq h(x) = x$.

So $a_\omega \leq x$. Hence a_ω is the least upper bound. ∎

in ωCPOs, $\omega_0 \xrightarrow{i} \omega_1 \xrightarrow{i} \omega_2 \xrightarrow{i} \cdots$ where $\omega_n = (0 \leq \cdots \leq n)$

has a least upper bound $\omega + 1 = (0 \leq 1 \leq 2 \leq \cdots \leq \omega)$.

This is a colimit that does not come from the underlying set functor.

Exponentials

In Sets, $C^B = \{f : B \to C\}$

for $f : A \times B \longrightarrow C$, $\exists \bar{f} : A \longrightarrow C^B$

$\bar{f}(a)(b) = f(a,b)$ currying

$g : A \to C^B$, $\exists \bar{g} : A \times B \to C$

$\bar{g}(a,b) = g(a)(b)$ uncurrying

Want to mimic C^B in other categories.

eval : $C^B \times B \longrightarrow C$, eval$(g, b) = g(b)$.

~~eval ∘ (f̄ × b) = eval(f̄(a)(b)~~

$$\begin{tikzcd} C^B \times B \arrow[r, "eval = \varepsilon"] & C \\ A \times B \arrow[u, "\bar{f} \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

<u>Def:</u> If $\underline{C}$ has binary products, ~~the~~ <u>exponential</u> of B and C, C^B and arrow $\varepsilon : C^B \times B \longrightarrow C$ such that for any $f : A \times B \longrightarrow C$ there is a unique $\bar{f} : A \to C^B$ such that the diagram commutes

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Exponentials:

<u>Def:</u> if $\underline{C}$ has binary products, the exponential C^B of objects B, C if there is some $C^B \in \underline{C}$ and $\varepsilon : C^B \times B \longrightarrow C$ such that for any $f : A \times B \to C$ there is a unique $\bar{f} : A \longrightarrow C^B$, such that the below commutes.

$$\begin{tikzcd} C^B \times B \arrow[r, "\varepsilon"] & C \\ A \times B \arrow[u, "\bar{f} \times 1_B"] \arrow[ur, "f"'] & \end{tikzcd}$$

$$\mathrm{Hom}_{\underline{C}}(A \times B, C) \xrightarrow{\sim} \mathrm{Hom}_{\underline{C}}(A, C^B)$$

$$f \longmapsto \bar{f}$$

$$\bar{g} \longleftarrow g$$

"transposition"

Note $\bar{\bar{f}} = f$ and $\bar{\bar{g}} = g$.

Def: $\underline{C}$ is Cartesian closed category if $\underline{C}$ has finite products and all exponentials. (CCC)

Examples: $\underline{Sets}$, $\underline{Sets}_{fin}$, $\underline{Posets}$ $Q^P = \{f: P \to Q \text{ monotone}\}$ with the order $f \leq g$ iff $f(p) \leq g(p)$ for all $p \in P$.

$\varepsilon(f, p) = f(p)$.

Why should ε in $\underline{Posets}$ be monotone?

Recall $f \leq g$ iff $\forall p \in P$, $f(p) \leq g(p)$

and $(f, p) \leq (f', p')$ iff $f \leq f'$ and $p \leq p'$,

so $\varepsilon(f, p) = f(p) \leq f'(p) \leq f'(p') = \varepsilon(f', p')$.

Now, given $f: X \times P \to Q$ in $\underline{Posets}$, want $\bar{f}: X \to Q^P$ monotone: $x \leq_X x'$ takes $p \in P$,

$$\bar{f}(x)(p) = f(x, p) \leq f(x', p) = \bar{f}(x')(p).$$

Hence $\bar{f}(x) \leq \bar{f}(x')$.

$\underline{\omega CPOs}$ is CCC, but $\underline{\text{strict } \omega CPOs}$ is not.

meaning $\exists\, 0$ in any strict ωCPO, must map 0 to 0.

Because in strict ωCPOs,

$$\text{Hom}(1, Q^P) \cong 1 \cong \text{Hom}(1 \times P, Q) \cong \text{Hom}(P, Q).$$

$\underline{Graphs}$ is CCC.

H^G = graph with vertices $= \{\psi: V(G) \to V(H)\}$

edges are functions $\{\theta: E(G) \to E(H)\}$

such that the diagram commutes.

<u>Graphs</u> is CCC

$$\begin{tikzcd}
 & E(G) \arrow[d, phantom, "\shortparallel"] & \\
V(G) = G_v \arrow[d, "\phi"'] & G_e \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\theta"] & G_v = V(G) \arrow[d, "\psi"] \\
V(H) = H_v & H_e \arrow[l, "s"] \arrow[r, "t"] \arrow[d, phantom, "\shortparallel"] & H_v = V(H) \\
 & E(H) &
\end{tikzcd}$$

Transpose of $\varepsilon : B^A \times A \longrightarrow B$

$$\bar{\varepsilon} : B^A \longrightarrow B^A$$

is the unique map such that the following commutes:

$$\begin{tikzcd}
B^A \times A \arrow[dr, "\varepsilon"] & \\
 & B \\
B^A \times A \arrow[uu, "\bar{\varepsilon} \times 1_A"] \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

But 1_{B^A} is one such map, so $\bar{\varepsilon} = 1_{B^A}$.

<u>Prop</u> if $\underline{C}$ is a CCC, then $(-)^A$ is a functor $\underline{C} \to \underline{C}$ ~~such that~~ for any $A \in \underline{C}$.

<u>Proof</u>: given $B \xrightarrow{\beta} C$, want $B^A \xrightarrow{\beta^A} C^A$.

$$\begin{tikzcd}
C^A \times A \arrow[r, "\varepsilon"] & C \\
B^A \times A \arrow[u, "\beta^A \times id_A"] \arrow[r, "\varepsilon"'] & B \arrow[u, "\beta"']
\end{tikzcd}
\Longrightarrow (id_C)^A = id_{C^A}$$

Proof continued: $B \xrightarrow{\beta} C \xrightarrow{\gamma} D$

$$\begin{tikzcd}
D^A \times A \arrow[r, "\varepsilon"] & D \\
C^A \times A \arrow[u, "\gamma^A \times id_A"] \arrow[r, "\varepsilon"] & C \arrow[u, "\gamma"'] \\
B^A \times A \arrow[u, "\beta^A \times id_A"] \arrow[r, "\varepsilon"] \arrow[uu, bend left=60, "(\gamma\beta)^A \times id_A", "\cancel{(\beta^A \gamma^A) \times id_A}"'] & B \arrow[u]
\end{tikzcd}$$

Hence $\cancel{(\beta^A \gamma^A) \times id_A = (\beta\gamma)}$

$(\gamma^A \beta^A) \times id_A = \cancel{\gamma\beta}\ (\gamma\beta)^A \times id_A.$

So $(-)^A$ is a functor. $\blacksquare$

Transpose of $id_{A\times B} : A \times B \longrightarrow A \times B$

$$\eta = \overline{id_{A\times B}} : A \longrightarrow (A\times B)^A$$

$$\begin{tikzcd}
(A\times B)^B \times B \arrow[r, "\varepsilon"] & A \times B \\
A \times B \arrow[u, "\eta \times id_B"] \arrow[ur, "id_{A\times B}"']
\end{tikzcd}$$

Given $f : Z \times A \longrightarrow B$, what is $\bar{f} : Z \longrightarrow B^A$.

$$\begin{tikzcd}
B^A \times A \arrow[rr, "\varepsilon"] & & B \\
(Z\times A)^A \times A \arrow[u, "f^A \times id_A"] \arrow[r, "\varepsilon"] & Z \times A \arrow[ur, "f"'] & \\
Z \times A \arrow[u, "\eta \times id_A"] \arrow[ur, "id_{Z\times A}"'] \arrow[uu, bend left=60, "\bar{f} \times id_A"] & &
\end{tikzcd}$$

$$\Longrightarrow \quad \bar{f} = f^A \circ \eta$$

Uses of Exponentials

$$\begin{tikzcd} \mathrm{IPC} \arrow[r, bend left] & \mathrm{HA} \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \lambda\text{-calculus} \arrow[r, bend left] & \mathrm{CCCs}. \arrow[l, bend left] \end{tikzcd}$$

Def: A lattice is a poset with meet ($\wedge$) and join ($\vee$). (binary product + coproduct).

Alternatively, a set L with commutative & associative binary operators $\wedge, \vee$ such that

$$(a \wedge b) \vee a = a \qquad \text{(absorbtion laws)}$$
$$(a \vee b) \wedge a = a$$

define $a \leq b$ if and only if $a = b \wedge a$ if and only if $a \vee b = b$.

Bounded Lattice: has least and greatest elements 0 and 1.

alternatively, $a \vee 0 = a = a \wedge 1$.

Distributed Lattice: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, or

equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Heyting Algebras: a poset which is a CCC and has all finite coproducts,

$$(L, \wedge, \vee, 0, 1, \rightarrow) \qquad a \rightarrow b = b^a.$$

Boolean Algebra

= complimented distributive lattice, with operation $\overline{}$

$$(a \vee \bar{a}) = 1 \quad \text{and} \quad (a \wedge \bar{a}) = 0$$

= Heyting algebras with operation $\neg$, $(a \vee \neg a) = 1$.

= Algebras over $\mathbb{F}_2$.

$$\begin{tikzcd}
 & & \text{Boolean Algebra} \arrow[dl] \arrow[dr] & \\
 & \text{Heyting Algebra} \arrow[dl] \arrow[dr] & & \text{complimented} \arrow[dl] \\
\text{distributive lattice} \arrow[dr] & & \text{bounded lattice} \arrow[dl] & \\
 & \text{lattice} \arrow[d] & & \\
 & \text{poset} & &
\end{tikzcd}$$

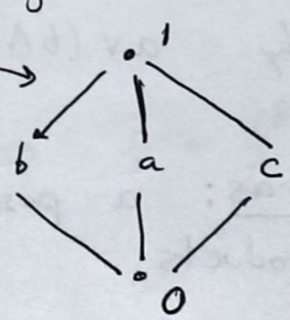
Examples: $P(X)$ is the powerset of X, is a boolean algebra.

$P_{fin}(X) = \{ Y \subseteq X \mid |Y| \text{ finite} \}$ is a lattice, but not in general bounded.

$(\mathbb{N}, 1)$ ordered by divisibility is a non-bounded lattice.

The topology τ on a space X, is a Heyting Algebra.

Note $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ in general, but the converse need not hold: eg this is not a distributive lattice



$0 = (a \wedge b) \vee (a \wedge c)$

$a = a \wedge (b \vee c)$

Def: A complete ~~lattice~~ poset is a ~~lattice~~ poset that is complete as a category (has all limits).

Prop: A poset is complete if and only if it is cocomplete.

Proof: $(\Longrightarrow)$ $J \subseteq P$, want $\bigvee J$.

Take $\bigvee J = \bigwedge \{ p \in P \mid p \geq j \text{ for all } j \in J \}$

$(\Longleftarrow)$ similary. ∎

Prop: A complete lattice is a Heyting algebra if and only if it satisfies the ∞ distributive law.

$$\left(\bigvee_i b_i\right) \wedge a = \bigvee_i (b_i \wedge a)$$

Proof: $(\Rightarrow)$ given a Heyting algebra, take any x.

$$\begin{aligned}
\left(\bigvee b_i\right) \wedge a \leq x &\iff \bigvee_i b_i \leq a \to x \\
&\iff \forall i,\ b_i \leq a \to x \\
&\iff \forall i,\ \cancel{b_i \wedge a \longrightarrow}\ b_i \wedge a \leq x \\
&\iff \bigvee_i (b_i \wedge a) \leq x .
\end{aligned}$$

Hence, $\left(\bigvee b_i\right) \wedge a = \bigvee_i (b_i \wedge a)$ $\quad \left(p = q \iff (\forall x,\ a \leq x \iff b \leq x) \right)$.

$(\Longleftarrow)$ let $a \to b := \bigvee \{ x \mid x \wedge a \leq b \}$

Show that it is an exponentral object.

if $y \wedge a \leq b$, $y \leq a \to b$ because $a \to b$ is $\bigvee \{ x \mid x \wedge a \leq b \}$

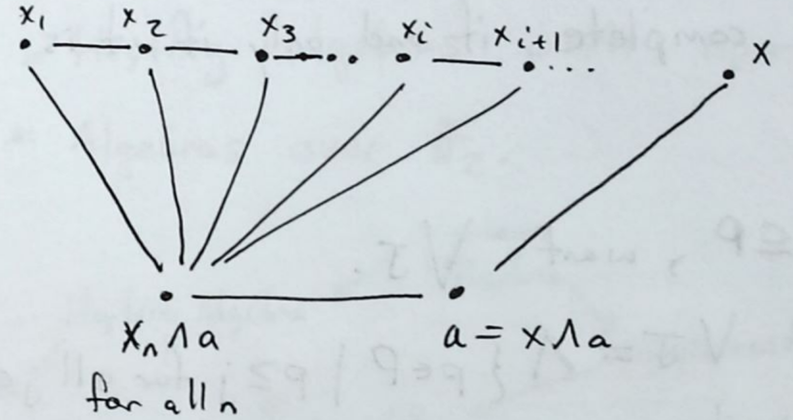
if $y \leq a \to b$, then $y \wedge a \leq \left(\bigvee \{ x \mid x \wedge a \leq b \} \right) \wedge a$

$= \bigvee \{ x \wedge a \mid x \wedge a \leq b \} = b$.

Counterexample on 46 from Homework:

$$\varinjlim F \times A \not\cong \varinjlim (F \times A)$$

$$(x_1 \le x_2 \le \dots x_i \le x_{i+1} \dots) \le x$$



02/23/15

Plan:

$$\begin{tikzcd} IPC \arrow[r, leftrightarrow] \arrow[d, squiggly] & HA \arrow[d, squiggly] \\ \lambda\text{-calc} \arrow[r, leftrightarrow] & CCCs \end{tikzcd}$$

Def: An IPC is a set Σ of atomic formulas

$$\mathrm{Prop}(\Sigma) \text{ is either } \begin{cases} a & \text{for } a \in \Sigma \\ \top & \text{true} \\ \bot & \text{false} \\ p \vee q & p, q \text{ propositions} \\ p \wedge q & \\ p \Rightarrow q & \end{cases}$$

A = set of axioms, $A \subseteq \mathrm{Prop}(\Sigma)$

$\mathcal{L}$ a theory in IPC is dependent on (Σ, A)

write $\mathcal{L} = \mathcal{L}(\Sigma, A)$.

A relation $\vdash_{\mathcal{L}} \subseteq \mathrm{Prop}(\Sigma) \times \mathrm{Prop}(\Sigma)$, that is the least relation such that

(1) $\top \vdash p$ for all $p \in A$

(2) $p \vdash p$, $\dfrac{p \vdash q \quad q \vdash r}{p \vdash r}$ $p \vdash q$ and $q \vdash r$ implies $p \vdash r$

(3) $p \vdash \top$ ($\top$ is greatest element in preorder)

$\bot \vdash p$ (F is least element)

over

(4) $p \vdash q$ and $p \vdash r$ iff $p \vdash (q \wedge r)$

$$\frac{p \vdash q \quad p \vdash r}{p \vdash q \wedge r}$$ } means that these statements are equivalent

(5) $p \vdash r$ and $q \vdash r$ iff $p \vee q \vdash r$

(6) $p \wedge q \vdash r$ iff $p \vdash q \Rightarrow r$.

Natural Deduction:

A set of premises entails single conclusion

$\Gamma = \{p_1, \dots, p_n\}$ $\qquad \Gamma \vdash q$

$\underbrace{\Gamma, p}_{= \Gamma \cup \{p\}} \vdash p$ (var) $\qquad \dfrac{\Gamma \vdash p \quad \Gamma, p \vdash q}{\Gamma \vdash q}$ (cut)

$$\frac{\Gamma \vdash p \quad \Gamma \vdash q}{\Gamma \vdash p \wedge q}$$

$\Gamma \vdash T$ (truth introduction)

$\dfrac{\Gamma \vdash \bot}{\Gamma \vdash p}$ (falsehood existence) ← for any p

$\dfrac{\Gamma \vdash p \wedge q}{\Gamma \vdash p}$ $(\wedge I_1)$ $\qquad \dfrac{\Gamma \vdash p \wedge q}{\Gamma \vdash q}$ $(\wedge I_2)$

$(\Rightarrow I)$ $\dfrac{\Gamma, p \vdash q}{\Gamma \vdash p \Rightarrow q}$

$\dfrac{\Gamma \vdash p}{\Gamma \vdash p \vee q}$ $(\vee I_1)$

$(\Rightarrow E)$ $\dfrac{\Gamma \vdash p \Rightarrow q \quad \Gamma \vdash p}{\Gamma \vdash q}$

$\dfrac{\Gamma \vdash q}{\Gamma \vdash p \vee q}$ $(\vee I_2)$

(axiom) $\Gamma \vdash p$ for p an axiom.

Lemma: if $p \vdash q$ in IPC, then $\{p\} \vdash q$ in Natural Deduction

Proof works by how we came to know p by the axioms and induction. Show that the natural deduction relation is at least as large as that of $\vdash$.

Lemma: If $\Gamma \vdash q$ in natural deduction, then $\bigwedge \Gamma \vdash q$ according to IPC.

Def. If $\mathcal{L}$ is a theory in IPC, we get a Heyting Algebra $HA(\mathcal{L})$, called the Lindenbaum-Tarski algebra of $\mathcal{L}$.

$\vdash_{\mathcal{L}}$ a preorder, take the corresponding poset

$$[p] = [q] \text{ iff } \underbrace{p \vdash q \text{ and } q \vdash p}_{\text{sometimes written } p \dashv\vdash q}$$

$$[p] \leq [q] \text{ iff } p \vdash q$$

top $1 = [\top]$

bottom $0 = [\bot]$

$[p \wedge q] = [p] \wedge [q]$

$[p] \vee [q] = [p \vee q]$

$[p] \Rightarrow [q] = [p \Rightarrow q]$

Thus, $1 = [p] \iff \top \vdash p$

$[p \Rightarrow q] = 1$ iff $p \vdash q$ (always true in Heyting algebra)

Def: An interpretation of Σ ~~is a function~~ in a Heyting algebra H is a function $[\![-]\!] : \Sigma \longrightarrow H$.

It can be extended to Prop (Σ) by

$$[\![\top]\!] = 1_H, \quad [\![\bot]\!] = 0_H$$

$$[\![p \wedge q]\!] = [\![p]\!] \wedge_H [\![q]\!] \text{ , etc}$$

$[\![-]\!]$ is a model of $\mathcal{L} = \{\Sigma, A\}$ iff $[\![a]\!] = 1_H$ for all $a \in A$
"all axioms are true".

Models in Heyting algebras are sound and complete for theorems in IPC.

Soundness: if $\top \vdash_{\mathcal{L}} p$, then $[\![p]\!] = 1_H$ in all models

Completeness: if $[\![p]\!] = 1_H$ in all models, then $\top \vdash_{\mathcal{L}} p$.

$\Longrightarrow$ in $HA(\mathcal{L})$, $[p] = 1_H \Rightarrow \top \vdash_{\mathcal{L}} p$

Universal Property for $HA(\mathcal{L})$:

if $[\![-]\!] : \Sigma \longrightarrow H$ is a model for $\mathcal{L}$, then there is a unique HA homomorphism such that the diagram commutes.

$$\begin{tikzcd} HA(\mathcal{L}) \arrow[rr] & & H \\ & \Sigma \arrow[ul, "{[-]}"] \arrow[ur, "{[\![-]\!]}"'] & \end{tikzcd}$$

Everything we did for HAs and IPC, we can do for smaller objects called HA^-s and positive fragments of IPCs.

Theory in λ-calculus:

Σ set of base types

(all types $A, B := X$ for $X \in \Sigma$
$1,\ A \times B,\ A + B$)

C set of constants with types $a : A$ (like the set of axioms)

Rules as normal for λ-calculus.

$\mathcal{E}$ set of equations between terms.

Note that $\langle\rangle = t$ for any $t : 1$ (all things of type 1 are the same)

Category $\mathcal{C}(\lambda)$ which has $\begin{cases} \text{objects} = \text{types} \\ \text{arrows} = \text{closed terms } a : A \to B \text{ up to equality} \end{cases}$

has products ✓

has terminal object 1 ✓

$A \longrightarrow B$ is the exponential object, with evaluation map

$\varepsilon = \lambda z.\ fst(z)(snd(z)) : B^A \times A \longrightarrow B.$

<u>Def</u>: If $\mathcal{L} = (\Sigma, C, \mathcal{E})$ is a λ-calculus theory, a <u>model</u> of $\mathcal{L}$ in a CCC $\underline{C}$ is given by

for $X \in \Sigma$, $[\![X]\!] \in \underline{C}$.

(for $b : A \to C$, $[\![b]\!] : [\![A]\!] \to [\![B]\!]$ in $\underline{C}$)

in general, if $a : A$, $[\![a]\!] : 1 \to [\![A]\!]$

such that if $s = t$ in $\mathcal{E}$, then $[\![s]\!] = [\![t]\!]$.

There is an inverse too! For any CCC $\underline{C}$, get a theory in λ-calculus $\mathcal{L}(\underline{C})$ with base types the objects of $\underline{C}$

constants of type $A \to B$ the arrows of $\underline{C}$.

equations
$$\lambda(x).\, fst(x) = p_1 \qquad \lambda x . x = 1_A . \text{ if } x : A.$$
$$\lambda(x).\, snd(x) = p_2$$
$$\lambda y .\, f\langle x, y\rangle = \tilde{f}$$
$$g(f(x)) = (g \circ f)(x)$$
$$\cancel{\lambda x:}$$

From here, get an isomorphism of categories

$$\underline{C}(\mathcal{L}(\underline{C})) \cong \underline{C}.$$

03/02/15

Naturality

Def: A natural transformation $\eta: F \to G$, where $F, G: \underline{C} \to \underline{D}$ are functors, is a family of arrows $\eta_C: FC \to GC$ for C an object of $\underline{C}$ such that for every arrow $f: C \to D$ in $\underline{C}$,

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FD \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_D} \\ GC & \xrightarrow{Gf} & GD \end{array}$$

commutes.

Defines a category $\underline{\text{Fun}}(\underline{C}, \underline{D})$ whose arrows are natural transformations and objects are functors $\underline{C} \to \underline{D}$.

The exponential $\underline{D}^{\underline{C}}$ in $\underline{\text{Cat}}$.

Category Theory of Cat:

has finite products, finite coproducts, and actually all small products/coproducts.

Can construct equalizers as a category $\underline{E}$.

over →

$$\begin{tikzcd} \underline{E} \arrow[r, "E"] & \underline{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \underline{D} \end{tikzcd}$$ is an equalizer diagram,

$$\underline{E}_0 = \{ C \in \underline{C}_0 \mid FC = GC \}$$

$$\underline{E}_1 = \{ f \in \underline{C}_1 \mid Ff = Gf \}.$$

Hence, $\underline{Cat}$ has all small limits (dually, colimits).

Def: $F : \underline{C} \longrightarrow \underline{D}$ is

(1) <u>injective on objects</u> iff $F_0 : \underline{C}_0 \to \underline{D}_0$ injective

(2) <u>surjective on objects</u> iff F_0 is surjective

(3) <u>injective on arrows</u> iff F_1 injective

(4) <u>surjective on arrows</u> iff F_1 surjective

(5) <u>faithful</u> iff for all $A, B \in \underline{C}_0$, the restriction

$$F : \mathrm{Hom}_{\underline{C}}(A, B) \longrightarrow \mathrm{Hom}_{\underline{D}}(FA, FB) \text{ is injective}$$

(6) <u>full</u> iff all restrictions

$$F : \mathrm{Hom}_{\underline{C}}(A, B) \longrightarrow \mathrm{Hom}_{\underline{D}}(FA, FB)$$

is surjective.

Warning: injective on arrows $\neq$ faithful

$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[dr, "id_{\underline{C}}"'] & \underline{C} + \underline{C} \arrow[d, "\nabla = [id_{\underline{C}}, id_{\underline{C}}]"] & \underline{C} \arrow[l] \arrow[dl, "id_{\underline{C}}"] \\ & \underline{C} & \end{tikzcd}$$

The codiagonal ∇ is faithful but <u>not</u> injective on arrows.

Example $\underline{\text{Sets}}_{\text{finite}} \longrightarrow \underline{\text{Sets}}$ is full and faithful

$\underline{\text{Groups}} \longrightarrow \underline{\text{Cat}}$ is full and faithful

$\left.\begin{array}{l} \underline{\text{Pos}} \longrightarrow \underline{\text{Cat}} \\ \underline{\text{Sets}} \longrightarrow \underline{\text{Cat}} \end{array}\right\}$ also fully faithful

$U : \underline{\text{Groups}} \longrightarrow \underline{\text{Sets}}$ faithful but not full

Def: An object in $\underline{C}$ is a generator for $\underline{C}$ iff

$$\text{Hom}_{\underline{C}}(C, -) : \underline{C} \to \underline{\text{Sets}} \quad \text{is faithful}$$

in particular, if $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is a diagram in $\underline{C}$, then

for any $x : C \to X$, $fx = gx \implies f = g$.

Examples: in $\underline{\text{Sets}}$, any singleton

in $\underline{\text{Groups}}$, $\mathbb{Z}$ = free group on 1 generator, because

$$\begin{tikzcd}
\underline{\text{Hom}}_{\underline{\text{Groups}}}(\mathbb{Z}, G) \arrow[r, "\sim"] \arrow[d, "h_*"'] & UG \arrow[d, "Uh"] \\
\underline{\text{Hom}}_{\underline{\text{Groups}}}(\mathbb{Z}, H) \arrow[r, "\sim"] & UH
\end{tikzcd} \qquad h : G \to H$$

If G is a group object in $\underline{C}$, then $\underline{\text{Hom}}_{\underline{C}}(-, G) : \underline{C}^{op} \to \underline{\text{Groups}}$ is a functor: for any $X \in \underline{C}$, take pointwise operations on $\underline{\text{Hom}}_{\underline{C}}(X, G)$.

$X \longrightarrow 1 \xrightarrow{u} G$ unit $\quad X \xrightarrow{f} G \xrightarrow{i} G$ inverse

$X \xrightarrow{\langle f, g \rangle} G \times G \xrightarrow{m} G$ multiplication

Check the unit law:

$$\begin{tikzcd}
 & G & \\
X \arrow[ur, bend left, "f"] \arrow[r, "\langle f, u\rangle"] \arrow[d] & G \times G \arrow[u, "\pi_1"'] \arrow[r, "m"] \arrow[d, "\pi_2"] & G \\
\underline{1} \arrow[r, "u"] & G &
\end{tikzcd}$$

This is exactly one of the group axiom!

$\mathbb{R} \in \underline{Top}$ is a ring object in $\underline{Top}$

for $X \in \underline{Top}$, $\underline{Hom}_{\underline{Top}}(X, \mathbb{R})$ is a ring! $C^1(X, \mathbb{R})$.

for $f : X \to Y$, get the pullback map

$$\underline{Hom}_{\underline{Top}}(X, \mathbb{R}) \xrightarrow{f^*} \underline{Hom}_{\underline{Top}}(Y, \mathbb{R})$$

Stone Duality

The two object set $2 = \{0, 1\}$ is both a boolean algebra object in $\underline{Sets}$ and a boolean algebra.

$$Hom_{\underline{sets}}(-, 2) : \underline{Sets}^{op} \to \underline{Boolean}\ \underline{Algebras}$$

$f : Y \to X$

$$\begin{tikzcd}
Hom_{\underline{sets}}(X, 2) \arrow[r, phantom, "\cong"] \arrow[d, "f^*"'] & P(X) \arrow[d, "f^{-1}"] \\
Hom_{\underline{sets}}(Y, 2) \arrow[r, phantom, "\cong"] & P(Y)
\end{tikzcd}$$

$\leftarrow$ also a boolean algebra with union / intersection

$1 = X$ $\qquad \neg U = X \setminus U$.

$0 = \emptyset$

$U \wedge V = U \cap V$

$U \vee V = U \cup V$

$\mathrm{Hom}_{\underline{BA}}(B, \underline{2})$ is isomorphic to the ultrafilters on B, $\mathrm{Ult}(B)$

$\mathrm{Hom}_{\underline{BA}}(-, 2) : BA^{op} \longrightarrow Sets$

$U \subseteq B$ is an ultrafilter if it is a maximal proper filter

$$\text{proper filter} \begin{cases} 1 \in U,\ 0 \notin U \\ U \ni x \leq y \implies y \in U \\ x, y \in U \implies x \wedge y \in U \end{cases}$$

maximal: if $U \subseteq U'$ are filters, then $U = U'$.

An ultrafilter corresponds to the preimage of 1 in B, for some $f : B \longrightarrow \{0,1\}$.

An equivalent condition to being an ultrafilter is that for all $x \in B$, either $x \in U$ or $\neg x \in U$.

$$\begin{tikzcd} \underline{Sets} \arrow[r, "P"] \arrow[rr, bend right, "\mathcal{U}"'] & \underline{BA}^{op} \arrow[r, "\mathrm{Ult}"] & \underline{Sets} \end{tikzcd}$$

Want a natural transformation $\eta : 1_{\underline{sets}} \to \mathcal{U}$.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) \\ \psi & & \psi \\ x & \longmapsto & \uparrow\{x\} = \{V \subseteq X \mid V \ni x\}. \end{array}$$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathcal{U}(X) \arrow[d, "\mathcal{U}(f)."] \\ Y \arrow[r, "\eta_Y"] & \mathcal{U}(Y) \end{tikzcd}$$

$\mathcal{U}(f) : \mathcal{U}(X) \longrightarrow \mathcal{U}(Y)$

$$\begin{pmatrix} \psi \\ U \\ \text{represented by } \chi_U : P(X) \to 2 \end{pmatrix} \longmapsto \begin{pmatrix} P(Y) \xrightarrow{f^{-1}} (P(X)) \\ \downarrow \chi_U \\ 2 \end{pmatrix}$$

$$\left(\mathcal{U}(f) \circ \eta_X\right)(x) = \{ V \subseteq Y \mid \chi_{\eta_X(x)}(f^{-1}(V)) = 1 \}$$
$$= \{ V \subseteq Y \mid x \in f^{-1}(V) \}$$
$$= \{ V \subseteq Y \mid f(x) \in V \} = \eta_Y(f(x))$$

Hence, η is really a natural transformation. Similarly, get

$$\underline{BA} \xrightarrow{Ult} \underline{Sets}^{op} \xrightarrow{P^{op}} \underline{BA}$$

<u>Prop:</u> (Stone's representation theorem)

$$\phi : 1_{BA} \longrightarrow P(Ult(-))$$

has injective components. In particular, ~~the~~ each boolean algebra is a subalgebra of the powerset of some set.

<u>More natural transformations:</u>

$X \in \underline{Sets}$, $M(X) \in \underline{Mon}$ the free monoid

$U : \underline{Mon} \to \underline{Sets}$, $X \xrightarrow{\eta_X} UM(X)$ insertion of generators.

This is natural; $\eta : id_{\underline{Sets}} \longrightarrow UM$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UM(X) \arrow[d, "UM(f)"] \\
Y \arrow[r, "\eta_Y"'] & UM(Y)
\end{tikzcd}$$

if $\underline{C}$ has binary products, there is a natural isomorphism between $F : (A, B, C) \longmapsto A \times (B \times C)$ and

$G : (A, B, C) \longmapsto (A \times B) \times C$

03/04/15

Natural Transformations:

Def: $\text{Fun}(\underline{C}, \underline{D})$ is a category with objects functors $F: \underline{C} \longrightarrow \underline{D}$ and arrows natural transformations.

Example: let $\underline{\text{Vec}}(\mathbb{R})$ be the category of real vector spaces. Define $V^* = \underline{\text{Hom}}_{\underline{\text{Vec}}(\mathbb{R})}(V, \mathbb{R})$.

So $(-)^*$ is a functor $\underline{\text{Vect}}(\mathbb{R})^{op} \longrightarrow \underline{\text{Vect}}(\mathbb{R})$.

V is naturally isomorphic to V^{**} for any vector space V over $\mathbb{R}$, if V is finite dimensional.

Have a natural transformation $\eta: (-)^{**} \longrightarrow id_{\underline{\text{Vec}}(\mathbb{R})}$.

Prop: $\alpha: F \to G$ is an iso in $\underline{\text{Fun}}(\underline{C}, \underline{D})$ if and only if all components α_c are isos in $\underline{D}$.

Proof: $(\Longrightarrow)$ α^{-1} exists, so componentwise,

$\alpha_c \circ \alpha_c^{-1} = (\alpha \circ \alpha^{-1})_c = (id_G)_c = id_{Gc}$

$\alpha_c^{-1} \circ \alpha_c = (\alpha^{-1} \circ \alpha)_c = (id_F)_c = id_{Fc}$.

$(\Longleftarrow)$ Define $\alpha^{-1}: G \to F$ by $\alpha_c^{-1} = (\alpha_c)^{-1}$; the components $(\alpha_c)^{-1}$ exist b/c α componentwise iso.

Check naturality: $f: A \to B$

$$\begin{array}{ccc} GA & \xrightarrow{(\alpha_A)^{-1}} & FA \\ {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ GB & \xrightarrow{(\alpha_B)^{-1}} & FB \end{array}$$

$Ff \circ (\alpha_A)^{-1} = (\alpha_B)^{-1} \circ Gf$

$\Updownarrow$

$\alpha_B \circ Ff = Gf \circ \alpha_A$ ← we know b/c α natural.

$\underline{\text{Cat}}$ is CCC with $\underline{D}^{\underline{C}} = \underline{\text{Fun}}(\underline{C}, \underline{D})$

Lemma: (bifunctor lemma)

Let $\underline{A}, \underline{B}, \underline{C}$ be categories, and given functions

$$F_0 : \underline{A}_0 \times \underline{B}_0 \longrightarrow \underline{C}_0$$
$$F_1 : \underline{A}_1 \times \underline{B}_1 \longrightarrow \underline{C}_1$$

Then (F_0, F_1) is a functor $\underline{A} \times \underline{B} \longrightarrow \underline{C}$ iff

(1) F is functorial in each variable

(2) (interchange law)

for $\begin{cases} A \xrightarrow{\alpha} A' & \text{in } A \text{ and} \\ B \xrightarrow{\beta} B' & \text{in } B, \text{ the square} \end{cases}$

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(\mathrm{id}_A, \beta)"] \arrow[d, "F(\alpha, \mathrm{id}_A)"'] & F(A,B') \arrow[d, "F(\alpha, \mathrm{id}_{B'})"] \\
F(A',B) \arrow[r, "F(\mathrm{id}_{A'}, \beta)"'] & F(A',B')
\end{tikzcd}$$

commutes.

Proof: $(\Longrightarrow)$ in $\underline{A} \times \underline{B}$, we have the diagram

$$\begin{tikzcd}
 & A' \times B \arrow[dr, "\mathrm{id}_{A'} \times \beta"] & \\
A \times B \arrow[ur, "\alpha \times \mathrm{id}_B"] \arrow[rr, "\alpha \times \beta"] \arrow[dr, "\mathrm{id}_A \times \beta"'] & & A' \times B' \\
 & A \times B' \arrow[ur, "\alpha \times \mathrm{id}_{B'}"'] &
\end{tikzcd}$$

, so (2) holds.

(1) is clearly necessary; we must have

$F(-, B)$ and $F(A, -)$ are functors

over $\longrightarrow$

<u>proof</u> (continued):

$(\Longleftarrow)$ F preserves composition:

let $A \times B \xrightarrow{\alpha \times \beta} A' \times B' \xrightarrow{\alpha' \times \beta'} A'' \times B''$, then

~~F(A,B) F(A,B')~~

$$\begin{tikzcd}
 & & F(A,B'') \arrow[dr] & & \\
 & F(A,B') \arrow[ur] \arrow[dr] & & F(A',B'') \arrow[dr] & \\
F(A,B) \arrow[ur] \arrow[dr] & & F(A',B') \arrow[ur] \arrow[dr] & & F(A'',B'') \\
 & F(A',B) \arrow[ur] \arrow[dr] & & F(A'',B') \arrow[ur] & \\
 & & F(A'',B) \arrow[ur] & &
\end{tikzcd}$$

The ~~outer edges~~ smaller squares of the above diagram commute by the condition (2), and the whole square does as well, also by (2).

∎

Prop: $\underline{D}^{\underline{C}} = \underline{\mathrm{Fun}}(\underline{C}, \underline{D})$ is an exponential object.

"proof" $(\varepsilon_0, \varepsilon_1) : \underline{D}^{\underline{C}} \times \underline{C} \longrightarrow \underline{D}$ Define the evaluation map

$$\theta : F \longrightarrow G, \quad f : C \longrightarrow C'$$

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\theta_C"'] \arrow[dr, "\varepsilon_1(\theta, f)"] & FC' \arrow[d, "\theta_{C'}"] \\ GC \arrow[r, "Gf"'] & GC' \end{tikzcd}$$

just a naturality square!

$\varepsilon_0 : F : \underline{\mathbb{C}} \longrightarrow \underline{D}, \; C \in \underline{C}, \; \varepsilon_0(F, C) = FC.$ ☒

Examples: $\mathbb{C}^{\mathbb{1}} = \mathbb{C}$ $\qquad \mathbb{C}^{\mathbb{1}+\mathbb{1}} = \mathbb{C} \times \mathbb{C}$

$\mathbb{C}^{\mathbb{2}} = \mathbb{C}^{\rightarrow}$ $\qquad \mathbb{C}^{I} = \prod_{i \in I} \mathbb{C}$, for I a discrete category.

Now, because arrows $\mathbb{1} \longrightarrow \mathbb{D}^{\mathbb{C}}$ are in bijection with arrows $\mathbb{C} \longrightarrow \mathbb{D}$, and arrows $\mathbb{2} \longrightarrow \mathbb{D}^{\mathbb{C}}$ are in bijection with arrows $\mathbb{C} \times \mathbb{2} \longrightarrow \mathbb{D}$, and with arrows $\mathbb{C} \longrightarrow \mathbb{D}^{\mathbb{2}}$.

So natural transformations are maps from?

$\mathbb{2} \longrightarrow \mathbb{D}^{\mathbb{C}}$ a natural transformation $F \xrightarrow{\theta} G$

is the same as a map $\mathbb{C} \longrightarrow \mathbb{D}^{\mathbb{2}}$

$$C \longmapsto FC \xrightarrow{\theta_C} GC.$$

Another way to think about cones:

Given an index category $\mathbb{J}$, and

$$\mathbb{C} \times \mathbb{J} \xrightarrow{P_1} \mathbb{C}, \text{ get}$$

$$\overline{P_1} = \Delta : \mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{J}}$$

$\Delta(C)$ is constant functor $\mathbb{J} \to \mathbb{C}$ at C.

$f : A \to B$ ~~for cons~~

$\Delta(f)$ is constant natural transformation at f,

$$\Delta(A) \xrightarrow{\Delta(f)} \Delta(B).$$

Fix $D : \mathbb{J} \longrightarrow \mathbb{C}$.

$\theta : \Delta(C) \longrightarrow D$ in $\underline{\mathrm{Fun}}(\mathbb{J}, \mathbb{C})$

$i \in \mathbb{J}$, $\theta_i : C \longrightarrow D_i$

$\alpha : i \longrightarrow j$ in $\mathbb{J}$:

$$\begin{tikzcd} C \arrow[r, "\theta_i"] \arrow[d] & D_i \arrow[d, "D_\alpha"] \\ C \arrow[r, "\theta_j"'] & D_j \end{tikzcd} \iff \begin{tikzcd} & C \arrow[dl, "\theta_i"'] \arrow[dr, "\theta_j"] & \\ D_i \arrow[rr, "D_\alpha"'] & & D_j \end{tikzcd}$$

A cone over D is exactly the data of C and θ.

Make the cones into a category by

$(C, \theta) \longrightarrow (C', \theta')$ is an arrow $f : C \longrightarrow C'$

such that

$$\begin{tikzcd} \Delta(C) \arrow[rr, "\Delta(f)"] \arrow[dr, "\theta"'] & & \Delta(C') \arrow[dl, "\theta'"] \\ & D & \end{tikzcd}$$

commutes,

in particular, for $i \in \mathbb{J}$,

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\theta_i"'] & & C' \arrow[dl, "\theta_i'"] \\ & D_i & \end{tikzcd}$$

commutes.

03/16/15

<u>Recall</u>: <u>Cat</u> is a CCC with $\mathbb{D}^{\mathbb{C}} = \mathrm{Fun}(\mathbb{C}, \mathbb{D})$.

We have bijections

$$\left\{\begin{array}{c}\text{Natural} \\ \text{transformations} \\ \text{in } \mathbb{D}^{\mathbb{C}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{functors} \\ \mathbf{2} \times \mathbb{C} \longrightarrow \mathbb{D}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{functors} \\ \mathbb{C} \longrightarrow \mathbb{D}^{\mathbf{2}}\end{array}\right\}$$

$$\updownarrow$$

$$\left\{\begin{array}{c}\text{functors} \\ \mathbf{2} \longrightarrow \mathbb{D}^{\mathbb{C}}\end{array}\right\}$$

<u>Equivalence of Categories</u>

<u>Def</u>: An equivalence of categories $\mathbb{C}, \mathbb{D}$ consists of functors $F: \mathbb{C} \longrightarrow \mathbb{D}$, $G: \mathbb{D} \longrightarrow \mathbb{C}$ and natural isomorphisms

$$\alpha: \mathrm{id}_{\mathbb{C}} \xrightarrow{\sim} G \circ F$$

$$\beta: \mathrm{id}_{\mathbb{D}} \xrightarrow{\sim} F \circ G$$

We write $\mathbb{C} \simeq \mathbb{D}$.

<u>Examples</u>:

- if $\mathbb{C} \cong \mathbb{D}$, then $\mathbb{C} \simeq \mathbb{D}$
- <u>Ord</u>$_{fin}$: $\begin{cases} \text{objects} & 0 = \emptyset,\ 1 = \{\emptyset\},\ 2 = \{0, 1\}, \ldots \\ \text{arrows} & \text{functions between these} \end{cases}$

<u>Ord</u>$_{fin}$ $\simeq$ <u>Sets</u>$_{fin}$ via the functors

i : <u>Ord</u>$_{fin} \longrightarrow$ <u>Sets</u>$_{fin}$ inclusion

$|-|$: <u>Sets</u>$_{fin} \longrightarrow$ <u>Ord</u>$_{fin}$ cardinality

Prop: for $F : \mathbb{C} \longrightarrow \mathbb{D}$, TFAE

(1) F is part of an equivalence of categories

(2) F is full and faithful and

F is essentially surjective on objects

(for each $D \in \mathbb{D}_0$, there is $C \in \mathbb{C}_0$, $FC \cong D$).

Proof: (1) $\Rightarrow$ (2)

Given $G : \mathbb{D} \to \mathbb{C}$, $\alpha : 1_{\mathbb{C}} \longrightarrow G \circ F$, $\beta : 1_{\mathbb{D}} \longrightarrow F \circ G$

then for $f : C \longrightarrow C'$, we have

$$\begin{tikzcd} C \arrow[r, "\alpha_C", "\sim"'] \arrow[d, "f"'] & GF(C) \arrow[d, "GF(f)"] \\ C' \arrow[r, "\sim", "\alpha_{C'}"'] & GF(C') \end{tikzcd}$$

$f = \alpha_{C'}^{-1} \circ GF(f) \circ \alpha_C$

So each arrow f is determined by $GF(f)$.

if $F(f) = F(f')$, for $f, f' : C \to C'$

then $GF(f) = GF(f') \Longrightarrow f = f'$

So F is faithful, and by a symmetric argument G is also faithful.

Now given $h : FC \longrightarrow FC'$, we get a diagram:

$$\begin{tikzcd} GF(C) \arrow[r, "G(h)"] & GF(C') \\ C \arrow[u, "\alpha_C"] \arrow[r] & C' \arrow[u, "\wr\ \alpha_{C'}"'] \end{tikzcd}$$

define $f := \alpha_{C'}^{-1} \circ G(h) \circ \alpha_C$, and we have that

$\underset{\text{(as before)}}{f = \alpha_{C'}^{-1} \circ GF(f) \circ \alpha_C} \Longrightarrow GFf = Gh \underset{G \text{ faithful}}{\Longrightarrow} Ff = h \Longrightarrow F$ is ~~faithful.~~ full.

Proof: (2) $\Rightarrow$ (1)

For each object $D \in \mathbb{D}_0$, choose $G(D) \in C$ together with an isomorphism $\beta_D : D \xrightarrow{\sim} FG(D)$.

given $h : D \to D'$ in $\mathbb{D}$,

$$\begin{tikzcd} D \arrow[r, "\beta_D", "\sim"'] \arrow[d, "h"'] & FG(D) \arrow[d, "\beta_{D'} \circ h \circ \beta_D^{-1}"] \\ D' \arrow[r, "\sim", "\beta_{D'}"'] & FG(D') \end{tikzcd}$$

Gives arrow $G(D) \xrightarrow[G(h)]{} G(D')$ which we take as the definition of $G(h)$; these data give a functor G.

Likewise, take $\alpha_C : C \xrightarrow{\sim} GF(C)$ to be $F^{-1}(\beta_{FC})$.

Can check that F, G, α, β form an equivalence of categories. ■

Prop: If $\mathbb{C} \simeq \mathbb{D}$ and $\mathbb{C}$ has J-limits, then $\mathbb{D}$ has J-limits.

Proof: Let. $D : J \to \mathbb{D}$.

We have
$$\begin{cases} F : \mathbb{C} \to \mathbb{D} \\ G : \mathbb{D} \to \mathbb{C} \\ \alpha : 1_C \xrightarrow{\sim} G \circ F \\ \beta : 1_D \to F \circ G \end{cases}$$

Compose: $GD : J \to \mathbb{C}$ has (L, c_i) the limiting cone

$$\begin{tikzcd} & L \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ GDi \arrow[rr, "GD\alpha"'] & & GD_j \end{tikzcd}$$

$\longrightarrow$

Get a cone in $\mathbb{D}$: $(FL, \ \beta_{D_i}^{-1} \circ Fc_i)$ over the diagram D.

$$\begin{tikzcd}
& FL \arrow[dl, "Fc_i"'] \arrow[dr, "Fc_j"] \arrow[ddl, bend right=60, "\beta_{D_i}^{-1}\circ Fc_i"'] \arrow[ddr, bend left=60, "\beta_{D_j}^{-1}\circ Fc_j"] & \\
FG(D_i) \arrow[rr, "FGD_\alpha"'] & & FG(D_j) \\
D_i \arrow[u, "\beta_{D_i}", "\wr"'] \arrow[rr, "D_\alpha"'] & & D_j \arrow[u, "\beta_{D_j}"', "\wr"]
\end{tikzcd}$$

Want to show that this cone is terminal.

Given any other cone (X, x_i) over D, apply G.

$$\begin{tikzcd}
& X \arrow[ddl, bend right, "x_i"'] \arrow[ddr, bend left, "x_j"] & \\
& FL \arrow[dl] \arrow[dr] & \\
D_i \arrow[rr, "D_\alpha"'] & & D_j
\end{tikzcd}
\xrightarrow[\text{L is a limit.}]{\text{apply } G,}
\begin{tikzcd}
& GX \arrow[d, "u"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& L \arrow[dl] \arrow[dr] & \\
D_i \arrow[rr] & & D_j
\end{tikzcd}$$

Now we have a map $u: GX \longrightarrow L$ and so get $Fu: FG(X) \longrightarrow FL$, so have

$$Fu \circ \beta_X : X \xrightarrow{\sim} FG(X) \longrightarrow FL$$

that commutes as needed. It is unique because F is fully faithful. ∎

Stone Duality

Prop: $\underline{BA}_{fin} \underset{Ult}{\overset{P}{\rightleftarrows}} \underline{Sets}_{fin}^{op}$ (baby version of Stone duality)

We pick instead of Ult a functor $A : \underline{BA}_{fin}^{op} \to \underline{Sets}$

$$A(B) := \{ a \in B \mid 0 < a \text{ and } \forall b,\ b < a \to b = 0 \}$$ "atoms"

elements nonzero, yet closest to zero.

Lemma: $A \cong Ult$ for $\underline{BA}_{fin}^{op}$.

$A \to Ult : \quad a \longmapsto \uparrow(a) = \{ b \in B \mid b \geq a \}$ $\longrightarrow$ this is an ultrafilter:

filter $(a \leq b, b' \Rightarrow a \leq b \wedge b')$

ultra $\begin{cases} \text{for } b \in B, \text{ either } b \wedge a = a, \\ \quad \text{or } b \wedge a = 0 \quad \text{b/c } a \text{ is atom} \\ \text{if } b \wedge a = a,\ b \geq a, \text{ so } b \in \uparrow(a) \\ \text{if } b \wedge a = 0 \\ \quad \neg b \wedge a = a \Rightarrow \neg b \geq a, \\ \quad\quad \neg b \in \uparrow(a) \end{cases}$

$Ult \to A : \quad U \longmapsto \bigwedge_{b \in U} b \in U$

here we use the fact that our boolean algebra is finite

if $b_0 < \bigwedge_{b \in U} b$, $b_0 \notin U$, then

$\neg b_0 \in U$, $b_0 < \neg b_0$, $b_0 = b_0 \wedge (\neg b_0) = 0$. ✓

~~$b \in A$~~

So to prove stone duality, we show that P and A give an equivalence of categories.

$$\alpha_X : X \xrightarrow{\sim} AP(X) \qquad \alpha_X(x) = \{x\}$$

$$\beta_X : B \xrightarrow{\sim} P(A(B)) \qquad \beta_B(b) = \{ a \in A(B) \mid a \leq b \}$$

$\longrightarrow$

Lemma: 1) $b = \bigvee \{a \in A(B) \mid a \leq b\}$

2) $a \in A(B)$ and $a \leq b \vee b' \implies a \leq b$ or $a \leq b'$.

Proof of 2: if $a \nleq b$, $a \nleq b'$, then

$$a \wedge b = 0, \quad a \wedge b' = 0 \implies a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b') = 0$$

but $a \leq b \vee b'$, so $a \wedge (b \vee b') = a$

but $a \in A(B)$, so $a \neq 0$ $\not\ast$.

Prop $\underline{\text{caBA}} \simeq \underline{\text{Sets}}^{op}$

$$\begin{tikzcd} \text{caBA} \arrow[d, hook] & \simeq & \text{Sets}^{op} \arrow[ll, bend right, "\phi"'] \arrow[dll, bend left, "\text{Ult}"] \\ \text{BA} \arrow[urr, bend left, "P"] & & \end{tikzcd}$$

complete atomic Boolean algebra

"discrete Stone Duality"

$$B \xrightarrow{\phi_B} P(\text{Ult}(B))$$

$$\phi_B(b) = \{V \in \text{Ult}(B) \mid b \in V\}$$

always injective, but only surjective when BA's are complete and atomic.

Theorem (Stone Duality) $\underline{\text{BA}} \cong \underline{\text{Stone}}^{op}$

$\uparrow$ stone spaces: a certan family of topological spaces.

03/18/15

The Yoneda Lemma:

functors from $\mathbb{C}$ to $\underline{\text{Sets}}$ / $\mathbb{C}$-shaped diagrams in $\underline{\text{Sets}}$

- if $\mathbb{C} = P$ is a poset

  variable sets $A : P \longrightarrow \underline{\text{Sets}}$

  - a family $(A_i)_{i \in P}$ a set
  - transition functions $A_i \longrightarrow A_j$ for $i \leq j$

- if $\mathbb{C} = M$ is a monoid,

  $\underline{\text{Sets}}^M$ are sets with an action of M

  $$A \circlearrowleft M$$

- if $\mathbb{C} = G$ is a group, $\underline{\text{Sets}}^G$ is $G\,G\,A$, a group action on a set A.

- if $\mathbb{C} = \mathbb{C}(\bullet \rightrightarrows \bullet)$, the free category on $\bullet \rightrightarrows \bullet$,

  then $\underline{\text{Sets}}^{\mathbb{C}} \cong \underline{\text{Graphs}}$.

We have an evaluation functor $ev : \underline{\text{Sets}}^{\mathbb{C}} \times \mathbb{C} \longrightarrow \underline{\text{Sets}}$

Yoneda Embedding

$\underline{\text{Sets}}^{\mathbb{C}}$ has special elements, the representable functors

covariant $\quad \mathrm{Hom}(C, -) : \mathbb{C} \longrightarrow \text{Sets}$

contravariant $\quad \mathrm{Hom}(-, C) : \mathbb{C}^{op} \longrightarrow \text{Sets}$

Also a bifunctor

$$\mathrm{Hom}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \underline{\text{Sets}} .$$

gives

$$k : \mathbb{C}^{op} \longrightarrow \underline{\text{Sets}}^{\mathbb{C}}$$

$$y : \mathbb{C} \longrightarrow \underline{\text{Sets}}^{\mathbb{C}^{op}} =: \widehat{\mathbb{C}}$$

$k\, C = \mathrm{Hom}_{\mathbb{C}}(C, -)$

$y\, C = \mathrm{Hom}_{\mathbb{C}}(-, C)$

$\widehat{\mathbb{C}}$ is category of set-valued presheaves.

y is the "Yoneda Embedding", because it is fully faithful and injective on objects.

improved version of the Cayley theorem

$$\mathbb{C} \xrightarrow{\text{faithful}} \overline{\mathbb{C}} \ : \ \text{objects } \overline{C} = \bigcup_{X \in \mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(X, C)$$

for $C \in \mathbb{C}$.

(e.g. Cayley's theorem when $G = \mathbb{C}$, is a group.

$G \hookrightarrow S_{|G|}$

$g \longmapsto (x \mapsto gx)$)

Can also do yoneda embedding for groups:

$$G \hookrightarrow \underline{Sets}^{G^{op}} = \text{Sets with right-action of } G$$

$$X \circlearrowleft G$$

every such action is a homomorphism between G and S_X ← symmetric group of X.

<u>Theorem</u> (Yoneda Lemma) We have two functors:

$$E(X,Y) = \mathrm{Hom}_{\hat{\mathbb{C}}}(yX, Y) : \mathbb{C}^{op} \times \hat{\mathbb{C}} \longrightarrow \underline{Sets}^*$$

$$ev : \mathbb{C}^{op} \times \hat{\mathbb{C}} \longrightarrow \text{Sets}$$

↑ star b/c there may be some set theoretic concerns

ev and E are naturally isomorphic via some $\eta : ev \longrightarrow E$, with components

<u>Proof</u>: $C \in \mathbb{C}$, $H \in \hat{\mathbb{C}} = \underline{Sets}^{\mathbb{C}^{op}}$

$$\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, H) \underset{\eta_{C,H}}{\cong} HC = ev(C, H)$$

We define what η does on natural transformations

$\theta \in \mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F)$ is a map $\theta : yC \longrightarrow F$

$$\theta_C : yC(C) \longrightarrow FC$$
$$\| $$
$$\mathrm{Hom}_{\mathbb{C}}(C,C) \ni id_C$$

Define $x_\theta = \eta_{C,F}(\theta) := (\theta_C)(id_C) \in FC$

Now given $a \in FC$, want $\theta_a : yC \longrightarrow F$

Fix $C' \in \mathbb{C}$, $\quad yC(C') \longrightarrow FC'$

$$yC(C') = \mathrm{Hom}_{\mathbb{C}}(C', C)$$

take any $h : C' \longrightarrow C$.

$$a \in FC \xrightarrow{Fh} FC'$$

$$(\theta_a)_{C'}(h) := F(h)(a)$$

Check that θ_a is natural. Let $f : C'' \longrightarrow C$ in $\mathbb{C}$.

$$\begin{tikzcd}
h \in \arrow[d, mapsto] \arrow[rrr, bend left, mapsto] & yC(C') = \mathrm{Hom}_{\mathbb{C}}(C', C) \arrow[r, "(\theta_a)_{C'}"] \arrow[d] & FC' \arrow[d, "Ff"] & \ni F(h)(a) \arrow[d, mapsto] \\
hf \in \arrow[rrr, bend right, mapsto] & yC(C'') = \mathrm{Hom}_{\mathbb{C}}(C'', C) \arrow[r, "(\theta_a)_{C''}"'] & FC'' & \ni F(f)(F(h)(a)) = F(hf)(a)
\end{tikzcd}$$

Now we want to check that these are mutually inverse.

So for $\theta : yC \longrightarrow F$ a natural transformation,

we want to know that $\theta_{x_\theta} = \theta$. Let $h : C' \longrightarrow C$

$$(\theta_{x_\theta})_{C'}(h) = F(h)(x_\theta) = F(h)(\theta_C(\mathrm{id}_C))$$
$$= (F(h) \circ \theta_C)(\mathrm{id}_C)$$

~~Use~~ Use the naturality square for θ

$$\begin{tikzcd}
yC(c) = \mathrm{Hom}(c, C) \arrow[r, "\theta_c"] \arrow[d, "yC(h) = h^*"'] & Fc \arrow[d, "F(h)"] \\
yC(c') = \mathrm{Hom}(c', C) \arrow[r, "\theta_{c'}"] & Fc'
\end{tikzcd}$$

$$\begin{aligned}
(\theta_{x_\theta})_{c'}(h) = F(h)(x_\theta) &= F(h)(\theta_c(id_c)) \\
&= (F(h) \circ \theta_c)(id_c) \\
&= (\theta_{c'} \circ yC(h))(id_c) \\
&= \theta_{c'}(h^*(id_c)) \\
&= \theta_{c'}(h)
\end{aligned}$$

Hence $\theta_{x_\theta} = \theta$.

Conversely, for $a \in Fc$

$$x_{\theta_a} = (\theta_a)_c(id_c) = F(id_c)(a) = id_{Fc}(a) = a$$

So x and θ are inverses. ✓

Finally, we check that ξ and η are natural.

$$\begin{tikzcd}[column sep=large]
\theta_a & & a \arrow[ll, mapsto] \\
\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F) \arrow[rr, shift right, "x = \eta_{C,F}"'] & \cong & FC \in \underline{\mathbf{Sets}} \arrow[ll, shift right, "\xi_{C,F}"'] \\
\theta \arrow[rr, mapsto] & & x_\theta
\end{tikzcd}$$

Suffices to show that one of them is natural, say η.

For fixed C, check naturality in F.

$\phi : F \longrightarrow F'$

$$\begin{tikzcd}
\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F) \arrow[rr, "\sim", "\eta_{F,C}"'] \arrow[ddd, "\phi_*"'] & & FC \arrow[ddd, "\phi_C"] \\
\theta \arrow[r, mapsto] \arrow[dd, mapsto] & \theta_C(\mathrm{id}_C) \arrow[d, mapsto] & \\
& \phi_C(\theta_C(\mathrm{id}_C)) = (\phi\circ\theta)_C(\mathrm{id}_C) & \\
\phi\circ\theta \arrow[ur, phantom] & & \\
\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F') \arrow[rr, "\sim", "\eta_{F',C}"'] & & F'C
\end{tikzcd}$$

$\phi\circ\theta \longmapsto (\phi\circ\theta)_C(\mathrm{id}_C)$

For fixed F, vary C. Let $h : C' \longrightarrow C$

$$\begin{tikzcd}
\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F) \arrow[rr, "\eta_{F,C}", "\sim"'] \arrow[dd] & & FC \arrow[dd, "F(h)"] \\
\theta \arrow[r, mapsto] \arrow[d, mapsto] & \theta_C(\mathrm{id}_C) \arrow[d, mapsto] & \\
\theta\circ yh \arrow[r, mapsto] & (F(h)\circ\theta_C)(\mathrm{id}_C) = (\theta\circ yh)_{C'}(\mathrm{id}_C) & \\
\mathrm{Hom}_{\hat{\mathbb{C}}}(yC', F) \arrow[rr, "\sim", "\eta_{F,C'}"'] & & FC'
\end{tikzcd}$$

$\theta_{C'}(\underline{h})$

Theorem: y is Full, Faithful, and injective on objects.

Proof: Let $C, D \in \mathbb{C}$.

$$\mathrm{Hom}_{\mathbb{C}}(C, D) = y(D)(C) \cong \mathrm{Hom}_{\hat{\mathbb{C}}}(yC, yD)$$

if $yC = yD$, then

$$id_C \in yC(C) = yD(C) = \mathrm{Hom}(C, D)$$

so $$id_C : C \longrightarrow D \implies D = C. \quad \blacksquare$$

Remark: if $F : \mathbb{C} \longrightarrow \mathbb{D}$ is fully faithful, and $FC \cong FC'$, then $C \cong C'$.

$$F(f \circ g) = id_{F'C} = F(id_{C'}) \implies f \circ g = id_{C'}.$$

Example: $(A^B)^C \cong A^{B \times C}$

$$\mathrm{Hom}(X, (A^B)^C) \cong \mathrm{Hom}(X \times C, A^B) \cong \mathrm{Hom}(X \times C \times B, A)$$
$$\cong \mathrm{Hom}(X, A^{B \times C})$$

So by Yoneda, $(A^B)^C \cong A^{B \times C}$.

(Morally, should check that these are natural in X)

3/23/15

Yoneda's Lemma

$y : \mathbb{C} \longrightarrow \underline{Sets}^{\mathbb{C}^{op}}$

transpose of $\mathrm{Hom}_{\mathbb{C}}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \underline{Sets}$

$$yC = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$$

We have a natural transformation between

$$\mathrm{Hom}_{\mathbb{C}}(y-,-) \quad \text{and} \quad (C,F) \longmapsto FC.$$

in particular, $\mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F) \cong FC$.

Yoneda Principle: if $C, D \in \mathbb{C}$, $yC \cong yD$ in $\hat{\mathbb{C}}$, then $C \cong D$ in $\mathbb{C}$

ex if $\mathbb{C}$ is a CCC with coproducts,

$$\begin{aligned}
\mathrm{Hom}(A \times (B+C), X) &\cong \mathrm{Hom}(B+C, X^A) \\
&\cong \mathrm{Hom}(B, X^A) \times \mathrm{Hom}(C, X^A) \\
&\cong \mathrm{Hom}(A\times B, X) \times \mathrm{Hom}(A \times C, X) \\
&\cong \mathrm{Hom}(A\times B + A\times C, X).
\end{aligned}$$

All natural in $X \Longrightarrow \quad A\times(B+C) \cong A\times B + A\times C$.

Lemma: if $\mathbb{D}$ complete, then $\mathbb{D}^{\mathbb{C}}$ complete, limits are computed pointwise, for $C \in \mathbb{C}$, $ev_C : \mathbb{D}^{\mathbb{C}} \to \mathbb{D}$ preserves limits

Proof sketch:

Take $F : J \to \mathbb{D}^{\mathbb{C}}$ (a bifunctor $J \times \mathbb{C} \to \mathbb{D}$)

Fix $C \in \mathbb{C}$, get limit $\varprojlim_j F_j C = (\varprojlim_j F)(C)$

if $C \xrightarrow{f} C'$ in $\mathbb{C}$

$i \xrightarrow{\alpha} j$ in J

$$\begin{tikzcd} F_i C \arrow[r, "F_i f"] \arrow[d, "F_\alpha C"'] & F_i C' \arrow[d, "F_\alpha C'"] \\ F_j C \arrow[r, "F_j f"'] & F_j C \end{tikzcd}$$

$\varprojlim_j F_j$ is a functor, $\mathbb{C} \to \mathbb{D}$, and its action on maps is to induce a map between limits from maps between diagrams given by $f : C \to C'$.

$$\begin{tikzcd} \varprojlim_j F_j C \arrow[r] \arrow[d] & \varprojlim_j F_j C' \arrow[d] \\ F_i C \arrow[r] & F_i C' \end{tikzcd}$$

Corollary: (Dual) if $\mathbb{D}$ is cocomplete, then $\mathbb{D}^{\mathbb{C}}$ is cocomplete with colimits computed pointwise, preserved by ev_C.

Colimit in $\mathbb{D}^{\mathbb{C}}$ is limit in $\mathbb{D}^{op\,\mathbb{C}^{op}} = (\mathbb{D}^{\mathbb{C}})^{op}$.

If $\mathbb{C}$ is small, then each presheaf is canonically a colimit of representables.

$$P \cong \varinjlim_j \mathrm{Hom}(-, C_j)$$

$$(J \text{ small}, \ \pi : J \to \mathbb{C}, \ P \cong \varinjlim y\pi)$$

This is furthermore natural in P.

<u>Proof + construction of (J, π):</u>

We say $J = \int_{\mathbb{C}} P$ = "category of elements"

$\int_{\mathbb{C}} P$ has $\begin{cases} \text{objects } (x, C) \text{ with } C \in \mathbb{C} \text{ and } x \in PC. \\ \text{arrows } (x', C') \longrightarrow (x, C) \text{ is an arrow } h : C' \to C \text{ such that } Ph(x) = x' \end{cases}$

Define π by

$$\int_{\mathbb{C}} P \xrightarrow{\pi} \mathbb{C} \xleftrightarrow{y} \widehat{\mathbb{C}}$$

π is projection $(x, C) \longmapsto C$

$$y\pi \xrightarrow{\text{"id"}} \Delta(P) = \text{constant functor } P \text{ in } \widehat{\mathbb{C}}$$

$$\begin{tikzcd} (x,C) & yC \arrow[r, "x"] & P \\ (x',C') \arrow[u, "h"] & yC' \arrow[u, "yh"] \arrow[ur, "x'"'] & \end{tikzcd}$$

for $x \in PC$

$Ph(x) = x'$

$\int_{\mathbb{C}} P \simeq$ full subcategory of $\widehat{\mathbb{C}}/P$ on $yC \xrightarrow{x} P$ (representable domain)

Given $y\pi \xrightarrow{\theta} \Delta(Q)$, want $P \xrightarrow{u} Q$ such that

$$\begin{tikzcd}
yC \arrow[dr, "x"] \arrow[drr, bend left, "\theta_{(x,c)}"] & & \\
 & P \arrow[r, dashed, "u"] & Q \\
yC' \arrow[uu, "yh"] \arrow[ur, "x'"'] \arrow[urr, bend right, "\theta_{(x',c')}"'] & &
\end{tikzcd}$$

$u_c : PC \longrightarrow QC$

$u_c(x) = \theta_{(x,c)}$

Toposes

<u>Lemma</u>: Let $A : J \longrightarrow \hat{\mathbb{C}}$ for $\mathbb{C}$, J small. Let $B \in \hat{\mathbb{C}}$. Then we have a natural isomorphism

$$\varinjlim_j (A_j \times B) \cong \left(\varinjlim_j A_j\right) \times B .$$

(i.e. $(-)\times B : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ preserves colimits).

<u>Proof</u>: the cocone $A_i \times B \xrightarrow{\theta_i \times 1_B} \left(\varinjlim A_i\right) \times B$

gives a map $\varinjlim (A_i \times B) \longrightarrow \left(\varinjlim A_i\right) \times B$

where $\theta_i : A_i \longrightarrow \varinjlim A_i$.

Need to check for $C \in \mathbb{C}$ that

$$\varinjlim (A_i(C) \times B(C)) = \varinjlim (A_i \times B)(C) \cong \left(\varinjlim A_i\right)(C) \times B(C)$$

coYoneda lemma in <u>Sets</u>

$$\begin{aligned}
\mathrm{Hom}\left(\varinjlim (A_i \times B), X\right) &\cong \varprojlim \mathrm{Hom}(A_i \times B, X) \\
&\cong \varprojlim \mathrm{Hom}(A_i, X^B) \\
&\cong \mathrm{Hom}\left(\varinjlim A_i, X^B\right) \cong \mathrm{Hom}\left(\varinjlim A_i \times B, X\right)
\end{aligned}$$

<u>Lemma:</u> $\hat{\mathbb{C}}$ has exponential objects

(Note: $(Q^P)(C) = Q(C)^{P(C)}$ is <u>not</u> functorial)

<u>Proof:</u> If Q^P exists, $Q^P(C) \cong \mathrm{Hom}_{\hat{\mathbb{C}}}(yC, Q^P)$

$$\cong \mathrm{Hom}_{\hat{\mathbb{C}}}(yC \times P, Q)$$

Take as the definition $\mathrm{Hom}_{\hat{\mathbb{C}}}(yC \times P, Q) = Q^P(C)$.

What is the transpose?

For $X \in \hat{\mathbb{C}}$, ~~[illegible]~~

but $X \cong \varinjlim_j yC_j$ by previous lemma, so

$$\begin{aligned}
\mathrm{Hom}(X, Q^P) &\cong \mathrm{Hom}(\varinjlim yC_j, Q^P) \\
&\cong \varprojlim \mathrm{Hom}(yC_j, Q^P) \\
&\overset{\text{by defn}}{\cong} \varprojlim \mathrm{Hom}(yC_j \times P, Q) \\
&\cong \mathrm{Hom}(\varinjlim (yC_j \times P), Q) \\
&\overset{\text{use a previous lemma}}{\cong} \mathrm{Hom}((\varinjlim yC_j) \times P, Q) \\
&\cong \mathrm{Hom}(X \times P, Q).
\end{aligned}$$

All of the above is natural in X. ∎

<u>Theorem:</u> $\hat{\mathbb{C}}$ is a CCC for small categories $\mathbb{C}$ and Yoneda $y: \mathbb{C} \hookrightarrow \hat{\mathbb{C}}$ preserves those limits which are already present in $\mathbb{C}$, and those exponentials already present in $\mathbb{C}$.

Proof: already showed that $\hat{\mathbb{C}}$ is CCC, so we check y preserves stuff.

if $B^A \in \mathbb{C}$, then

$$\begin{aligned} y(B^A)(C) = \mathrm{Hom}(C, B^A) &\cong \mathrm{Hom}(C \times A, B) \\ &\cong \mathrm{Hom}_{\hat{\mathbb{C}}}(yC \times yA, yB) \\ &\cong \mathrm{Hom}(yC, yB^{yA}) \\ &\cong (yB^{yA})(C). \end{aligned}$$

So $y(B^A) = yB^{yA}$.

$\blacksquare$

Def: Let $\mathcal{E}$ be a category with finite limits.

$\Omega \in \mathcal{E}$ and an arrow $1 \xrightarrow{t} \Omega$ is a subobject classifier for $\mathcal{E}$ iff for any mono $U \overset{m}{\rightarrowtail} E$, there is a unique arrow $u : E \to \Omega$ such that

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "t"] \\ E \arrow[r, dashed, tail, "u"] & \Omega \end{tikzcd}$$

is a pullback square.

e.g. in Sets, subsets are pullbacks of $1 \xrightarrow{\top} 2 = \{\top, \bot\}$

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, hook] & 1 \arrow[d, "\top"] \\ E \arrow[r] & \{\top, \bot\} \end{tikzcd}$$

A subobject classifier is equivalent to

$$\mathrm{Sub}_{\mathcal{E}}(-) : \mathcal{E}^{op} \longrightarrow \underline{\text{Sets}}$$
$$\text{\rotatebox{90}{$\cong$}}$$
$$\mathrm{Hom}_{\mathcal{E}}(-,\Omega) : \mathcal{E}^{op} \to \underline{\text{Sets}}.$$

The map $1 \longrightarrow \Omega$ is given by $1 \longmapsto \{E \xrightarrow{1_E} E\}$

<u>Def</u>: An (elementary) topos is a category $\mathcal{E}$ such that

(1) $\mathcal{E}$ has finite limits;

(2) $\mathcal{E}$ has a subobject classifier;

(3) $\mathcal{E}$ has exponential objects.

<u>Prop</u>: $\mathbb{C}$ small $\Longrightarrow$ $\widehat{\mathbb{C}}$ is an (elementary) topos.

<u>Proof</u> $1 \xrightarrow{t} \Omega$ in $\widehat{\mathbb{C}}$

$$\Omega(C) \cong \mathrm{Hom}_{\widehat{\mathbb{C}}}(yC, \Omega) \cong \mathrm{Sub}_{\widehat{\mathbb{C}}}(yC) \ni S$$

For each $C' \in \mathbb{C}$, $S(C') \subseteq \mathrm{Hom}(C', C)$ (right ideal)

<u>Def</u>: A <u>sieve</u> on C is a subset $S \subseteq \overline{C} = \bigcup_{C'} \mathrm{Hom}(C', C)$ such that $f \in S$, $g : C'' \to C' \Longrightarrow fg \in S$

$$\Omega(C) = \{S \subseteq \overline{C} \mid S \text{ a sieve on } C\}$$

$$\Omega(h)(S) = \{g \in \overline{C'} \mid hg \in S\}$$

for $h : C' \to C$ $\qquad C'' \xrightarrow{g} C' \xrightarrow{h} C$

Monoidal Categories

03/25/15

Recall: A monoid is a set M, a binary operation $*$, and a unit $u \in M$ such that

$$a * (b * c) = (a * b) * c$$
$$u * a = a = a * u \qquad \text{for all } a, b, c \in M$$

Def: A strict monoidal category is a category $\mathbb{C}$, a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$, and an object $I \in \mathbb{C}$ such that

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$
$$I \otimes A = A = A \otimes I \qquad \text{for all } A, B, C \in \mathbb{C}.$$

Examples:

- if $\mathbb{C}$ is small and discrete, then the strict monoidal category structure $I, \otimes$ on $\mathbb{C}$ makes it into a monoid
- if $\mathbb{C}$ is a preorder category, then $(\otimes, I) = (\wedge, 1)$ or $(\otimes, I) = (\vee, 0)$ are monoidal category structures
- endofunctor categories, $(\otimes, I) = (\cdot, id)$ where $f \cdot g = g \circ f$ and $id(A) = id_A$ makes $\mathbb{C}^{\mathbb{C}}$ into a monoidal category.

Note $f \otimes A = f \otimes id_A$

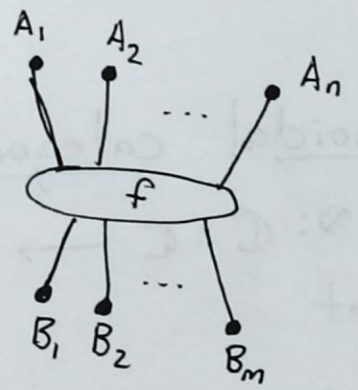
~~Strict Mono~~

String Diagrams

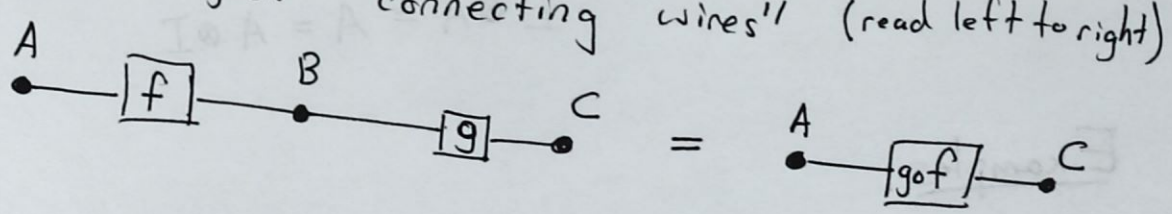

represent objects of $\mathbb{C}$ as points extruded into wires

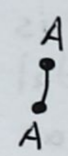
represent arrows $f : A_1 \otimes \cdots \otimes A_n \longrightarrow B_1 \otimes \cdots \otimes B_m$

as

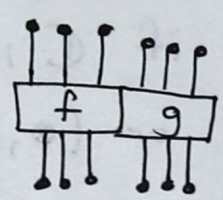


Composition is just "connecting wires" (read left to right)



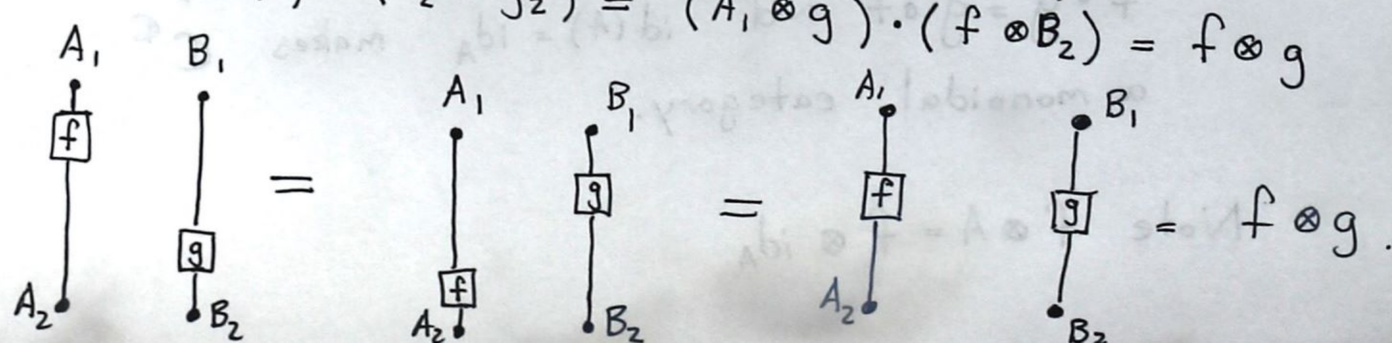
nullary composition is drawn 

Binary tensor is drawn as



Example if $f : A_1 \longrightarrow A_2$, $g : B_1 \longrightarrow B_2$, then

$$(f \otimes B_1) \cdot (A_2 \otimes g_2) = (A_1 \otimes g) \cdot (f \otimes B_2) = f \otimes g$$

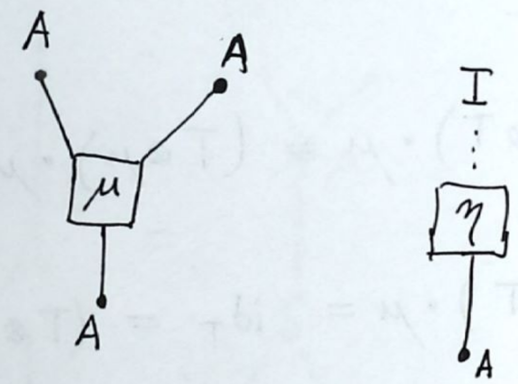


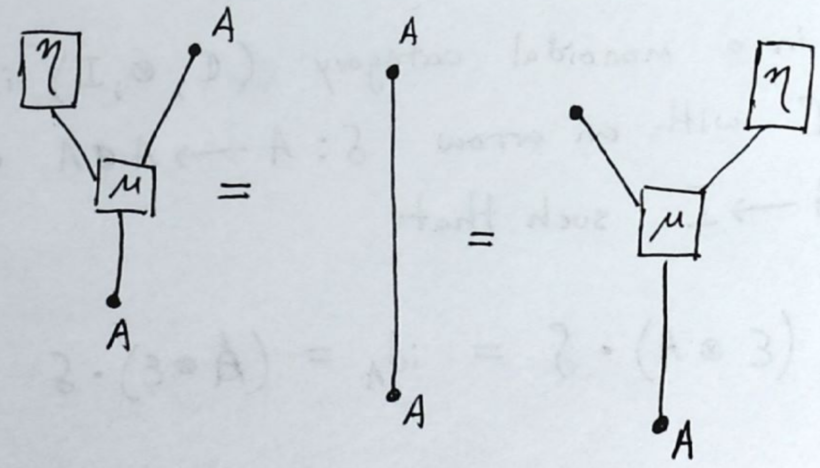
Def: A <u>monoidal object</u> in a monoidal category $(\mathbb{C}, \otimes, I)$ is an object $A \in \mathbb{C}$, and an arrow $\mu : A \otimes A \longrightarrow I$, and an arrow $\eta : I \longrightarrow A$

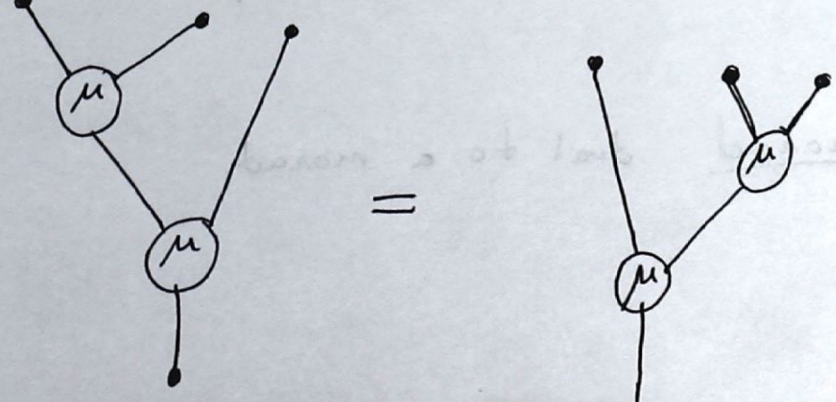
with associativity $(\mu \otimes A) \cdot \mu = (A \otimes \mu) \cdot \mu$

and units $(\eta \otimes A) \cdot \mu = id_A = (A \otimes \eta) \cdot \mu$

As diagrams:







Monads: A monad is a monoid in a monoidal category of endofunctors; that is

- an endofunctor $T : \mathbb{C} \longrightarrow \mathbb{C}$
- a natural transformation $\mu : T \otimes T \longrightarrow T$
- a natural transformation $\eta : id_{\mathbb{C}} \longrightarrow T$

such that

- $(\mu \otimes T) \cdot \mu = (T \otimes \mu) \cdot \mu$ $\quad \left[(\mu \otimes T)(A) := T(\mu_A) \right]$
- $(\eta \otimes T) \cdot \mu = id_T = (T \otimes \eta) \cdot \mu$

Def: A comonoid in a monoidal category $(\mathbb{C}, \otimes, I)$ is an object $A \in \mathbb{C}$ with an arrow $\delta : A \longrightarrow A \otimes A$ and an arrow $\varepsilon : A \longrightarrow I$ such that

- $$(\varepsilon \otimes A) \cdot \delta = id_A = (A \otimes \varepsilon) \cdot \delta$$
- $$\delta \cdot (\delta \otimes A) = \delta \cdot (A \otimes \delta)$$

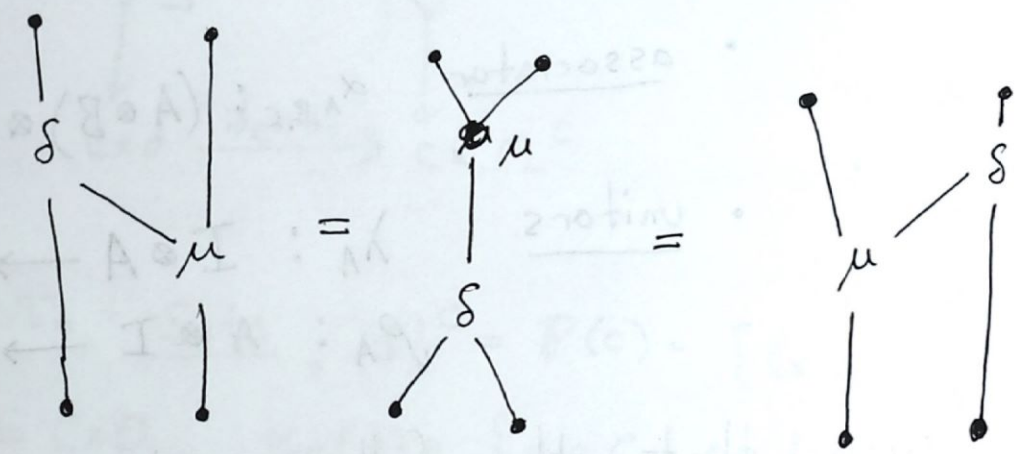
Can form a comonad dual to a monad.

Example: Frobenius Algebra

In a monoidal category $(\mathbb{C}, \otimes, I)$, an object $A \in \mathbb{C}$ with both monoid and comonoid structure satisfying

$$(\delta \otimes A) \cdot (A \otimes \mu) = \mu \cdot \delta = (A \otimes \delta) \cdot (\mu \otimes A)$$

as a map $A \otimes A \longrightarrow A \otimes A$



Monoidal Categories

Def: A monoidal category is a category $\mathbb{C}$, a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$, and an object $I \in \mathbb{C}$ with two natural isomorphisms (called coheritors):

- associator $\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ $\cancel{(A \otimes B) \otimes}\, C$
- unitors $\lambda_A : I \otimes A \longrightarrow A$
  $\rho_A : A \otimes I \longrightarrow A$

such that the following diagrams commute:

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha"] \arrow[d, "A \otimes \alpha"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha \otimes D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes B"] \\
 & A \otimes B &
\end{tikzcd}$$

Note: Every monoidal category is equivalent to a strict monoidal category.

Hence, string diagrams make sense for arbitrary monoidal categories.

Power Objects

Def: $c \in \mathcal{E}$ has a <u>power object</u> $\Omega^c \in \mathcal{E}$ together with a map $\varepsilon_c \rightarrowtail c \times \Omega^c$ such that for any $r \rightarrowtail c \times d$ there is a unique $\chi_r : d \longrightarrow \Omega^c$ and the following diagram commutes and is a pullback:

$$\begin{tikzcd}
r \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \varepsilon_c \arrow[d, tail] \\
c \times d \arrow[r, "id_c \times \chi_r"] & c \times \Omega^c
\end{tikzcd}.$$

Example: In <u>Sets</u>, $\Omega^c = \mathcal{P}(c) = \{(x, U) \mid x \in U\}$

for $R \subseteq C \times D$, $\chi_R(d) = \{c \in C \mid (c, d) \in R\}$.

Claim: subobject classifier + exponentials $\Longrightarrow$ power objects

proof: Let Ω^c be the exponential. Let $r \rightarrowtail c \times d$.

$$\begin{tikzcd}
r \arrow[d, tail] \arrow[r] \arrow[rr, bend left, "!"] & \varepsilon_c \arrow[r, "!"] \arrow[d, tail] & 1 \arrow[d, tail, "true"] \\
c \times d \arrow[r, "id_c \times \chi_r"'] \arrow[rr, bend right, "\chi_r"] & c \times \Omega^c \arrow[r, "eval"'] & \Omega
\end{tikzcd}$$

The ~~following~~ left square is a pullback because the front face and back faces are.

Claim: Power objects give subobject classifiers and exponentials.

Proof:

Set $\Omega := \Omega^1$

and $d^c = \{ f \in c \times d \mid \forall x \in c, \exists!\, y \in d, \langle x, y \rangle \in f \}$. ■

Example: $\underline{\text{Sets}}$, $\underline{\text{Sets}}_{fin}$, $\underline{\text{Sets}}^{\bullet \to \bullet}$, $\underline{\text{Sets}}^{\bullet \rightrightarrows \bullet}$, $\underline{\text{Sets}}^{P}$, $\underline{\text{Sets}}^{\mathbb{C}}$

presheaf toposes

Theorem: (Fundamental Theorem of Toposes)

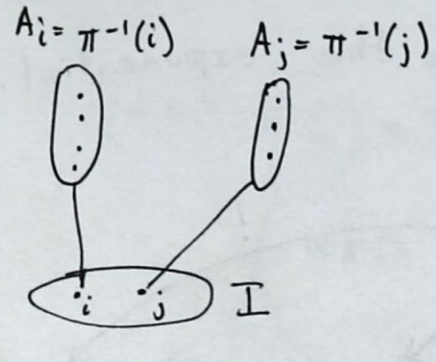
If $\mathcal{E}$ is a topos, C is an object, then $\mathcal{E}/_C$ is a topos.

Example:

Consider $\underline{\text{Sets}}/_I$ for a set I.

Def: $A_i = \pi^{-1}(i)$ is the fiber or stalk over i

$$\begin{array}{c} A \\ \Big\downarrow{\scriptstyle \pi} \\ I \end{array}$$



Def: an element of A_i is a germ at i

$(A \xrightarrow{\pi} I)$ is a bundle

A is the total space, stalk space, or l'espace étalé

Def: For $I \in \underline{\text{Top}}$, $Sh(I)$ is the topos of sheaves over I, with objects

- objects local homeomorphisms $A \xrightarrow{\pi} I$

  ($\forall x \in A$, $\exists$ a nbhd $U \ni x$ s.t. $\pi \restriction U : U \to \pi(U)$ is a homeomorphism)

- arrows any $f: A \to B$ such that
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & I & \end{tikzcd}$$
commutes.

Ω is $Sh(I)$ is the sheaf of germs of open sets.

$\Omega(i)$ = equivalence classes of open neighborhoods at i

Adjoints (unifies all universal properties)

Def: An adjunction of categories refers to the situation

$$\mathbb{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{D}$$ with a natural isomorphism

$$\phi_{C,D} : \mathrm{Hom}_{\mathbb{D}}(FC, D) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(C, UD).$$

We say F is left adjoint to U or U is right adjoint to F. Write $F \dashv U$.

<u>Prop</u>: If $F \dashv U$, $F \dashv V$, then $U \cong V$.

$$\mathrm{Hom}_{\mathbb{C}}(C, UD) \cong \mathrm{Hom}_{\mathbb{D}}(FC, D) \cong \mathrm{Hom}_{\mathbb{C}}(C, VD)$$

Thus $UD \cong VD$ naturally in $\mathbb{D}$.

So $U \cong V$ in $\mathbb{C}^{\mathbb{D}}$.

<u>Example</u>:
Free Monads and forgetful functor

$$\underline{\mathrm{Mon}} \underset{F}{\overset{U}{\rightleftarrows}} \underline{\mathrm{Sets}}$$

F is free monad
U is underlying set

$$\mathrm{Hom}_{\underline{\mathrm{Mon}}}(FX, M) \xrightarrow{\phi} \mathrm{Hom}_{\underline{\mathrm{Sets}}}(X, UM)$$

$$\phi(g) = U(g) \circ i_X$$

<u>Mon</u>

$$FX \xrightarrow{g} M$$

<u>Sets</u>

$$\begin{array}{ccc} UFX & \xrightarrow{U(g)} & UM \\ {\scriptstyle i_X}\uparrow & \nearrow & \\ X & & \end{array}$$

ϕ iso is the same as the universal property.

Prop: Given two functors $\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$, TFAE

(1) $F \dashv U$, with $\phi_{C,D} : \mathrm{Hom}_{\mathbb{D}}(FC, D) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(C, UD)$

(2) There is a natural transformation $\eta : \mathrm{id}_{\mathbb{C}} \longrightarrow U \circ F$ with the following universal property: for any $C \in \mathbb{C}$, $D \in \mathbb{D}$, $f : C \longrightarrow UD$, $\exists!\ g : FC \to D$ such that $f = U(g) \circ \eta_C$.

$\boxed{\eta \text{ called } \underline{\text{unit}}}$

Moreover $\phi_{C,D}(g) = U(g) \circ \eta_C$ and $\eta_C = \phi_{C,FC}(\mathrm{id}_{FC})$.

Proof: $((1) \Longrightarrow (2))$ ϕ is natural, meaning:

Given $h : D \longrightarrow D'$ in $\mathbb{D}$

$$\begin{tikzcd} FC \arrow[r, "g"] \arrow[dr, "h \circ g"'] & D \arrow[d, "h"] \\ & D' \end{tikzcd} \longrightarrow \begin{tikzcd} C \arrow[r, "\phi(g)"] \arrow[dr, "{\phi(h\circ g) = U(h)\circ\phi(g)}"'] & UD \arrow[d, "U(h)"] \\ & UD' \end{tikzcd}$$

Let $h : C' \longrightarrow C$ in $\mathbb{C}$

$$\begin{tikzcd} FC \arrow[r, "g"] & D \\ FC' \arrow[u, "Fh"] \arrow[ur, "g \circ Fh"'] & \end{tikzcd} \longrightarrow \begin{tikzcd} C \arrow[r, "\phi(g)"] & UD \\ C' \arrow[u, "h"] \arrow[ur, "{\phi(g)\circ h = \phi(g \circ Fh)}"'] & \end{tikzcd}$$

Let $\eta_C = \phi(\mathrm{id}_{FC})$. Let $h : C \longrightarrow C'$ in $\mathbb{C}$.

$$\begin{aligned} UF(h) \circ \eta_C &= UF(h) \circ \phi(\mathrm{id}_{FC}) \\ &= \phi(F(h)) = \phi(\mathrm{id}_{FC'} \circ F(h)) = \eta_{C'} \circ h. \end{aligned}$$

Similarly, given $FC \xrightarrow{g} D$, $U(g) \circ \eta_C = U(g) \circ \phi(\mathrm{id}_{FC}) = \phi(g)$.
Since ϕ is an iso, this gives universal property for η.

<u>Proof:</u> ((2) $\Rightarrow$ (1)) The universal property of η says each $\phi_{C,D}$ is an iso, so we just have to check naturality in C and D.

First check it's natural in $\mathbb{C}$. For $h : C' \to C$ in $\mathbb{C}$,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{D}}(FC, D) \arrow[rrr, "\phi_{C,D}"] \arrow[ddd, "(Fh)^*"'] & & & \mathrm{Hom}_{\mathbb{C}}(C, UD) \arrow[ddd, "h^*"] \\
& g \in \mathrm{Hom}_{\mathbb{D}}(FC,D) \arrow[r, mapsto] \arrow[d, mapsto] & U(g) \circ \eta_C \in \mathrm{Hom}_{\mathbb{C}}(C,UD) \arrow[d, mapsto] & \\
& g \circ F(h) \in \mathrm{Hom}_{\mathbb{D}}(FC',D) \arrow[r, mapsto] & U(g) \circ \eta_C \circ h = U(g \circ F(h)) \circ \eta_{C'} \in \mathrm{Hom}_{\mathbb{C}}(C',UD) & \\
\mathrm{Hom}_{\mathbb{D}}(FC', D) \arrow[rrr, "\phi_{C',D}"'] & & & \mathrm{Hom}_{\mathbb{C}}(C', UD)
\end{tikzcd}$$

Similarly, we can check naturality in D. ∎

<u>Corollary</u>: By duality ~~((1) is the same)~~ we also see that the following is equivalent to (1) and (2):

<u>Counit</u> (3) $\varepsilon : F \circ U \to \mathrm{id}_{\mathbb{D}}$ is a natural transformation such that for $C \in \mathbb{C}$, $D \in \mathbb{D}$, $g : FC \to D$, $\exists!\ f : C \to UD$ s.t. $g = \varepsilon_D \circ F(f)$.

Example: $\Delta : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$ the diagonal functor.

$\Delta(C) = (C, C)$ and $\Delta(f) = (f, f)$.

Does Δ have a right adjoint?

Such a functor would be $R : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ such that arrows $\Delta(C) \longrightarrow (X, Y)$ are in bijection with arrows $C \longrightarrow R(X,Y)$.

$R(X,Y)$ must be the product $X \times Y$.

The unit is $\eta_C : C \xrightarrow{\langle id_C, id_C \rangle} C \times C$

the counit is $\varepsilon_{(X,Y)} : (X \times Y, X \times Y) \longrightarrow (X, Y)$

$$\varepsilon_{(X,Y)} = (\pi_X, \pi_Y).$$

Similarly, coproduct is left adjoint to Δ.

More examples:

$(-) \times A$ has right adjoint $(-)^A$ if all exponentials exist.

Initial objects are left adjoint to $\mathbb{C} \longrightarrow \underline{1}$, terminal objects are right adjoint to the same.

More examples:

In general, let J be an index category,

$$\Delta_J : \mathbb{C} \longrightarrow \mathbb{C}^J \quad \text{a constant functor}$$

Δ_J has a right adjoint iff $\mathbb{C}$ has J-limits

a left adjoint iff $\mathbb{C}$ has J-colimits.

$$\varinjlim_J(-) \dashv \Delta_J \dashv \varprojlim_J(-) \;.$$

Polynomial Rings

Fix $R \in \underline{\text{Rings}}$, create $R[x]$.

Have $\eta_R : R \longrightarrow R[x]$ inclusion of constant.

This is the unit of the adjunction ~~⊣ (-)[x]~~

$(-)[x] \dashv \mathcal{U}$

$$\begin{tikzcd} \underline{\text{Rings}} \arrow[r, shift right] & \underline{\text{Rings}_*} \arrow[l, shift right, "\mathcal{U}"'] \end{tikzcd}$$

$$\mathcal{U} : (R, r) \longmapsto R$$

Counit is $\varepsilon_{(A,a)} = ev_a : \quad (A[x], x) \longmapsto (A, a)$.

Preorder Adjoints

Consider preorders P, Q with $P \underset{F}{\overset{U}{\rightleftarrows}} Q$, $F \dashv U$.

$$Fa \leq x \quad \text{iff} \quad a \leq Ux$$

called a Galois connection.

unit of $p \in P$;

Fp is some element such that $p \leq UF(p)$

by universal property, Fp is least $x \in Q$ such that $p \leq Ux$.

$$\begin{array}{llll} Q & & Fp \leq x & \\ \hline P & UF(p) & & Ux \\ & \text{VI} & \leq & \\ & p & & \end{array}$$

counit of $q \in Q$;

some $Ux \leq q$. if $Fy \leq q$ then $y \leq Uq$

Uq is greatest among those $y \in P$ such that $Fy \leq q$.

Example: $X \in$ Top $\quad \mathcal{O}(X) \underset{\text{inclusion}}{\overset{\text{interior}}{\rightleftarrows}} P(X)$ forms adjunction.

inclusion (U) is the least $A \subseteq X$ s.t. $U \subseteq \mathring{A}$ $(= U.)$

interior of A is greatest $U \in \mathcal{O}(X)$ s.t. $U \subseteq A$ ✓ $(= \mathring{A})$.

Example: $f: A \to B$ in Sets $\quad P(A) \underset{f}{\overset{f^{-1}}{\rightleftarrows}} P(B) \quad f \dashv f^{-1}$

Also $\quad P(B) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} P(A) \quad f^{-1} \dashv f_*$

$f_*(U)$ is greatest subset $V \subseteq B$ s.t. $f^{-1}(V) \subseteq U$.

$$f_*(U) = \{ b \in B \mid f^{-1}(b) \subseteq U \}$$

Theorem: (RAPL) right adjoints preserve limits.

(Dually, left adjoints preserve colimits (LAPC)).

Proof: Consider $F \dashv U$, $\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$ and

$D : J \longrightarrow \mathbb{D}$, $\varprojlim_J D \in \mathbb{D}$.

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(X, U(\varprojlim_J D)) &\cong \mathrm{Hom}_{\mathbb{D}}(FX, \varprojlim_J D) \\
&\cong \varprojlim_{J \ni j} \mathrm{Hom}_{\mathbb{D}}(FX, D_j) \\
&\cong \varprojlim_{j \in J} \mathrm{Hom}_{\mathbb{C}}(X, UD_j)
\end{aligned}$$

So by Yoneda, $U(\varprojlim_J D) \cong \varprojlim_J UD$. ∎

Free Cocompletion

<u>Prop:</u> Yoneda is the free cocompletion of a small category $\mathbb{C}$. That is, ...

For any cocomplete, locally small $\mathcal{E}$, and any functor $F : \mathbb{C} \longrightarrow \mathcal{E}$, there is a natural isomorphism $F_! : \hat{\mathbb{C}} \longrightarrow \mathcal{E}$ such that $F_! \circ y = F$, and $F_!$ is cocontinuous. Moreover, $F_!$ is unique up to natural isomorphism.

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "F_!"] & \mathcal{E} \\ \mathbb{C} \arrow[u, hook, "y"] \arrow[ur, "F"'] & \end{tikzcd}$$

<u>Proof:</u>

We take $F_!$ as the left adjoint of F^*. What is F^*?

For $E \in \mathcal{E}$, $C \in \mathbb{C}$,

$$\begin{aligned} F^*(E)(C) &\cong \mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F^*(E)) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(F_!(yC), E) \\ &\cong \mathrm{Hom}_{\mathcal{E}}(F(C), E). \end{aligned}$$

So $\mathrm{Hom}_{\mathcal{E}}(F(-),(-)) : \mathbb{C}^{op} \times \mathcal{E} \longrightarrow \underline{\mathbf{Sets}}$ is a bifunctor.

Let F^* be the transpose of the above,

$$F^* : \mathcal{E} \longrightarrow \underline{\mathbf{Sets}}^{\mathbb{C}^{op}}.$$

over $\longrightarrow$

Proof (continued): Let $P \in \hat{\mathbb{C}}$. We know $P \cong \varinjlim_{(x,C)\in\int P} yC$.

$$F_!(P) \cong \varinjlim_{(x,C)\in\int P} F_!(yC) \cong \underbrace{\varinjlim_{(x,C)\in\int P} FC}$$

we know that this always exists, so this is another def of $F_!$

Recall that colimit is adjoint to the diagonal.

If $\theta : P \to Q$, then maps $\left(\varinjlim_{(x,C)\in\int P} FC\right) \longrightarrow \left(\varinjlim_{(x',C')\in\int Q} FC'\right)$

are in bijection with collections of maps

$$\left(FC \longrightarrow \varinjlim_{(x',C')\in\int Q} FC'\right)_{(x,C)\in\int P}.$$

We have for $E \in \mathcal{E}$ and $P \in \hat{\mathbb{C}}$, $P \cong \varinjlim_{(x,C)\in\int P} yC$,

$$\begin{aligned}
\mathrm{Hom}_{\hat{\mathbb{C}}}(P, F^*(E)) &\cong \mathrm{Hom}_{\hat{\mathbb{C}}}(\varinjlim yC, F^*(E)) \\
&\cong \varprojlim \mathrm{Hom}_{\hat{\mathbb{C}}}(yC, F^*E) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{E}}(FC, E) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(\varinjlim FC, E) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(F_!(P), E)
\end{aligned}$$

So this shows the proposition. ∎

Corollary: If $f : \mathbb{C} \to \mathbb{D}$ in $\underline{\mathrm{Cat}}$, we have

$f^* : \hat{\mathbb{D}} \longrightarrow \hat{\mathbb{C}}$ given by $f^*(Q)(C) = Q(fC)$

$$f^*(Q) = Q \circ f^{op}.$$

Then: $f_! \dashv f^* \dashv f_*$

$$y_{\mathbb{D}} \circ f \cong f_! \circ y_{\mathbb{C}}$$

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[rr, bend left, "f_!"] \arrow[rr, bend right, "f_*"'] & & \hat{\mathbb{D}} \arrow[ll, "f^*"'] \\
\mathbb{C} \arrow[u, "y_{\mathbb{C}}"] \arrow[rr, "f"] & & \mathbb{D} \arrow[u]
\end{tikzcd}$$

Proof: let $F = y_D \circ f$. need $F^* = f^*$.

$$(F^*Q)(C) = \mathrm{Hom}_{\hat{\mathbb{D}}}(FC, Q) \cong \mathrm{Hom}_{\hat{\mathbb{D}}}((yD)(fC), Q)$$

$$\cong Q(f(C)) = (f^*Q)(C).$$

$$\begin{tikzcd}
 & \hat{\mathbb{C}} \arrow[d, bend right, "F_!"'] \\
\mathbb{C} \arrow[ur, "y_{\mathbb{C}}"] \arrow[r, "F = y_{\mathbb{D}} \circ f"'] & \hat{\mathbb{D}} \arrow[u, bend right, "F^*"']
\end{tikzcd}$$

$$(f^* \circ y_{\mathbb{D}})_! \dashv (f^* \circ y_{\mathbb{D}})^* =: f_*$$

$$\begin{tikzcd}
\hat{\mathbb{D}} \arrow[r, "f^*"] & \hat{\mathbb{C}} \\
\mathbb{D} \arrow[u, "y_{\mathbb{D}}"] \arrow[ur, "f^* \circ y_{\mathbb{D}}"']
\end{tikzcd}$$

$$f^*(\varinjlim Q_j)(C) = (\varinjlim Q_j)(fC) \cong \varinjlim Q_j(fC) \cong \varinjlim (f^*Q)\, C.$$

$$\cong (\varinjlim f^*Q)\, C \quad \blacksquare$$

Def: A locally <u>cartesian</u> <u>closed</u> <u>category</u> is a category $\mathcal{E}$ with terminal object 1 and for any $f: A \longrightarrow B$,

$$\Sigma_f : \mathcal{E}/A \longrightarrow \mathcal{E}/B \quad \text{has} \quad \Sigma_f \dashv f^* \dashv \Pi_f .$$

Prop: For any $\mathcal{E}$ with terminal object, TFAE:

(1) $\mathcal{E}$ is locally CCC

(2) For all $A \in \mathcal{E}$, $\mathcal{E}/A$ is CCC

[Proof in the book]

Corollary: if $\mathbb{C}$ is small, $P \in \hat{\mathbb{C}}$, then ~~[illegible]~~

$$\hat{\mathbb{C}}/P \simeq \hat{\mathbb{D}} \quad \text{where} \quad \mathbb{D} = \int_{\mathbb{C}} P.$$

$$\Phi\,(Q \xrightarrow{\theta} P)\,(yC \xrightarrow{x} P) = \mathrm{Hom}_{\hat{\mathbb{C}}/P}\left(\begin{matrix} yC \\ \downarrow x \\ P \end{matrix},\ Q \xrightarrow{\theta} P \right).$$

So $\hat{\mathbb{C}}$ is always locally CCC.

April 08, 2015

Adjoint Functor Theorem

An abstract tool used to show existence of adjoint functors.

- free group functor $\underline{\text{Sets}} \underset{U}{\overset{F}{\rightleftarrows}} \underline{\text{Groups}}$

  free group F

  forgetful U $\qquad F \dashv U$

- stone - čech compactification $\beta \dashv U$

  $\underline{\text{Top}} \underset{\beta}{\overset{U}{\leftrightarrows}} \underline{\text{CHaus}}$ = compact hausdorff spaces.

<u>Theorem</u> (Freyd's AFT):

$\mathbb{D}$ locally small, complete category

$\mathbb{X}$ any category

$U : \mathbb{D} \longrightarrow \mathbb{X}$ continuous functor

Then <u>TFAE</u>:

(1) U has a left adjoint

(2) U satisfies the <u>solution set condition</u> (SSC)

<u>Def</u>: For every $X \in \mathbb{X}$, if there is a (small) set I and I-indexed family $\{ f_i : X \longrightarrow U D_i \mid i \in I \}$ such that for any $f : X \longrightarrow UD$ there is some $\bar{f} : D_i \longrightarrow D$ and $f = U(\bar{f}) \circ f_i$. (Diagram on next page)

Solution Set Condition

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & UD_i \arrow[d, "U(\bar{f})"] \\ & UD \end{tikzcd} \qquad \begin{tikzcd} D_i \arrow[d, "\bar{f}"] \\ D \end{tikzcd}$$

given f, $\exists\ i \in I$, $\bar{f} : D_i \longrightarrow D$ in $\mathcal{D}$, such that the diagram above commutes.

<u>Aside</u>: Comma Categories

<u>Def</u>: Given categories $\mathbb{C}, \mathbb{D}, \mathbb{E}$ and functors

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "F"'] & & \mathbb{D} \arrow[dl, "G"] \\ & \mathbb{E} & \end{tikzcd},$$

define the <u>comma category</u> $(F \downarrow G)$ which has

objects triples $(C, D, h : FC \longrightarrow GD)$, $C \in \mathbb{C}$, $D \in \mathbb{D}$

arrows from (C, D, h) to (C', D', h') are pairs $(f : C \to C', g : D \longrightarrow D')$ such that

$$h' \circ Ff = Gg \circ h$$

Examples:

$$(X \downarrow Y) \cong \mathrm{Hom}_{\mathbb{C}}(X,Y)$$

$$\begin{tikzcd} \mathbb{1} \arrow[dr, "X"'] & & \mathbb{1} \arrow[dl, "Y"] \\ & \mathbb{C} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "1_{\mathbb{C}}"'] & & \mathbb{1} \arrow[dl, "X"] \\ & \mathbb{C} & \end{tikzcd}$$

$$(1_{\mathbb{C}} \downarrow X) \cong \mathbb{C}/X$$

$$(X \downarrow 1_{\mathbb{C}}) \cong X/\mathbb{C}$$

$$\begin{tikzcd} \mathbb{C} \arrow[dr, "1_{\mathbb{C}}"'] & & \mathbb{C} \arrow[dl, "1_{\mathbb{C}}"] \\ & \mathbb{C} & \end{tikzcd}$$

$$(1_{\mathbb{C}} \downarrow 1_{\mathbb{C}}) \cong \mathbb{C}^{\rightarrow}$$

Lemma: AFT for initial objects

$$U : \mathbb{D} \xrightarrow{!} \mathbb{1}$$

$\mathbb{D}$: locally small, complete category. TFAE:

(1) $\mathbb{D}$ has initial object

(2) $\mathbb{D}$ satisfies SSC: There is a set $(D_i)_{i \in I}$ of objects in $\mathbb{D}$ s.t. each $D \in \mathbb{D}$ has some arrow $D_i \to D$.

Proof: $(1 \to 2)$ take $\{0\}$ to be the solution set.

$(2 \to 1)$ $W = \prod_{i \in I} D_i \longrightarrow D$ has arrows to each object.

$$\begin{tikzcd} W = \prod_{i \in I} D_i \arrow[rr] \arrow[r] & D_i \arrow[r] & D \end{tikzcd}$$

over $\rightarrow$

proof (continued):

$W = \prod_{i \in I} D_i$ has a map to every object but these need not be unique...

So we form an equalizer

$$\begin{tikzcd} V \arrow[r, tail, "h"] & W \arrow[r, shift left, "\Delta"] \arrow[r, shift right, "\langle d \rangle"'] & \prod\limits_{d: W \to W} W \end{tikzcd}$$

$p_d \circ \Delta = id_W$

$p_d \circ \langle d \rangle = d$

(Note: for any $d: W \to W$, $d \circ h = h$) ✯

Now given two maps $f, g: V \longrightarrow D$ for any $D \in \mathbb{D}$, want to show that $f = g$. So consider the equalizer U, we have a map $W \xrightarrow{s} U$ b/c W is weakly initial.

$$\begin{tikzcd} U \arrow[r, tail, "e"] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, tail, "h"] & D \\ W \arrow[u, "s"] \arrow[r, "hes"'] & W & \end{tikzcd}$$

$hesh = h$ by ✯

$\Longrightarrow esh = 1_V$ b/c h monic

$\Longrightarrow$ e monic and split epic, so e is an iso.

<u>Def</u>: An object $X \in \mathbb{C}$ is <u>weakly initial</u> if for every $C \in \mathbb{C}$, there is a map $X \longrightarrow C$. Need not be unique.

<u>Theorem</u> (Adjoint functor theorem):

$$U : \mathbb{D} \longrightarrow \mathbb{X}$$

$\mathbb{D}$ locally small, complete category. Then TFAE

(1) U has left adjoint

(2) U satisfies SSC: for every $X \in \mathbb{X}$, there is a set I, objects $(c_i)_{i \in I}$ of $(X \downarrow U)$ that are jointly weakly initial.

<u>Lemma</u>: An adjunction $\mathbb{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{D}$ is completely determined by the functor U and for $c \in \mathbb{C}$, an initial object of $(c \downarrow U)$.

<u>Proof</u>: $(\Longrightarrow)$ Use the UMP for η_c, $c \in \mathbb{C}$.

Claim $(F(c), c \xrightarrow{\eta_c} UF(c))$ is an initial object.

$$\begin{tikzcd} & UF(c) \arrow[d, "U(\bar{f})"] \\ c \arrow[ur, "\eta_c"] \arrow[r, "f"'] & U(D) \end{tikzcd} \qquad \begin{tikzcd} F(c) \arrow[d, "\bar{f}"] \\ D \end{tikzcd}$$

$\mathbb{C}$ | $\mathbb{D}$

$(\Longleftarrow)$ Define $F_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ by $\eta_C : C \longrightarrow U(F_0 C)$.

Extends uniquely to a functor F s.t. $(\eta_C)_{C \in \mathbb{C}_0}$ composition of natural transformation $\mathbb{1}_{\mathbb{C}} \longrightarrow U \circ F$

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[d, "h"'] & U(F_0 C) \arrow[d, "UFh"] \\ C' \arrow[r, "\eta_{C'}"'] & U(F_0 C') \end{tikzcd} \qquad \begin{tikzcd} F_0 C \arrow[d, "\exists!\ Fh"] \\ F_0 C' \end{tikzcd}$$

■

Proof of adjoint functor theorem only needs the following lemma now:

<u>Lemma:</u> If $U : \mathbb{D} \longrightarrow \mathbb{X}$ preserves products/equalizers, then for $X \in \mathbb{X}$, the projection

$$P : (X \downarrow U) \longrightarrow D$$

creates products / equalizers

<u>Proof</u>: (1) products.

Let J a set, $\{ (D_j , f_j : X \longrightarrow UD_j) \mid j \in J \}$ collection of objects in $(X \downarrow U)$

$U\left(\prod_{j \in J} D_j\right) \xrightarrow{U(p_j)} U(D_j)$ product diagram in $\mathbb{X}$.

$$\begin{tikzcd} & U(\prod D_j) \arrow[d, "U(p_j)"] \\ X \arrow[ur, dashed, "\exists! f"] \arrow[r, "f_j"'] & U(D_j) \end{tikzcd}$$

(2) Equalizers

given a parallel pair in $(X \downarrow U)$, and an equalizer e

$$E \overset{e}{\rightarrowtail} B \underset{s_1}{\overset{s_0}{\rightrightarrows}} C \quad \text{in } \mathbb{D},$$

$$\begin{tikzcd}
UE \arrow[r, tail, "Ue"] & UB \arrow[r, shift left, "Us_0"] \arrow[r, shift right, "Us_1"'] & UC \\
& X \arrow[ul, "h"] \arrow[u, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Can check that this is an equalizer in the comma category $(X \downarrow U)$.

■

This completes the proof of the adjoint functor theorem.

Examples: What is the solution set for the forgetful functor $\underline{\text{Groups}} \xrightarrow{U} \underline{\text{Sets}}$?

for $X \in \underline{\text{Sets}}$, want a soln set $\{ f_i : X \to US_i \}$

$$\begin{tikzcd}
& US_i \arrow[d, "U(h)"] \\
X \arrow[ur, "f_i"] \arrow[r, "f"'] & UG
\end{tikzcd}
\qquad
\begin{tikzcd}
S_i \arrow[d, "h"] \\
G
\end{tikzcd}$$

~~Pick a set~~ $f(X)$ generates subgroup $S \subseteq G$ of size $\leq |X \cup \aleph_0|$

Pick a $\underline{\text{set}}$ of isomorphism classes of ~~of~~ groups of at most this size, with all functions $X \xrightarrow{f_i} US_i$.

Examples:

Corollary: If $\mathbb{D}$ is small + complete, then any continuous functor $U: \mathbb{D} \longrightarrow \mathbb{X}$, where $\mathbb{X}$ is locally small, U has a left adjoint.

Prop: $\mathbb{D}$ is small and complete $\Longrightarrow$ $\mathbb{D}$ is a preorder.

Monads and Algebras

Recall: T an equational theory

get an adjunction
$$\begin{tikzcd} \text{Sets} \arrow[r, bend right, "F"'] \arrow[r, phantom, "T"] & T\text{-alg} \arrow[l, bend right, "U"'] \end{tikzcd}$$

We reformulate adjunctions in equational terms.

Given an adjunction $U \vdash F$ $\quad \mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$,

$\eta : id_{\mathbb{C}} \to UF$, $\varepsilon : FU \to id_{\mathbb{D}}$

$$Hom_{\mathbb{D}}(FC, D) \underset{\tilde{\psi}}{\overset{\tilde{\phi}}{\rightleftarrows}} Hom_{\mathbb{C}}(C, UD)$$

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "\phi f = (Uf)\eta_C"'] & UFC \arrow[d, "Uf"] \\ & UD \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, "f"] \\ D \end{tikzcd} \quad \begin{tikzcd} C \arrow[d, "g"] \\ UD \end{tikzcd} \qquad \begin{tikzcd} FC \arrow[d, "Fg"'] \arrow[dr, "\psi g = \varepsilon_D(Fg)"] & \\ FUD \arrow[r, "\varepsilon_D"'] & D \end{tikzcd}$$

Triangle identities

$$\begin{tikzcd} F \arrow[rr, "id_F"] \arrow[dr, "\eta_U"'] & & F \\ & FUF \arrow[ur, "\varepsilon_F"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "id_U"] \arrow[dr, "\eta_U"'] & & U \\ & UFU \arrow[ur, "U_\varepsilon"'] & \end{tikzcd}$$

$$\eta_C = \phi(id_{FC}) \implies id_{FC} = \psi(\eta_C)$$
$$= \varepsilon_{FC}(F\eta_C)$$

<u>Prop</u>: TFAE

(1) $F \dashv U$ w/ unit η, counit ε

(2) Triangle identities hold

<u>Proof</u>: just did $(1) \Longrightarrow (2)$.

for $(2) \Longrightarrow (1)$, take $\phi f = (Uf)\eta_C$

$$\psi g = \varepsilon_D (Fg)$$

can check naturality. ∎

Now given
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "F"'] & \mathbb{D} \arrow[l, bend right, "U"'] \end{tikzcd}$$

$T = UF : \mathbb{C} \longrightarrow \mathbb{C}$

$\eta : 1_{\mathbb{C}} \longrightarrow T$

$\varepsilon_{FC} : FUFC \longrightarrow FC$

$U\varepsilon_{FC} : T^2 C \longrightarrow TC$

$\mu = U\varepsilon_F : T^2 \longrightarrow T$ "multiplication map"

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

This square commutes using naturality of ε

"associativity"

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, bend right, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, bend left, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

"unit"

$\mu\eta_T = (U\varepsilon_F)\eta_{UF} = \mathrm{id}_{UF} = \mathrm{id}_T$

$\mu(T\eta) = (U\varepsilon_F)(UF\eta)$

$= U(\varepsilon F\eta) = U\,\mathrm{id}_F = \mathrm{id}_{UF}$

Def: A <u>monad</u> on $\mathbb{C}$ is an endofunctor $T : \mathbb{C} \longrightarrow \mathbb{C}$ together with $\eta : id_{\mathbb{C}} \longrightarrow T$, $\mu : T^2 \longrightarrow T$ such that $\mu \circ \mu_T = \mu \circ T(\mu)$ and $\mu \circ \eta_T = id_T = \mu \circ T(\eta)$

equivalently, this is nothing more ~~than~~ or less than a monoid in $(\mathbb{C}^{\mathbb{C}}, \circ, id_{\mathbb{C}})$.

$$\begin{tikzcd}
T \otimes T \otimes T \arrow[r, "id_T \otimes \mu"] \arrow[d, "\mu \otimes id_T"'] & T \otimes T \arrow[d, "\mu"] \\
T \otimes T \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$\mu \circ \mu_T = \mu \circ (\mu \otimes id_T) =$

$\mu \circ (id_T \otimes \mu) = \mu \circ T(\mu)$.

$$\begin{tikzcd}
id_{\mathbb{C}} \otimes T = T \arrow[r, "\eta_T"'] \arrow[r, bend left, "\eta \otimes id_T"] \arrow[dr, "id_T"'] & T \otimes T \arrow[d, "\mu"] & T = T \otimes id_{\mathbb{C}} \arrow[l, "T(\eta)"] \arrow[l, bend right, "id_T \otimes \eta"'] \arrow[dl, "id_T"] \\
& T &
\end{tikzcd}$$

$\mu \circ \eta_T = \mu \circ (id_T \otimes \eta) = id_T$

$\mu \circ T(\eta) = \mu \circ (\eta \circ id_T) = id_T$

<u>Prop</u>: Every adjunction gives rise to ~~an~~ a monad UF.

<u>Prop</u>: Every monad arises from an adjunction.

Proven in the next few pages

Example: P : Sets $\longrightarrow$ Sets

covariant: $X \longrightarrow P(X)$

$$\left(X \xrightarrow{f} Y\right) \longmapsto \left(P(X) \xrightarrow[\mathrm{im}(f)]{} P(Y)\right)$$

$\eta : \mathrm{id}_{\text{Sets}} \longrightarrow P$ $\qquad$ $\mu : P^2 \longrightarrow P$

$\eta_X : X \longmapsto P(X)$ $\qquad$ $\mu_X : P(P(X)) \longrightarrow P(X)$

$\eta_X(x) = \{x\}$ $\qquad$ $\mu_X(\alpha) = \bigcup\limits_{a \in \alpha} a$.

Example: P poset, monad on P is a monotone map $T : P \longrightarrow P$

$$\overset{(\eta)}{x \leq Tx} \qquad \overset{(\mu)}{T^2x \leq Tx}$$

$$Tx \leq T^2x \qquad \therefore \quad T^2x = Tx$$

a Closure operator. (think topological closure)

$$\begin{tikzcd} P \arrow[r, "T"] \arrow[dr, "t"'] & P \\ & K = \{x \in P \mid Tx = x\} \arrow[u, "i"'] \end{tikzcd}$$

$p \leq ik \Longrightarrow tp \leq tk = k$

$tp \leq k \Longrightarrow p \leq itp \leq k$

$t \dashv i$, $T = it$

Example: a ring R, A an R-algebra

$$\left.\begin{array}{l} \eta : R \longrightarrow A \\ \cdot : A \otimes A \longrightarrow A \end{array}\right\} \text{ in } R\text{-}\underline{\text{mod}}$$

gives a monad on R-$\underline{\text{mod}}$

$$T(M) = A \otimes M$$

$$\begin{tikzcd} M = R \otimes M \arrow[r, "\eta \otimes id_M"'] \arrow[r, bend right, "\eta"'] & A \otimes M = T(M) \end{tikzcd}$$

Extension of scalars is the monad

$$A \otimes A \otimes M \xrightarrow[(\cdot) \otimes id_M]{} A \otimes M$$

Algebras for a monad

goal: $\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "u"'] & \mathbb{C}^T \arrow[l, bend right, "F"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$ ← category of T-algebras

$uF = T$

Def: the Eilenberg - Moore category $\mathbb{C}^T$ has

objects: pairs (A, α)

$$A \in \mathbb{C}, \quad \alpha : TA \longrightarrow A$$

such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T(\alpha)"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

Arrows in $\mathbb{C}^T$ from (A, α) to (B, β) are

$A \xrightarrow{h} B$ in $\mathbb{C}$ such that the following commutes.

$$\begin{tikzcd}
TA \arrow[r, "\alpha"] \arrow[d, "T(h)"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "h"] \\
TB \arrow[r, "\beta"'] & B
\end{tikzcd}$$

<u>Example</u>: $\underline{\mathbf{Sets}} \underset{F}{\overset{u}{\rightleftarrows}} \underline{\mathbf{Mon}}$

$TX = UFX =$ set of strings / lists / words over X

$\eta_X : X \longrightarrow TX \qquad \eta_X(x) = x.$

$\mu_X : T^2X \longrightarrow TX$

$\mu_X(\omega_1, \ldots, \omega_n) = \omega_1 \omega_2 \cdots \omega_n$ (concatenation)

T-algebra is a set M, arrow $TM \xrightarrow{\bullet} M$

$$\begin{tikzcd}
m \arrow[r, mapsto] & m \\
M \arrow[r] \arrow[dr] & TM \arrow[d, "\bullet"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 M \arrow[r, "T(\bullet)"] \arrow[d, "\text{concat}"'] & TM \arrow[d, "\bullet"] \\
TM \arrow[r, "\bullet"'] & M
\end{tikzcd}$$

~~Example: M an A module~~

Example:

$$
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[ddr, bend right, mapsto] & (1\otimes m) \arrow[dd, bend left, mapsto] \\
M \arrow[r] \arrow[dr] & A\otimes M \arrow[d, "\alpha"] \\
 & M \ni m = 1m
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "id_A \otimes \alpha"] \arrow[d] & A\otimes M \arrow[d, "\alpha"] \\
A\otimes M \arrow[r, "\alpha"'] & M
\end{tikzcd}
\qquad (ab)m = a(bm)
$$

Want to make an adjunction with $\mathbb{C}$ and $\mathbb{C}^T$

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend right, "F"'] & \mathbb{C}^T \arrow[l, bend right, "U"']
\end{tikzcd}
\qquad
\begin{aligned}
U(A,\alpha) &= A \\
U(h) &= h
\end{aligned}
$$

$T = UF$

$$FC = (TC, \mu_C)$$

Monad laws guarantee that this is an algebra!

$$
\begin{tikzcd}
T^3C \arrow[r, "T(\mu_C)"] \arrow[d, "\mu_{TC}"'] & T^2C \arrow[d, "\mu_C"] \\
T^2C \arrow[r, "\mu_C"'] & TC
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2C \arrow[r, "\mu_C"] & TC \\
TC \arrow[u, "\eta_{TC}"] \arrow[ur, "id_{TC}"'] &
\end{tikzcd}
$$

$F(h) = T(h)$ is an arrow of T-algebras.

Finally, show $F \dashv U$.

η (monad) $= \eta$ (adjunction)

μ (monad) $= U(\varepsilon_F)$

$\uparrow$ counit

Need $\varepsilon : FU \longrightarrow id_{\mathbb{C}^T}$

(A,α), $FU(A,\alpha) = (TA, \mu_A)$

$$\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "T(\alpha)"'] & TA \arrow[d, "\alpha"] \\
TA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

$$\begin{array}{ccc}
\alpha : (TA, \mu_A) & \longrightarrow & (A,\alpha) \\
\shortparallel & & \shortparallel \\
FU(A,\alpha) & & id_{\mathbb{C}^T}(A,\alpha)
\end{array}$$

Define $\varepsilon_{(A,\alpha)} := \alpha$.

Def: $U : \mathbb{D} \to \mathbb{C}$ is <u>monadic</u> (strictly monadic) if $F \dashv U$ such that Φ is an equivalence of categories.
(isomorphism)

<u>Theorem</u> (Beck's monadicity theorem):

$U : \mathbb{D} \to \mathbb{C}$ is monadic iff

(1) U has a left adjoint

(2) U creates coequalizers of <u>U-split</u> pairs.

<u>Def</u>: $D \overset{f}{\underset{g}{\rightrightarrows}} D'$ in $\mathbb{D}$ is <u>U-split</u> iff

$UD \overset{Uf}{\underset{Ug}{\rightrightarrows}} UD'$ extends to a split coequalizer diagram in $\mathbb{C}$.

<u>Theorem</u>: Let $T_0 : \mathbb{C}_0 \to \mathbb{C}_0$ $\quad (\eta_C : C \to T_0 C)_{C \in \mathbb{C}_0}$

Then (1) this extends to a monad (T, η, μ)

iff (2) there is $(\beta_{A,B} : Hom(A, TB) \to Hom(TA, TB))_{A,B \in \mathbb{C}_0}$

such that

(i) $\beta_{A,B}(f) \circ \eta_A = f$ for $A \xrightarrow{f} TB$

(ii) $\beta_{AA}(\eta_A) = 1_{TA}$

(iii) $\beta_{B,C}(g) \circ \beta_{A,B}(f) = \beta(\beta(g) \circ f)$

$A \xrightarrow{f} TB \quad B \xrightarrow{g} TC$

$$\begin{tikzcd} TA \arrow[r, "\beta(f)"] \arrow[rr, bend right, "\beta(\beta(g)\circ f)"'] & TB \arrow[r, "\beta(g)"] & TC \end{tikzcd}$$

Def: Comonads

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "R"] \arrow[r, bend right, "L"'] & \mathbb{D} \arrow[l, "U"'] \end{tikzcd}$$

$L \dashv U \dashv R$

$T = UL$ monad

$G = UR$ comonad

$T \dashv G$

Def: Given an endofunctor $P : S \longrightarrow S$, a P-Algebra(S) category has ~~Rte~~ $(P\text{-}Alg(S))$

objects $\quad (A, \alpha) : A \in S, \; PA \xrightarrow{\alpha} A$ in S

arrows $\quad (A, \alpha) \longrightarrow (B, \beta) :$

$$A \xrightarrow{h} B \qquad \begin{tikzcd} PA \arrow[r, "\alpha"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ PB \arrow[r, "\beta"'] & B \end{tikzcd}$$

Examples: $\quad P(X) = 1 + X + X \times X$ in Sets

$$1 + X + X \times X \xrightarrow{[u, i, m]} X$$

$P\text{-}Alg(\text{Sets})$: group structures (w/out group laws)

$$P(X) = \underset{\{0,1\}}{\underset{\uparrow}{2}} + X + \underset{2 \times X \times X}{\underset{\uparrow}{2X^2}}$$

two nullary operations
one unary operation
two binary operations

More Monads!

$$\left.\begin{array}{l} T : \mathbb{C} \to \mathbb{C} \\ \eta : 1 \to T \\ \mu : T^2 \to T \end{array}\right\} \text{monoid in } (\mathbb{C}^{\mathbb{C}}, \circ, 1_{\mathbb{C}})$$

Every monad comes from an adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "U"] & \mathbb{C}^T \arrow[l, shift left, "F"] \end{tikzcd} \qquad \begin{array}{l} T = UF \\ \eta = \eta \\ \mu = U(\varepsilon_F) \end{array}$$

$\mathbb{C}^T$ is Eilenberg-Moore algebra

objects $(A, \alpha : TA \to A)$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T(\alpha)"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

$U(A, \alpha) = A$, $U(h) = h$

$FC = (TC, \mu_C : T^2C \to TC)$

To show that $F \dashv U$, check Δ laws

$$\begin{tikzcd} U \arrow[r, "1_U"] \arrow[dr, "\eta_U"'] & U \\ & UFU \arrow[u, "U(\varepsilon)"'] \end{tikzcd}$$

at (A, α), $(U(\varepsilon) \circ \eta_U)_{(A,\alpha)} = \alpha \circ \eta_A = 1_A$

$$\begin{tikzcd} F \arrow[rr, "1_F"] \arrow[dr, "F(\eta)"'] & & F \\ & FUF \arrow[ur, "\varepsilon_F"'] & \end{tikzcd}$$

$$\left(\varepsilon_F \circ F(\eta)\right)_A \overset{\text{def of } \mu_A,\ T(\eta_A)}{=} \mu_A \circ T(\eta_A) \overset{\text{monad laws.}}{=} 1_{TA} = 1_{FA} \text{ in } \mathbb{C}^T$$

So this is in fact an adjunction.

Now, we should also be able to recover μ from U, F and check its the same one we started with:

$$\left(U(\varepsilon_F)\right)_C = U(\mu_C) = \mu_C \ \checkmark.$$

Now, given <u>any</u> adjunction $\mathbb{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{D}$, get $\Phi: \mathbb{D} \to \mathbb{C}^T$

$T = UF$ is a monad

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "\Phi"] \arrow[dr, bend right, "U"'] & & \mathbb{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathbb{C} \arrow[ul, bend right, "F"'] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$$

defined by $\Phi(D) = (UD, U(\varepsilon_D))$

$\Phi F = F^T$

$U^T \Phi = U$

<u>Example</u>: in <u>Sets</u>, have adjunction w/ <u>Mon</u>

$$\underline{\text{Sets}} \underset{\text{free}}{\overset{\text{forgetful}}{\rightleftarrows}} \underline{\text{Mon}} \qquad \underline{\text{Sets}}^T \simeq \underline{\text{Mon}}$$

But not always an equivalence:

for ~~U: Sets~~ $\underline{\text{Sets}} \underset{U}{\overset{F}{\rightleftarrows}} \underline{\text{Posets}}$

$T = 1_{\underline{\text{sets}}}$ and $\underline{\text{Sets}}^T \simeq \underline{\text{Sets}} \not\simeq \underline{\text{Posets}}$.

To generalize,

$$P(X) = C_0 + C_1 \times X + \dots + C_n \times X^n$$

(finitary) polynomial functor for finite sets C_i

$$P(X) = \sum_{i \in I} C_i \times X^{B_i}$$ general polynomial functor

<u>Initial</u> <u>Algebras</u>:

an initial object in $P\text{-}Alg(S)$

recursion principle:

let $P(X) = 1 + X^2$

given I, $[0,m] : 1 + I^2 \to I$ initial algebra

$0 \in I$, $m : I \times I \to I$

for any X, $a \in X$, $* : X \times X \to X$

there is a unique $I \xrightarrow{u} X$

$$\begin{tikzcd} 1+I^2 \arrow[r, "{[0,m]}"] \arrow[d, "1+u^2"'] & I \arrow[d, "u"] \\ 1+X^2 \arrow[r, "{[a,*]}"] & X \end{tikzcd}$$

$u(0) = a$

$u(m(i,j)) = u(i) * u(j)$

<u>Def</u>: A <u>natural</u> <u>numbers</u> <u>object</u> (NNO) is an initial algebra for $P(X) = 1 + X$.

Lambek's Lemma: $P: S \to S$ with initial algebra I, $PI \xrightarrow{i} I$, then i is an isomorphism.

Proof:

$$\begin{tikzcd} P^2 I \arrow[r, "Pi"] \arrow[d, "P(u)"'] & PI \arrow[d, "u"] \\ P^2 I \arrow[r, "Pi"] \arrow[d, "Pi"'] & PI \arrow[d, "i"] \\ PI \arrow[r, "i"] & I \end{tikzcd}$$

initial algebra $\Longrightarrow \exists!\, u$ so the diagram commutes.

the diagram commutes

$iu = id_I$

$ui = P(iu) = id_{PI}$

$PI \cong I$. ∎

Hence, an initial algebra is a least fixed point of P.

Corollary: $P: \underline{Sets} \longrightarrow \underline{Sets}$

covariant powerset functor

has no initial algebra! (would contradict Cantor).

Prop: If S has an initial object 0, ω-colimits, and P is an endofunctor which preserves ω-colimits, then P has an initial algebra.

Proof

$$I := \varinjlim \left(0 \xrightarrow[u_0]{!} P0 \xrightarrow[u_1]{P(!)} P^2 0 \xrightarrow[u_2]{} P^3 0 \longrightarrow \cdots\right)$$

$$\cancel{PI :=} \quad PI \cong \varinjlim \left(P0 \to P^2 0 \to \cdots\right) \cong I$$

Prop: Let S have finite coproducts, $P: S \to S$.
Then TFAE

(1) There's a monad (T, η, μ) on S

and
$$\begin{tikzcd} P\text{-}Alg(S) \arrow[rr, "\simeq"] \arrow[dr, "U"'] & & S^T \arrow[dl, "U^T"] \\ & S & \end{tikzcd}$$

(2) $U : P\text{-}Alg(S) \longrightarrow S$ has a left adjoint $F \dashv U$.

(3) For every $A \in S$, $P_A(X) := A + P(X)$ has an initial algebra.

Proof: (1) $\Rightarrow$ (2) Clear b/c U is a monoidal functor

(2) $\Rightarrow$ (3) $P_A\text{-}Alg(S) \simeq (A \downarrow U)$

$$\Psi$$

Why? $\left(A + P(X) \xrightarrow{\gamma} X\right) = \left(\begin{array}{c} PX \\ \downarrow \beta \\ A \xrightarrow[\alpha]{} X \end{array} \right)$

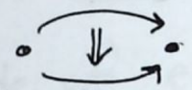
$$= A \xrightarrow{\alpha} U(X, \beta)$$
in $P\text{-}Alg(S)$

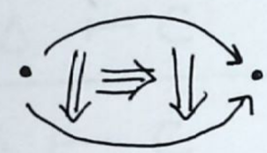
and $U(X, \beta) \cong$ Objects of $(A \downarrow U)$

(3) $\Rightarrow$ (1) Use Beck's monadicity theorem.

need to check that U creates U-split coequalizers.

Fun Stuff: Monoidal Categories and linear logic

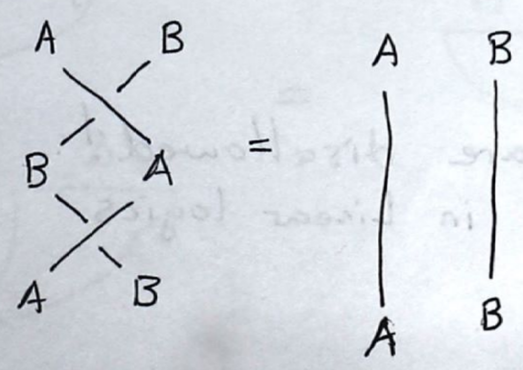
2-category 

3-category 

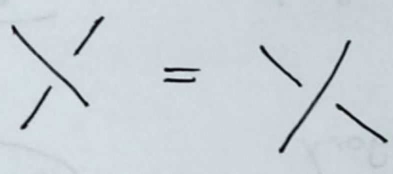
Periodic table of
k-tuply monoidal n-categories

$k \downarrow$ \ $\vec{n}$	0	1	2
0	pointed set	pointed category	pointed 2-category
1	monoid	monoidal category	monoidal 2-category
2	abelian monoid	braided monoidal category	braided monoidal 2-category
3	stabilizes	symmetric monoidal category	sylleptic monoidal 2-category
4		stabilizes	symmetric monoidal 2-category

Braided Monoidal Category: Monoidal category with a symmetry operator $A \otimes B \xrightarrow{\sim} B \otimes A$ such that



Symmetric Monoidal Category: Braided monoidal category such that



~~All~~ Along diagonals, we see that things to the lower left are special versions of things to upper right.

Linear logic

Resource interpretation

$A, A \vdash A$

have cake $\nvdash$ eat cake

have cake $\vdash$ have cake

have cake, have cake $\vdash$ eat cake $\wedge$ have cake

contraction $\Longrightarrow$ have cake $\vdash$ eat cake $\wedge$ have cake

exactly what is disallowed in linear logic.

$$\frac{B, B, \Gamma \vdash A}{B, \Gamma \vdash A} \text{ (contraction)} \qquad \frac{\Gamma \vdash A}{B, \Gamma \vdash A} \text{ (~~weaking~~ (weakening))}$$

Both are disallowed!

in Linear logics

Intuitionistic Linear logics:

Structural rule of exchange: we have a list of assumptions instead of a set

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ (\text{exchange})$$

Formulas

$$A, B, C ::= \overbrace{X \mid 1 \mid A \otimes B}^{\text{multiplicative}} \mid \overbrace{A \multimap B \mid A \,\&\, B \mid \top}^{\text{additive}} \mid \overset{\text{"of course"}}{!A}$$

X: atomic formula; 1: monoidal identity; $A \otimes B$: monoidal product; $A \multimap B$: linear implication; $A \,\&\, B$: "with"; $\top$: true

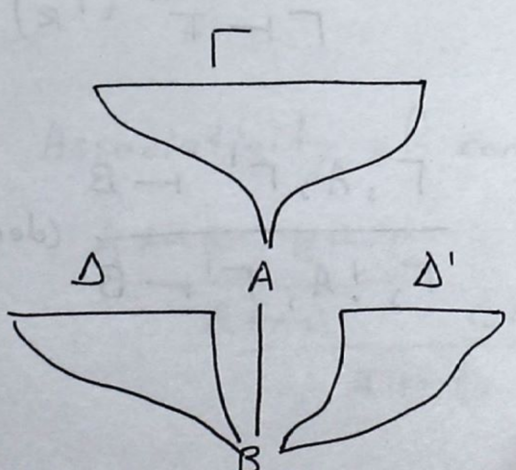
$A \,\&\, B$ is like cartesian product, with unit $\top$

Rules

$$\frac{}{A \vdash A}\ (\text{Axiom})$$

$$\frac{\Gamma \vdash A \qquad \Delta, A, \Delta' \vdash B}{\Delta, \Gamma, \Delta' \vdash B}\ (\text{cut})$$

String diagram for cut:



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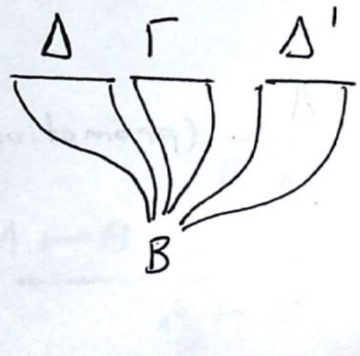


More rules: (for tensors)

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, A \otimes B, \Gamma' \vdash C}\ (\otimes_L) \qquad \frac{\Gamma, \Gamma' \vdash A}{\Gamma, 1, \Gamma' \vdash A}\ (1_L)$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_R) \qquad \frac{}{\underset{\substack{\uparrow \\ \text{empty list.}}}{\varepsilon} \vdash 1}\ (1_R)$$

$$\frac{\Gamma \vdash A \qquad \Delta, B, \Delta' \vdash C}{\Delta, \Gamma, A \multimap B, \Delta' \vdash C}\ (\multimap_L)$$

Note:

$(A \otimes -) \dashv (A \multimap -)$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_R)$$

$$\frac{\Gamma, A, \Gamma' \vdash C}{\Gamma, A \& B, \Gamma' \vdash C}\ (\&_{L,1}) \qquad \frac{\Gamma, B, \Gamma' \vdash C}{\Gamma, A \& B, \Gamma' \vdash C}\ (\&_{L,2})$$

$$\frac{T \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ (\&_R) \qquad \frac{}{\Gamma \vdash \top}\ (\top_R)$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (\text{promotion}) \qquad \frac{\Gamma, A, \Gamma' \vdash B}{\Gamma, !A, \Gamma' \vdash B}\ (\text{dereliction})$$

Yet more rules:

$$\frac{\Gamma, \Gamma' \vdash B}{\Gamma, !A, \Gamma' \vdash B} \qquad \frac{\Gamma, !A, !A, \Gamma' \vdash B}{\Gamma, !A, \Gamma' \vdash B}$$

Okay we're done with rules.

Theorem: Cut elimination.

If $\pi : \Gamma \vdash A$ with cuts, then $\exists\, \mathcal{E}(\pi) : \Gamma \vdash A$ without cuts.

What does this have to do with monoidal categories?

Get invariants for proofs using monoidal categories.

should be invariant under cut-elimination

$$\underbrace{[\pi]}_{\text{proof interpretation}} = [\mathcal{E}(\pi)]$$

will interpret proofs as arrows in monoidal categories, objects are statements

$$[\text{Axiom}] = id_{[A]} : [A] \to [A] \qquad \frac{}{A \vdash A}$$

$$[\pi] = [\pi_2] \circ [\pi_1] : [A] \to [C] \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

Associativity of composition

$$\frac{\dfrac{A \vdash B \quad B \vdash C}{A \vdash C} \quad C \vdash D}{A \vdash D} = \frac{A \vdash B \quad \dfrac{B \vdash C \quad C \vdash D}{B \vdash D}}{A \vdash D}.$$

$$\dfrac{\dfrac{A_1 \vdash A_2 \qquad B_1 \vdash B_2}{A_1, B_1 \vdash A_2 \otimes B_2}}{A_1 \otimes B_1 \vdash A_2 \otimes B_2} \qquad [\pi] = [\pi_1] \otimes [\pi_2]$$

in general, $\pi : A_1, \ldots, A_n \vdash B$

$\rightsquigarrow$ interpret

$$[\pi] : [A_1] \otimes [A_2] \otimes \cdots \otimes [A_n] \longrightarrow [B] .$$

<u>More fun stuff</u>: Coherence

- commutation criterion
  (simplest: all diagrams (of a certain kind) commute)

- strictification
  (every weak thing is equivalent to a strong thing)

  $\downarrow$

  monoidal cat.
  left/right closed cat
  braided/symmetric monoidal categories
  thing with duals

<u>Unbiased Definitions</u>

biased def of monoid : a set M, with unit u, operation $\cdot$ such that ...

unbiased def of monoid : elt of $\underline{Sets^T}$ where T is the free monoid monad

Categorify that:

biased: $\underline{\text{Mon Cat}}$ has elements $(\mathbb{C}, 1, \otimes)$ that follow some rules

unbiased: $\underline{\text{Cat}}^T$ where T is the list^2 monad on $\underline{\text{Cat}}$.

$$\otimes_n : \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{n} \longrightarrow \mathbb{C}$$

$$\otimes_n \big(\otimes_{k_1}(\cdots), \ \cdots, \ \otimes_{k_n}(\cdots) \big) \cong \otimes_{k_1 + \cdots + k_n} (\cdots)$$

<u>Recall</u>: A monoidal category is a category $\mathbb{C}$ with object 1 and bifunctor $\otimes$, isomorphisms

$$\alpha : (x \otimes y) \otimes z \xrightarrow{\sim} (x \otimes (y \otimes z))$$

$$\lambda : 1 \otimes x \xrightarrow{\sim} x$$

$$\rho : x \otimes 1 \longrightarrow x$$

Such that

$$\begin{tikzcd}
 & (x \otimes y) \otimes (z \otimes w) \arrow[dr] & \\
(x \otimes (y \otimes (z \otimes w))) \arrow[ur] \arrow[d] & & (((x \otimes y) \otimes z) \otimes w) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr] & & x \otimes ((y \otimes z) \otimes w) \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & & (x \otimes (1 \otimes y)) \arrow[dl, "\lambda"] \\
 & x \otimes y &
\end{tikzcd}$$

<u>Claim:</u>

$$\begin{tikzcd} 1 \otimes 1 \arrow[r, bend left, "\lambda_1"] \arrow[r, bend right, "\rho_1"'] & 1 \end{tikzcd} \qquad \lambda_1 \cong \rho_1$$

<u>lemma:</u>

$$\begin{tikzcd} (1 \otimes x) \otimes y \arrow[rr, "\alpha"] \arrow[dr, "\lambda_x \otimes id_y"'] & & 1 \otimes (x \otimes y) \arrow[dl, "\lambda_{x \otimes y}"] \\ & x \otimes y & \end{tikzcd}$$

commutes

<u>Proof:</u>

$$\begin{tikzcd} (1 \otimes 1) \otimes 1 \arrow[r, "\alpha", "\sim"'] \arrow[rr, bend right=60, "\lambda"'] & 1 \otimes (1 \otimes 1) \arrow[r, bend left, "1 \otimes \lambda"] \arrow[r, bend right, "1 \otimes \rho"'] \arrow[d, phantom, "\Delta\text{-law}"] & 1 \otimes 1 \\ & {} & \end{tikzcd}$$

$\blacksquare$

<u>Proving ~~MacClone~~ MacLane Coherence:</u>

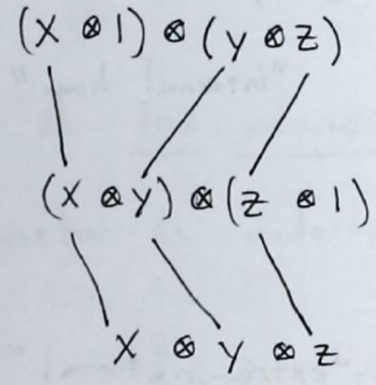
idea: $S, T ::= 1 \mid A \mid S \otimes T$, $n(T)$ = number of variables ~~with no repeat~~s

G : category w/ objects formulas with no repeated variables up to renaming

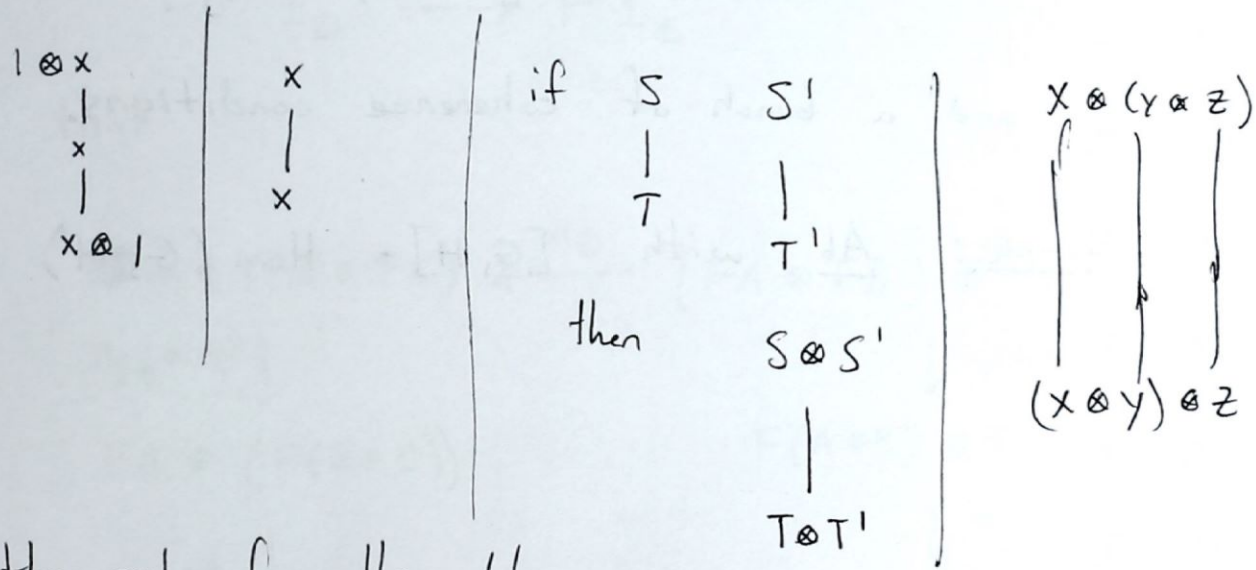
arrows $S \longrightarrow T$ iff $n(S) = n(T)$.

Graphical Representations of Arrows in $\mathcal{G}$

No lines may cross



<u>Def:</u> least set of arrows closed under

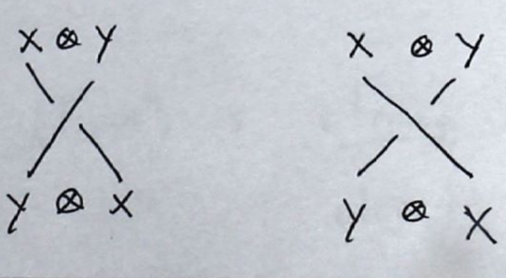


is the set of <u>allowable</u> arrows.

<u>Theorem</u>: If $S \xrightarrow{f} T$, $T \xrightarrow{g} R$ is allowable, then so is $S \xrightarrow{g \circ f} R$

<u>Theorem</u>: Every allowable arrow ~~to~~ corresponds to a unique (thing)?

Variations: allow arrows to be braids... add these guys.



Axiomaticize the notion of "internal hom"

Def: <u>closed</u> <u>category</u> is category $\mathbb{C}$ with

$$[-,-] : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{"internal hom"}$$

$$1 \in \mathbb{C}$$

$$id_{\mathbb{C}} \xrightarrow{\sim} [1, -]$$

$$j_X : 1 \longrightarrow [X, X] \qquad \text{"extra-natural"}$$

$$L : [X, Z] \longrightarrow \big[[X,Y], [Y,Z]\big]$$

and a bunch of coherence conditions

Example: $\underline{Ab}$ with $[G, H] =$ Hom (G, H) as an abelian group.

Monoidal Adjunctions and (co) monads (in any 2category)

Def: A lax monoidal functor $(\mathbb{C}, \otimes_{\mathbb{C}}, 1_{\mathbb{C}}) \longrightarrow (\mathbb{D}, \otimes_{\mathbb{D}}, 1_{\mathbb{D}})$ is (F, m), $F : \mathbb{C} \longrightarrow \mathbb{D}$ functor on underlying categories together with

$$m^2_{A,B} : FA \otimes_{\mathbb{D}} FB \longrightarrow F(A \otimes_{\mathbb{C}} B)$$

$$m^0 : 1_{\mathbb{D}} \longrightarrow F 1_{\mathbb{C}}$$

such that

$$\begin{tikzcd}
FA \otimes (FB \otimes FC) \arrow[d, "id_{FA} \otimes m^2"'] & (FA \otimes FB) \otimes FC \arrow[l, "\alpha_{\mathbb{D}}"'] \arrow[d, "m^2 \otimes id_{FC}"] \\
FA \otimes (F(B \otimes C)) \arrow[d, "m^2"'] & F(A \otimes B) \otimes FC \arrow[d, "m^2"] \\
F(A \otimes (B \otimes C)) & F((A \otimes B) \otimes C) \arrow[l, "F(\alpha_{\mathbb{C}})"']
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes 1_{\mathbb{D}} \arrow[r, "\rho_{\mathbb{D}}"] \arrow[d, "FA \otimes m^0"'] & FA \\
FA \otimes F1_{\mathbb{C}} \arrow[r, "m^2"] & F(A \otimes 1_{\mathbb{C}}) \arrow[u, "F(\rho_{\mathbb{C}})"']
\end{tikzcd}$$

Def: (F, m) is strong if m^2, m^0 are isos
strict if m^2, m^0 are identities.

Prop: A lax monoidal functor (F, m) sends monoids in $\mathbb{C}$ to monoids in $\mathbb{D}$.

Proof:

$$(A, \mu_A, i_A) \text{ in } \mathbb{C} \rightsquigarrow (FA, F(\mu_A), F(i_A)) \text{ in } \mathbb{D}$$

$$A \otimes A \xrightarrow{\mu_A} A \quad \rightsquigarrow \quad FA \otimes FA \xrightarrow{m^2} F(A \otimes A) \xrightarrow{F(\mu_A)} FA$$

$$1_{\mathbb{C}} \xrightarrow{i_A} A \quad \rightsquigarrow \quad 1_{\mathbb{D}} \xrightarrow{m^0} F1_{\mathbb{C}} \xrightarrow{F(i_A)} FA$$

$\blacksquare$

~~Mon(F, m)~~

Defines a functor between category of monoids in $\mathbb{C}$, $\mathrm{Mon}(\mathbb{C})$, to category of monoids in $\mathbb{D}$

$$\mathrm{Mon}(F, m) : \mathrm{Mon}(\mathbb{C}) \longrightarrow \mathrm{Mon}(\mathbb{D})$$

Example: $\mathrm{Mon}(\mathbb{C}) \cong \mathrm{LaxMonCat}(\mathbb{1}, \mathbb{C})$.

Def: **LaxMonCat** is a large 2-category with

objects monoidal categories

1-arrows lax monoidal functors

2-arrows monoidal natural transformations.

Given monoidal categories $\mathbb{C}, \mathbb{D}$, form $\mathbb{B}\mathbb{D}, \mathbb{B}\mathbb{C}$ 2-categories with one object, arrows objects of $\mathbb{C}/\mathbb{D}$, 2-arrows the arrows of $\mathbb{C}$ or $\mathbb{D}$

$F : \mathbb{C} \to \mathbb{D}$ lax-monoidal $\longleftrightarrow$ $\mathbb{B}F : \mathbb{B}\mathbb{C} \to \mathbb{B}\mathbb{D}$ lax 2-functor

Def: An <u>oplax</u> <u>monoidal</u> <u>functor</u>

$$(\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \cdot, u)$$

has natural transformations

$$\eta^2_{A,B} : F(A \otimes B) \to FA \cdot FB$$

$$\eta^0 : Fe \longrightarrow u$$

that obey the diagrams for lax monoidal functors.

<u>Examples</u>: $\mathbb{C}, \mathbb{D}$ cartesian monoidal categories

Every $F : \mathbb{C} \to \mathbb{D}$ is oplax monoidal in a unique way

Strong lax = strong oplax

Def: A <u>monoidal</u> <u>natural</u> <u>transformation</u> $(F, m) \to (G, n)$ is a natural transformation $\theta : F \to G$ such that

$$\begin{tikzcd}
FA \cdot FB \arrow[r, "\theta_A \cdot \theta_B"] \arrow[d, "m^2"'] & GA \cdot GB \arrow[d, "n^2"] \\
F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& u \arrow[dl, "m^0"'] \arrow[dr, "n^0"] & \\
Fe \arrow[rr, "\theta e"'] & & Ge
\end{tikzcd}$$

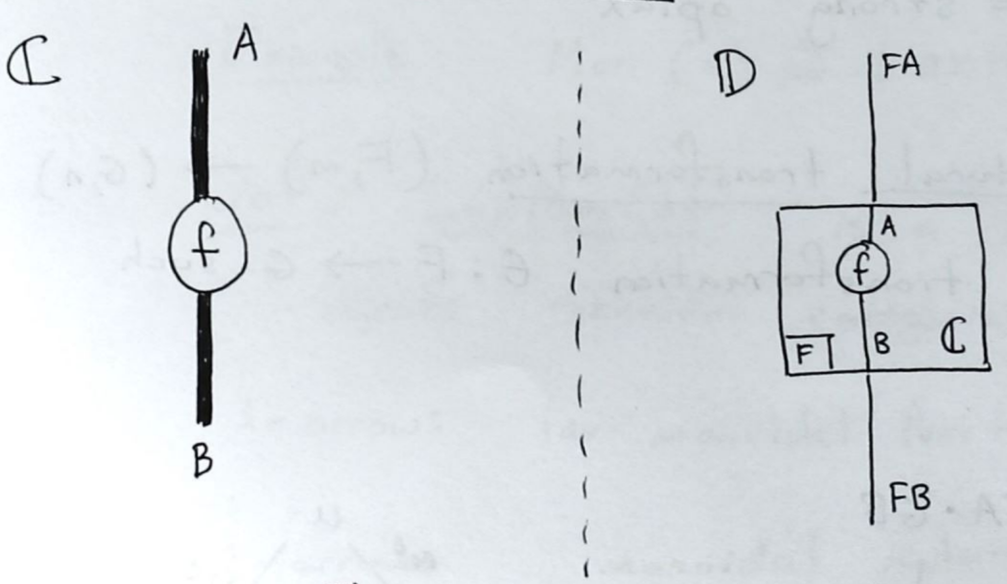
Def: A <u>symmetric</u> <u>lax</u> <u>monoidal</u> functor $(\mathbb{C}, \otimes, e)$ to $(\mathbb{D}, \bullet, u)$ (symmetric monoidal cats.) ~~functor~~

is a lax monoidal functor (F, m) such that

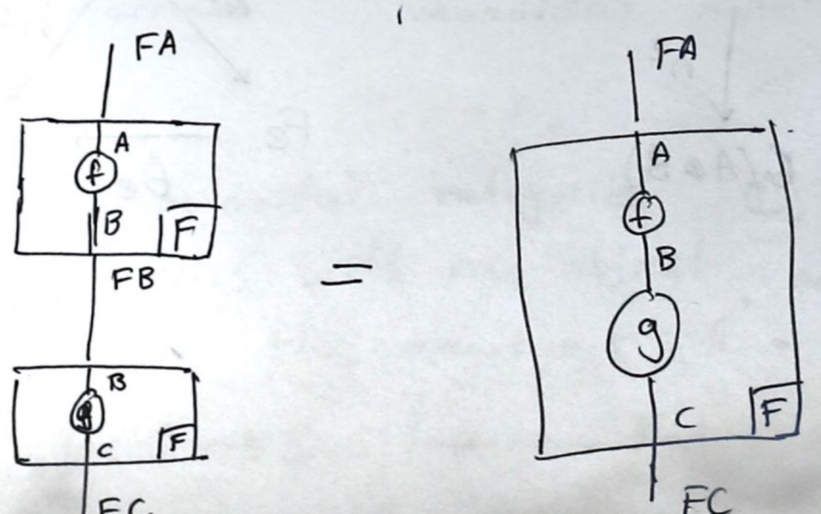
$$\begin{tikzcd}
FA \bullet FB \arrow[r] \arrow[d, "m^2_{A,B}"'] & FB \bullet FA \arrow[d, "m^2_{B,A}"] \\
FA \otimes FB \arrow[r] & F(B \otimes A)
\end{tikzcd}$$

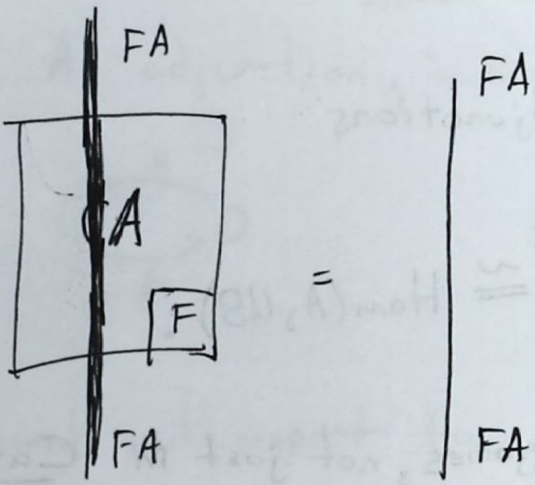
<u>Example</u>: Any $F : \mathbb{C} \to \mathbb{D}$ of cartesian monoidal categories is symmetric oplax.

<u>String Diagrams!</u> How to draw functors



← box is a "window" into the $\mathbb{C}$-world, specifically the one that comes from the functor F

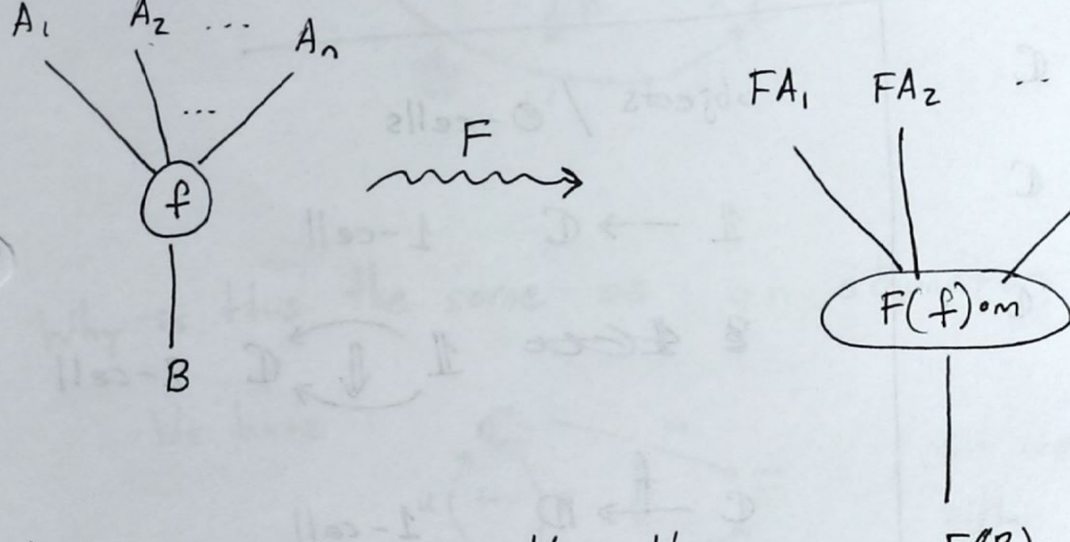




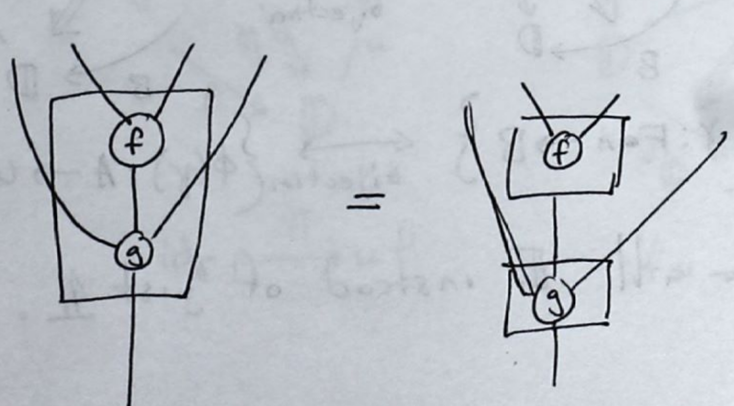
If F is lax monoidal,

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n \xrightarrow{f} B$$

$$F \left(\quad FA_1 \bullet FA_2 \bullet \cdots \bullet FA_n \xrightarrow{Ff \circ m} FB \right.$$



Lax monoidal is exactly the condition that we're allowed to split up $F(A \otimes B)$ into $FA \otimes FB$.



Monoidal Categories and Adjunctions

$F \dashv U$, $\mathbb{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{D}$, $\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(A, UB)$.

Want adjunctions in arbitrary 2-categories, not just in <u>Cat</u>

Should specialize to a normal adjunction in the 2-category <u>Cat</u>.

Concept in <u>Cat</u>	2-category version
category $\mathbb{C}$	objects / 0-cells
object of $\mathbb{C}$	$\mathbb{1} \longrightarrow \mathbb{C}$ 1-cell
arrow of $\mathbb{C}$	~~$\mathbb{1}$ Cat~~ $\begin{tikzcd} \mathbb{1} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbb{C} \end{tikzcd}$ 2-cell
functor	$\mathbb{C} \xrightarrow{f} \mathbb{D}$ 1-cell
$\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(A, UB)$	see below

$$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A"] \arrow[dr, bend right, "B"'] & \mathbb{C} \arrow[d, "F"] \arrow[dl, phantom, "\gamma \Downarrow"] \\ & \mathbb{D} \end{tikzcd} \underset{\text{bijection}}{\overset{\phi}{\longleftrightarrow}} \begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A"] \arrow[dr, bend right, "B"'] & \mathbb{C} \arrow[dl, phantom, "\Downarrow \phi(\gamma)"] \\ & \mathbb{D} \arrow[u, "U"'] \end{tikzcd}$$

$$\{\gamma : F \circ A \to B\} \underset{\text{bijection}}{\longleftrightarrow} \{\phi(\gamma) : A \to U \circ B\}$$

In fact, the last holds for all $\mathbb{E}$ instead of just $\mathbb{1}$.

Def: An adjunction between fixed objects $C, D \in \mathcal{C}$ in a 2-category $\mathcal{C}$ is triple (f, u, ϕ)

$$\begin{tikzcd} C \arrow[r, bend right, "f"'] & D \arrow[l, bend right, "u"'] \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dd, bend left, "f"] \\ E \arrow[ur, bend left, "a"] \arrow[dr, bend right, "b"'] \arrow[r, phantom, "\Downarrow\gamma"] & \\ & D \end{tikzcd} \xrightarrow{\phi_{E,a,b}} \begin{tikzcd} & C \\ E \arrow[ur, "a"] \arrow[dr, "b"'] \arrow[r, phantom, "\Downarrow\phi(\gamma)"] & \\ & D \arrow[uu, "u"'] \end{tikzcd}$$

Natural with respect to

$$\begin{tikzcd} & & C \\ F \arrow[urr, bend left, "a'"] \arrow[r, "h"] \arrow[drr, bend right, "b'"'] & E \arrow[ur, "a"] \arrow[dr, "b"] \arrow[u, phantom, "\Downarrow\alpha"] \arrow[d, phantom, "\Downarrow\beta"] & \\ & & D \end{tikzcd}$$

Why is this the same as an adjunction in $\underline{\mathrm{Cat}}$?

We have $\begin{tikzcd} \mathbb{C} \arrow[dd, bend left, "F"] \arrow[drr, "a"] & & \\ \arrow[r, phantom, "\dashv"] & \Downarrow & \mathbb{E} \\ \mathbb{D} \arrow[uu, bend left, "u"] \arrow[urr, "b"'] & & \end{tikzcd}$, but replacing $\mathbb{E}$ with either $\mathbb{C}$ or $\mathbb{D}$ recovers the unit/counit

$$\begin{tikzcd} \mathbb{C} \arrow[r, "id_{\mathbb{C}}"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=4, "\eta\Downarrow"] & \mathbb{C} \\ & \mathbb{D} \arrow[u, "u"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C} \arrow[d, "f"] \\ \mathbb{D} \arrow[ur, "u"] \arrow[r, "id_{\mathbb{D}}"'] \arrow[r, phantom, shift left=4, "\Downarrow\varepsilon"] & \mathbb{D} \end{tikzcd}$$

$id_{\mathbb{C}} \xrightarrow{\eta} uf \qquad fu \xrightarrow{\varepsilon} id_{\mathbb{D}}$

Prop: Equal axiomatizations of adjunctions in a 2-cat

(1)
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[rr, bend left=60, "\mathrm{id}_C"] & D \arrow[r, "u"] \arrow[rr, bend right=60, "\mathrm{id}_D"'] \arrow[r, phantom, bend left=30, "\Downarrow\eta"] & C \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & D \\ & & {} & \end{tikzcd} = \begin{tikzcd} C \arrow[r, "f"] \arrow[rr, bend left=50, "\mathrm{id}_C"] & D \arrow[r, phantom, "\Downarrow \mathrm{id}_f"] \arrow[rr, bend right=50, "\mathrm{id}_D"'] & C \arrow[r, "f"] & D \end{tikzcd}$$

(2)
$$\begin{tikzcd} C \arrow[d, bend right, "u"'] \arrow[dr, "a"] & \\ D \arrow[u, bend right, "F"'] \arrow[u, phantom, "\vdash"] \arrow[r, "b"'] & E \end{tikzcd}$$

Prop:
$$\begin{tikzcd} E_1 \arrow[r, "a_1"] \arrow[dr, "b_1"'] \arrow[r, phantom, shift right=4, "\gamma\Downarrow"] & C \arrow[d, "f"] \arrow[r, "a_2"] \arrow[r, phantom, shift right=4, "\psi(\delta)\Downarrow"] & E_2 \\ & D \arrow[ur, "b_2"'] & \end{tikzcd} = \begin{tikzcd} E_1 \arrow[r, "a_1"] \arrow[dr, "b_1"'] \arrow[r, phantom, shift right=4, "\phi(\gamma)\Downarrow"] & C \arrow[r, "a_2"] \arrow[r, phantom, shift right=4, "\Downarrow\delta"] & E_2 \\ & D \arrow[u, "u"'] \arrow[ur, "b_2"'] & \end{tikzcd}$$
.

Given an adjunction $\mathbb{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbb{D}$ ($\top$) between monoidal categories.

<u>Prop</u>: Every lax monoidal structure on U, p, induces oplax monoidal structure on F

$$\begin{tikzcd}
F(A \otimes B) \arrow[rr, dashed, "m^2"] \arrow[d, "F(\eta_A \otimes \eta_B)"'] & & FA \bullet FB \\
F(UF(A) \otimes UF(B)) \arrow[rr, "F(p^2)"] & & FU(FA \bullet FB) \arrow[u, "\varepsilon FA \bullet FB"']
\end{tikzcd}$$

$$Fe \xrightarrow{F(p)} FU(u) \xrightarrow{\varepsilon} u \ .$$

We also have the converse, giving a bijection between lax monoidal structures and oplax monoidal structures.

(Try drawing the string diagram, it's fun!)

<u>Facts</u>: $\mathbb{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$ ($\perp$) $\Longrightarrow$ F strong (in lax. mon. cats)

$\mathbb{C}$ symmetric monoidal $\Longrightarrow$ Mon ($\mathbb{C}$) & Comon($\mathbb{C}$) symmetric monoidal

04/29/15

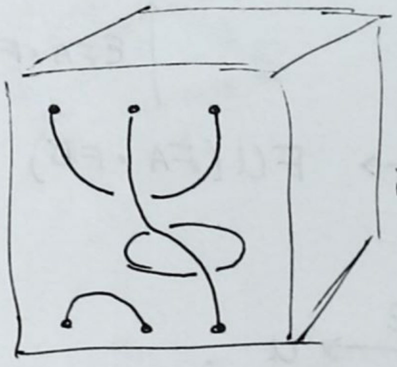
k-tuply monoidal n-category stabilization

for $n=1$, consider the category of tangles

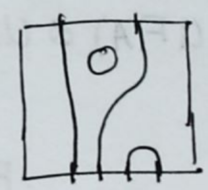
$\underline{Tang_k}$ with objects finite sets of points in $[0,1]^k$

arrows are finite sets of arcs and loops embedded in $(k+1)$ cube and terminate on top/bottom face, if two objects have same parity in number of pts

Example:



in $\underline{Tang_2}$



in $\underline{Tang_1}$

k	$\underline{Tang_k}$
0	pointed category
1	monoidal cat
2	braided monoidal
3	symmetric monoidal category
⋮	⋮

(Co)monads in a 2-category

<u>Mon</u> in the standard sense $\equiv$ monoid in $\mathrm{End}(\mathbb{C})$

$$\mathrm{End}(\mathbb{C}) = \left(\mathrm{Hom}_{\underline{Cat}}(\mathbb{C},\mathbb{C}), \circ, 1\right)$$

if $\mathcal{C}$ is any 2-category,

C an object, $\mathcal{C}(C,C)$ is the category of 1-cells

objects: 1-cells $\quad C \xrightarrow{f} C$

arrows: 2-cells
$$\begin{tikzcd} C \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & C \arrow[from=U, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

This is a strict monoidal category

Monoids in $\mathcal{C}(C,C)$ are monads.

<u>Concretely</u>: a comonad on C in $\mathcal{C}$ is

$C \xrightarrow{k} C$ with 2-cells

$$\begin{tikzcd} & C \arrow[dr, "k"] \arrow[d, phantom, "\Uparrow\delta"] & \\ C \arrow[ur, "k"] \arrow[rr, "k"] \arrow[rr, bend right=60, "id_C"'] & {} \arrow[d, phantom, "\Downarrow\varepsilon"] & C \\ & {} & \end{tikzcd}$$

such that

$$\begin{tikzcd} C \arrow[r, "k"] & C \arrow[d, "k"] \\ C \arrow[u, "k"] \arrow[ur, "k"] \arrow[r, "k"'] \arrow[u, phantom, shift left=4, "\Uparrow\delta"] & C \arrow[u, phantom, shift right=4, "\Uparrow\delta"] \end{tikzcd} \quad \equiv \quad \begin{tikzcd} C \arrow[r, "k"] \arrow[dr, "k"] & C \arrow[d, "k"] \arrow[d, phantom, shift right=4, "\Uparrow\delta"] \\ C \arrow[u, "k"] \arrow[r, "k"'] \arrow[u, phantom, shift left=4, "\Uparrow\delta"] & C \end{tikzcd}$$

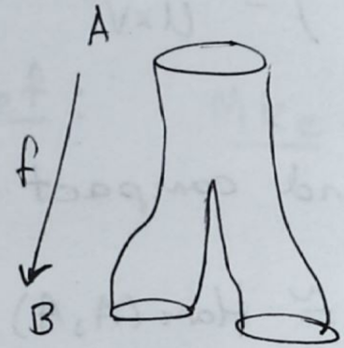
and

$$
\begin{tikzcd}
C \arrow[r, "k"'] \arrow[rr, bend left=50, "k"] \arrow[r, bend right=60, "\mathrm{id}_C"'] \arrow[r, phantom, shift right=4, "\Downarrow\varepsilon"] & C \arrow[r, "k"'] \arrow[u, phantom, shift left=3, "\Downarrow\delta"] & C
\end{tikzcd}
\equiv
\begin{tikzcd}
C \arrow[r, bend left=40, "k"] \arrow[r, bend right=40, "k"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_k"] & C
\end{tikzcd}
\equiv
\begin{tikzcd}
C \arrow[r, "k"] \arrow[rr, bend left=50, "k"] & C \arrow[r, "k"] \arrow[r, bend right=60, "\mathrm{id}_C"'] \arrow[r, phantom, shift right=4, "\Downarrow\varepsilon"] \arrow[u, phantom, shift right=3, "\Downarrow\delta"] & C
\end{tikzcd}
$$

Exercise: Every adjunction $\begin{tikzcd} C \arrow[r, bend right, "f"'] \arrow[r, phantom, "\top"] & D \arrow[l, bend right, "u"'] \end{tikzcd}$ in $\mathcal{C}$ gives rise to a monad on $C \circlearrowleft uf$ and a comonad on $D \circlearrowleft fu$.

Representation Theory & TQFTs

- Talk about category of spans.



Can think of a cobordism as an arrow in some category

$$\begin{tikzcd} & \mathcal{F}(T) \arrow[dl] \arrow[dr] & \\ \mathcal{F}(A) & & \mathcal{F}(B) \end{tikzcd}$$

← a span

given a field on the pants T, restrict to field on A or field on B.

Let $\mathcal{E}$ be a "nice" category

- $\underline{Span}(\mathcal{E})$ is category with objects same as in $\mathcal{E}$ and arrows are (isomorphism classes) of spans.

An arrow between U, V are

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ U & & V \end{tikzcd}$$

equivalent when

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd, "\wr"] & \\ U & & V \\ & S' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes

<u>identity</u>

$$\begin{tikzcd} & U \arrow[dl, "id_U"'] \arrow[dr, "id_U"] & \\ U & & U \end{tikzcd}$$

- <u>composition</u>

$$\begin{tikzcd} & & S \times_V T \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ U & & V & & W \end{tikzcd}$$

pullback

$\underline{\mathrm{Span}}(\mathcal{E})$ is monoidal category with $\times$

$$\left(\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ U & & U' \end{tikzcd} \right) \times \left(\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ V & & V' \end{tikzcd} \right) = \begin{tikzcd} & S\times T \arrow[dl] \arrow[dr] & \\ U\times V & & U'\times V' \end{tikzcd}$$

Moreover, $\underline{\mathrm{Span}}(\mathcal{E})$ is symmetric and compact closed.

- Has duals $\quad \mathrm{Hom}(A \otimes A^*, 1) \cong \mathrm{Hom}(A, A)$
- $\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, V \otimes W)$

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ U\times V & & W \end{tikzcd} \Longleftrightarrow \begin{tikzcd} & S \arrow[dl] \arrow[d] \arrow[dr] & \\ U & V & W \end{tikzcd} \Longleftrightarrow \begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ U & & V\times W \end{tikzcd}.$$

- The initial object of $\mathcal{E}$ is a zero object in $\underline{\mathrm{Span}}(\mathcal{E})$

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr, "!"] & \\ 0 & & U \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "!"'] \arrow[dr, equal] & \\ U & & 0 \end{tikzcd}$$

- addition of spans

$$\begin{tikzcd} & S \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ U & & V \end{tikzcd} + \begin{tikzcd} & T \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & \\ U & & V \end{tikzcd} = \begin{tikzcd} & S+T \arrow[dl, "s_1+t_1"'] \arrow[dr, "s_2+t_2"] & \\ U+U \arrow[d, "\nabla"] & & V+V \arrow[d, "\nabla"] \\ U & & V \end{tikzcd}$$

- $\underline{\mathrm{Span}}(\mathcal{E})$ is enriched over commutative monoids

- Two ways to embed $\mathcal{E}$ in $\underline{\mathrm{Span}}(\mathcal{E})$

$$U \xrightarrow{f} V \qquad \begin{tikzcd} & U \arrow[dl, "1"'] \arrow[dr, "f"] & \\ U & & V \end{tikzcd} \qquad \begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "1"] & \\ V & & U \end{tikzcd}$$

If G is a finite set, fix k a field

$\mathcal{E}$ = topos of finite G-sets, $\underline{\mathrm{Sets}}^G_{fin}$

<u>Def</u>: $\underline{\mathrm{Mk}}e(G)$ is the category of Maschke functors,

$$\equiv \left[\underline{\mathrm{Span}}(\mathcal{E}), \underline{\mathrm{Vect}}_k\right]$$

coproduct preserving functors

$$\underline{\mathrm{Span}}(\mathcal{E}) \longrightarrow \underline{\mathrm{Vect}}_k .$$

Useful in rep theory.

<u>Def</u>: Green functor $\equiv$ monoid in $\underline{\mathrm{Mk}}e(G)$

Let $\mathcal{R} = \underline{\mathrm{Rep}}_k(G)$:

$$k : \mathcal{E} \longrightarrow \mathcal{R}, \quad kX = k^X$$

$$\mathcal{T} = \underline{\mathrm{Span}}(\mathcal{E})$$

$$k_* : \mathcal{T}^{op} \longrightarrow \mathcal{R}$$

$$\left(X \xleftarrow{u} S \xrightarrow{v} Y \right) \longmapsto \begin{array}{c} kY \longrightarrow kX \\ y \longmapsto \displaystyle\sum_{s : v(s) = y} u(s) \end{array}$$

Strong monoidal functor

Kan Extension gives a map

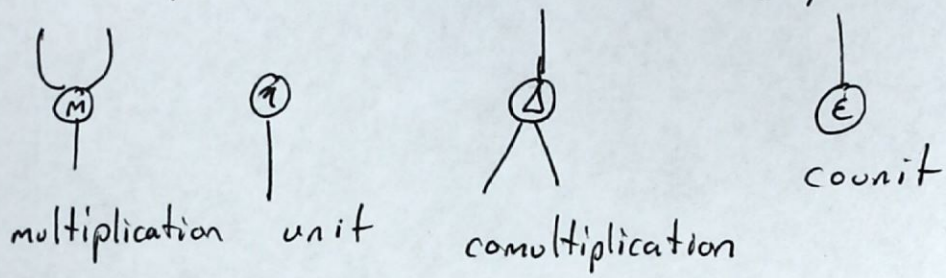
$$\widetilde{k_*} : \mathcal{R} \longrightarrow \underline{\mathrm{Mke}}(G)_{\text{fin. dim.}}$$

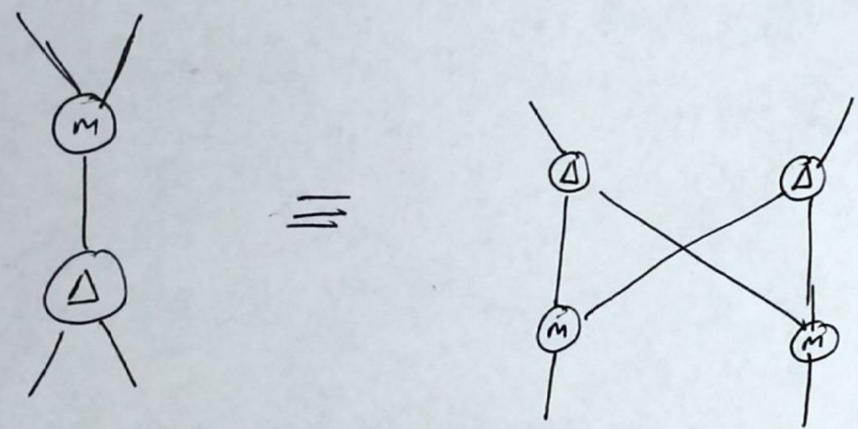
fully faithful

Has a left adjoint $\operatorname{colim}(-, k_*)$ which is part of a monoidal adjunction.

Hopf Algebras in Symmetric monoidal category

bialgebra





antipode