Complex Analysis

Let $\mathcal{U} \subseteq \mathbb{C}$ open, $f : \mathcal{U} \to \mathbb{C}$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad z \in \mathcal{U}$$

Examples:

Holomorphic Functions $\mathbb{C} \to \mathbb{C}$

$f(z) = z^k \quad f'(z) = k z^k$ (for $k > 0$)

f, g holomorphic, then $f+g$, fg holomorphic as well.

$p \in \mathbb{C}[x]$ are holomorphic

Let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.

$$\limsup r_n = \lim_{n \to \infty} \left(\sup_{m \geq n} r_m \right)$$

Assume $r_n \geq 0$ for all n. If (r_n) is bounded, $\left(\sup_{m \geq n} r_m \right)$ is decreasing, and bounded below $\implies$ converges, so

$\limsup r_n \in [0, \infty)$

Fact: Let $\limsup r_n = r$. If $\bar{r} > r$, then $r_n < \bar{r}$ for all large n.

If $\bar{r} < r$, $\bar{r} < r_n$ for infinitely many n.

Weierstrass M-Test: If $\sum_{n=0}^{\infty} a_n$ is a convergent series of non-negative reals, and $|z_n| \leq M|a_n|$ for (z_n), $z_n \in \mathbb{C}$, then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent

Consider complex power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n, z_n \in \mathbb{C}$. Like real power series are convergent on interval $[-R, R]$, these are convergent in $B(0,R) \subseteq \mathbb{C}$.

$$1/R = \limsup |a_n|^{1/n}$$

<u>Claim 1</u>: If $0 < \rho < R$, then $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $B(0,\rho) = \{z : |z| < \rho\}$.

<u>Proof:</u> Find ρ' such that $\rho < \rho' < R$, so $\frac{1}{\rho'} > \frac{1}{R} = \limsup |a_n|^{1/n}$
Then by previous fact $\frac{1}{\rho'} > |a_n|^{1/n}$ for all large n.
For all $z \in B(0,\rho)$, $|a_n z^n| \leq |a_n||z|^n \leq \left(\frac{\rho}{\rho'}\right)^n$
And $\rho/\rho' < 1$, so we get uniform convergence for $\sum_{n=0}^{\infty} a_n z^n$ on $B(0,\rho)$.

<u>Claim 2</u>: If $|z| > R$, $|a_n z^n| \not\to 0$, so $\sum a_n z^n$ diverges.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in B(0,R)$. Introduce another complex power series $f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Radius of convergence for f_1 is
$$1/R_1 = \lim_{n\to\infty} |n a_n|^{1/n} \quad \text{and } n^{1/n} \to 1 \text{ as } n \to \infty$$
so $R_1 = R$.

Theorem: Let f be Holomorphic on $B(0,R)$ and $f' = f_1$, where f, f_1 are as before.

Proof: Let $f(z) = S_n(z) + R_n(z)$ where $S_n(z) = \sum_{i<n} a_i z^i$

$R_n(z) = \sum_{i \geq n} a_i z^i$

Note that $S_n \in \mathbb{C}[x]$, so S_n' exists and is a polynomial. Let $z, z_0 \in B(0,\rho)$, $\rho < R$. $z \neq z_0$

$$\lim_{n\to\infty} \left(\frac{f(z) - f(z_0)}{z - z_0} - f_1(z) \right)$$

$$= \underbrace{\left(\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z) \right)}_{\text{small b/c } S_n \text{ is differentiable}} + \underbrace{\left(S_n'(z) - f_1(z) \right)}_{\text{small b/c } f_1 \text{ is convergent}} + \underbrace{\left(\frac{R_n(z) - R_n(z_0)}{z - z_0} \right)}_{\text{Takes some work}}$$

Some work: Let $k \geq n$. $\left| \frac{z^k - z_0^k}{z - z_0} \right| = |z^{k-1} + z^{k-2} z_0 + \cdots + z_0^{k-1}|$
$\leq k\rho^{k-1}$

Compare $\frac{R_n(z) - R_n(z_0)}{z - z_0}$ to $\sum_{k \geq n} k a_n \rho^{k-1}$

Since $\rho < R$, then $f_1(\rho)$ has tail $\sum_{k \geq n} k a_n \rho^{k-1}$, which becomes arbitrarily small as $n \to \infty$.

Hence $f'(z) = f_1(z)$. ∎

$\Rightarrow$ Complex Power Series are infinitely differentiable.
(Analytic $\Rightarrow$ Holomorphic)

Later: If f is holomorphic on U, $a \in U$, then
$f = \sum a_n (z-a)^n$ in a nbhd of U for some choice of a_n.
(Holomorphic $\Rightarrow$ Analytic)

The Exponential Function:

$\exp : \mathbb{C} \to \mathbb{C}$

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Radius of convergence:

$$\frac{1}{R} = \limsup \left|1/n!\right|^{1/n} = 0$$
$$\Rightarrow R = \infty$$

Defn: A function f which is holomorphic on $\mathbb{C}$ (everywhere) is called an entire function.

E.g. exp

$\exp'(z) = \exp(z)$
$\exp(0) = 1$

Show that $\exp(a)\exp(b) = \exp(a+b)$:

$$\frac{d}{dz}\left(\exp(z)\exp(c-z)\right) = \exp(z)\exp(c-z) - \exp(z)\exp(c-z)$$
$$= 0$$

So $\exp(z)\exp(c-z)$ is constant, and in particular

$\exp(z)\exp(c-z) = \exp(0)\exp(c-0) = \exp(c)$. ∎

Also $\exp(x+iy) = \exp(x)\exp(iy)$
$= e^x \operatorname{cis}(y)$

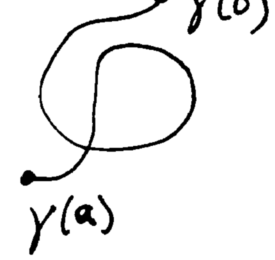
exp is periodic: $\exp(2\pi i) = \mathrm{cis}(2\pi) = 1$

$$\exp(z + 2\pi i) = \exp(z)$$

Can similarly define sin and cos via power series.

Complex Integration:

Line integrals over $\gamma : [a,b] \longrightarrow \mathbb{C}$, $[a,b] \subseteq \mathbb{R}$.



Fact: γ is continuous iff $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are cts. Typically, γ will be C^1. γ will be called differentiable if each of $\mathrm{Re}(\gamma)$, $\mathrm{Im}(\gamma)$ are differentiable.

Defn: γ is piecewise C^1 if $a = a_0 < a_1 < \ldots < a_n = b$ and γ is C^1 in (a_i, a_{i+1}) and derivative exists at a_i ~~foon~~ for each i from both left and right. Only from left for a_n, only from right for a_0.

Recall: (From 3D Calc)

Let $\Omega \subseteq \mathbb{R}^2$ be an open disk. Suppose $\gamma : [a,b] \to \Omega$ is continuous, and $p, q : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Then

$$\int_\gamma p\,dx + q\,dy = \int_a^b \left(p(\gamma_1(t), \gamma_2(t)) \frac{\partial \gamma_1}{\partial t} + q(\gamma_1(t), \gamma_2(t)) \frac{\partial \gamma_2}{\partial t} \right) dt$$

$$= \int_a^b \begin{pmatrix} p(\gamma(t)) \\ q(\gamma(t)) \end{pmatrix} \cdot \nabla \gamma \, dt$$

<u>Question</u>: When does $\int_\gamma p\,dx + q\,dy$ depend only on the endpoints a and b?

<u>Answer</u>: Exactly when there is $u : \Omega \to \mathbb{R}$ such that

$u_x = p$ and $u_y = q$

($p\,dx + q\,dy$ is an exact differential form)

If u exists, then along any curve from a to b

$$\int_\gamma p\,dx + q\,dy = u(b) - u(a)$$

<u>Complex Integration</u>:

$\gamma : [a,b] \to \Omega$ continuously differentiable, $\Omega \subseteq \mathbb{C}$ open

$f : \Omega \to \mathbb{C}$ continuous

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

<u>Change of Variable</u>:

$\phi : [a', b'] \to [a,b]$ ϕ strictly increasing, C^1.

$\phi(a') = a$, $\phi(b') = b$

$\delta = \gamma \circ \phi : [a', b'] \to [a,b]$

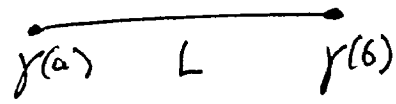
$$\int_\delta f(z)\,dz = \int_\gamma f(z)\,dz$$

<u>Examples</u>:

If $\delta(t) = \gamma(b + a - t)$

$$\int_\delta f(z)\,dz = -\int_\gamma f(z)\,dz$$

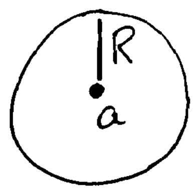
If γ parameterizes a line of length L



$$\left| \int_\gamma f(z)\,dz \right| \leq L \max_{z \in \text{line}} |f(z)|$$

Let $R \in \mathbb{R}$, $R > 0$, $a \in \mathbb{C}$, $\gamma : [0, 2\pi] \to \mathbb{C}$

$$\gamma(\theta) = a + R\,\mathrm{cis}(\theta)$$



$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{Ri\exp(i\theta)}{R\,\mathrm{cis}\,\theta}\, d\theta = 2\pi i$$

<u>Complex analysis version of Fundamental Theorem of Calculus:</u>

$\Omega \subseteq \mathbb{C}$ open, $\gamma : [a,b] \to \Omega$ piecewise C^1

Say $f : \Omega \to \mathbb{C}$ is continuous, and further that there is $F : \Omega \to \mathbb{C}$ holomorphic such that $F' = f$, then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

Note that the value only depends on the endpoints of γ.

<u>Proof:</u> Let $F(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$

$$F'(x+iy) = f(x+iy) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$

$$\int_a^b f(\gamma(t))\,dt = \int_a^b u_x(\gamma_1(t), \gamma_2(t)) + iv_x(\gamma_1(t), \gamma_2(t)) \frac{d\gamma}{\partial t}\, dt$$

$$= \int_a^b v_y(\gamma_1(t), \gamma_2(t)) - iu_y(\gamma_1(t), \gamma_2(t)) \frac{d\gamma}{\partial t}\, dt$$

where $\dfrac{d\gamma}{dt} = \gamma_1'(t) + i\gamma_2'(t)$.

Proof continued: work from other side

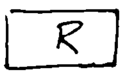
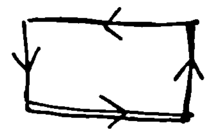
$$F(\gamma(t)) = u(\gamma_1(t), \gamma_2(t)) + iv(\gamma_1(t), \gamma_2(t))$$

$$\frac{d}{dt} F(\gamma(t)) = \Big(u_x(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t) + u_y(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t) \Big) + i\Big(v_x(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t) + v_y(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t) \Big)$$

Pair up terms to see that

$$\frac{d}{dt} F(\gamma(t)) = \text{integrand}. \qquad \blacksquare$$

Defn: Let $X \subseteq \mathbb{C}$ be arbitrary, $f : X \to \mathbb{C}$. f is holomorphic if there is open $\mathcal{U} \supseteq X$ open such that there is $g : \mathcal{U} \to \mathbb{C}$ holomorphic and $g \restriction X = f$.

Theorem (Cauchy 1): Let R be a closed rectangle in $\mathbb{C}$.  Let γ define the boundary of R in a natural way, counterclockwise. 

Let f be a holomorphic function $f : R \to \mathbb{C}$.

Then $$\int_\gamma f(z)\,dz = 0.$$

Converse to the Cauchy-Riemann Equations:

If u_x, u_y, v_x, v_y are ontinuous on $\mathcal{U}$ and satisfy the Cauchy-Riemann Equations, then f is holomorphic and $f' = u_x + iv_x = v_y - iu_y$.

~~Holomorphic~~

Proof: Let $f(x+iy) = u(x,y) + iv(x,y)$.

Let $(a,b) \in U$. In a neighborhood around (a,b), for r,s small

Let $D = u(a+r, b+s) - u(a,b) + iv(a+r, b+s) - iv(a,b)$

$= \nabla u(a,b) \cdot \binom{r}{s} + i \nabla v(a,b) \cdot \binom{r}{s}$ + "error term" (bounded by $\varepsilon(r^2+s^2)$).

Consider $\dfrac{D}{r+is} = (r-is)\dfrac{\left(\nabla u(a,b)\cdot\binom{r}{s} + i\nabla v(a,b)\cdot\binom{r}{s} + \text{error}\right)}{r^2+s^2}$

Using the fact that u and v satisfy Cauchy-Riemann equations, simplify to get that as $r+is \longrightarrow 0$

difference quotient $\longrightarrow u_x(a,b) + iv_x(a,b)$

$= v_y(a,b) + -u_y(a,b)$.

QED?

Theorem: (Cauchy #1)

R rectangle, f holomorphic on R (i.e. some open set $U \supseteq R$).

$\int_\gamma f(z)\,dz = 0$ where γ parameterizes boundary of R.

Proof: Define a sequence of Rectangles R_n where $R_0 = R$ and R_{n+1} will be a "quadrant" of R_n such that the integral around the boundary of R_{n+1} is the largest of the integrals around the quadrants:

$$\left|\oint_{\partial R_{n+1}} f(z)\,dz\right| \geq 1/4\left|\oint_{\partial R_n} f(z)\,dz\right|$$

Let $\{z^*\}$ be the intersection of the R_n's: $\{z^*\} = \bigcap_{n\in\mathbb{N}} R_n$.

Since f is differentiable at z^*, for any $\varepsilon > 0$ there is $\delta > 0$ such that $\left|\dfrac{f(z)-f(z^*)}{z-z^*} - f'(z^*)\right| \leq \varepsilon$ for $z \in B(z^*, \delta)\setminus\{z^*\}$

$\longrightarrow$

Proof continued:

$$\left| f(z) - \left(f(z^*) + (z-z^*) f'(z^*) \right) \right| \le \varepsilon |z-z^*| \quad \forall z \in B(z^*, \delta)$$

Recall: if $F' = f$ on U, $\gamma : [a,b] \to U$, f cts then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, for closed γ (i.e. $\gamma(b) = \gamma(a)$), $\oint_\gamma f(z)\,dz = 0$.

Also, as before, linear polynomials have antiderivatives, so

$$\oint_\psi \left(f(z^*) + (z-z^*) f'(z^*) \right) dz = 0 \quad \text{for all closed } \psi \quad (1)$$

If we fix a small ball $B(z^*, \delta)$, then for n large enough $R_n \subseteq B(z^*, \delta)$. For some constant C, perimeter of R_n has length $2^{-n} C$, where C = perimeter of R.

For some other constant D, $\max_{z \in \partial R_n} |z - z^*| \le 2^{-n} D$, where D is the diameter of R.

Combining all these estimates, given $\varepsilon > 0$, choose δ small so

$$\left| f(z) - \left(f(z^*) + (z-z^*) f'(z^*) \right) \right| \le \frac{\varepsilon}{CD} (z - z^*) \quad \text{for } z \in B(z^*, \delta)$$

and choose n large so that $R_n \subseteq B(z^*, \delta)$.

$$\left| \oint_{\partial R_n} f(z)\,dz \right| = \left| \oint_{\partial R_n} f(z)\,dz \right| - \underbrace{\left| \oint_{\partial R_n} \left(f(z^*) + (z-z^*) f'(z^*) \right) \right|}_{=0 \text{ by } (1)}$$

$$\le \left| \oint_{\partial R_n} \left(f(z) - f(z^*) + (z-z^*) f'(z^*) \right) dz \right|$$

$$\le \oint_{\partial R_n} \left| \frac{\varepsilon}{CD} (z - z^*) \right| dz \le \frac{1}{4^n} \varepsilon \implies \left| \oint_{\partial R_n} f(z)\,dz \right| \le \varepsilon \quad \blacksquare$$

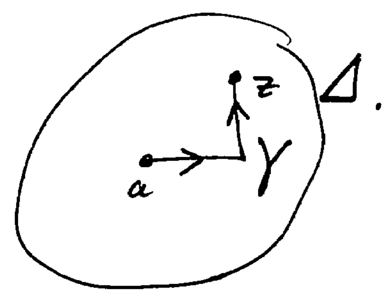
Corollary: Let f be holomorphic on an open disk Δ.

Then (a) f has a holomorphic antiderivative, that is there is F holomorphic on Δ s.t. $F' = f$.

(b) $\oint_\gamma f(z)\,dz = 0$ for all closed γ in Δ.

"Proof": Fix $a \in \Delta$. Define $F(z) = \int_\gamma f(t)\,dt$ where $\gamma : [0,1] \to \Delta$ has $\gamma(0) = a$, $\gamma(1) = z$ and

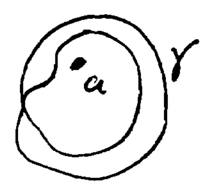
Calculation will show that $F' = f$.



Winding Number:

Given $a \in \mathbb{C}$, γ a closed curve around a. Winding # is number of times γ goes around a.

γ is closed in $\mathbb{C} \setminus \{a\}$



Consider: $\int_\gamma \frac{dz}{z-a}$

Remark: Since γ is cts, $[a,b]$ compact, then image of γ is compact (closed + bounded).

Reparameterize $\gamma : [0,1] \to \mathbb{C}\setminus\{a\}$, $\gamma(0) = \gamma(1)$

Define $f(x) = \int_0^x \frac{\gamma'(t)}{\gamma(t)-a}\,dt$ $\qquad \frac{df}{dt} = \frac{\gamma'(t)}{\gamma(t)-a}$

$g(t) = \exp(-f(t))\,(\gamma(t)-a)$

$$\begin{aligned} g'(t) &= -f'(t)\exp(-f(t))(\gamma(t)-a) + \exp(-f(t))(\gamma'(t)) \\ &= \frac{-\gamma'(t)}{\cancel{(\gamma(t)-a)}}\cancel{(\gamma(t)-a)}\exp(-f(t)) + \gamma'(t)\exp(-f(t)) \\ &= 0. \end{aligned}$$

Hence, since $g'(t)=0$, then g is constant, and $f(0)=0$.
$g(0)=g(1)$ and $\gamma(0)=\gamma(1)$, so $\exp(-f(1))=1$

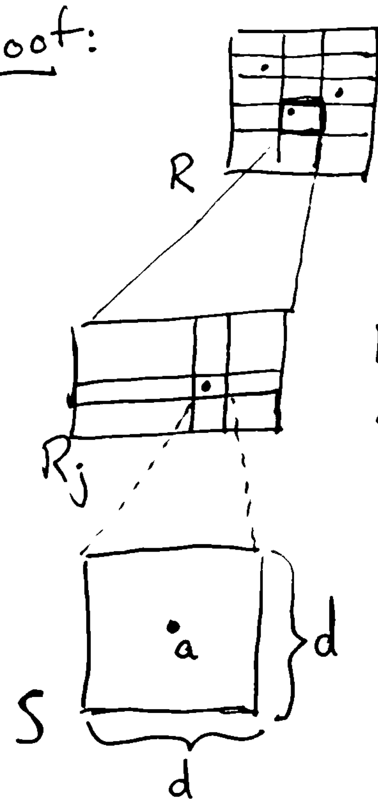
if $\exp(x+iy)=1$, then $e^x(\cos(y)+i\sin(y))=1$
So $x=0$ and $y=2\pi k$ $\exists$ $k\in\mathbb{Z}$.
Hence $f(1)$ is an integer multiple of $2\pi i$
So $$f(1)=\int_\gamma \frac{dz}{z-a} \in 2\pi i\mathbb{Z}.$$

Hence, winding number is integer.

<u>Cauchy's Theorem part 2:</u>
Let R be a rectangle. Let $a_1,\dots,a_k\in\mathring{R}$. Let f be holomorphic on $R\setminus\{a_1,\dots,a_k\}$. For each i,
$\lim_{z\to a_i} f(z)(z-a_i)=0$. (Holds when f is bounded, but can get slightly worse f than bdd).

Then $$\int_{\partial R} f(z)\,dz=0.$$

<u>Proof:</u>



Break R into smaller rectangles, so that each contains at most one a_i.
$\int_{\partial R} f=\sum_j \int_{\partial R_j} f$. WLOG, $k=1$ and only one bad point.

Now put the bad point a in a square, and center the bad point in square

Near a, $|f(z)(z-a)|<\varepsilon$
So $|f(z)|<\dfrac{\varepsilon}{|z-a|}<\dfrac{2\varepsilon}{d}$ (1) $\longrightarrow$ $\left|\int_{\partial S} f(z)dz\right|\le 8\varepsilon$.
length$(\partial S)<4d$ (2) $\nearrow$

$\blacksquare$

Δ open disk

Corollary: If f is holomorphic on $\Delta \setminus \{a_1, \dots, a_k\}$, and $\forall i$, $\lim_{z \to a_i} f(z)(z-a_i) = 0$, then f has an antiderivative in $\Delta \setminus \{a_1, \dots, a_k\}$. So $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in $\Delta \setminus \{a_1, \dots, a_k\}$.

Proof: Same as before, paths avoid all of the bad points.

Theorem (Cauchy Integral Formula): (And derivation)

Δ an open disk, γ a closed contour in Δ, $a \notin \operatorname{im}(\gamma)$, $a \in \Delta$, f holo on Δ.

Consider $g(z) = \dfrac{f(z) - f(a)}{z-a}$. This function is holomorphic in $\Delta \setminus \{a\}$

$$\lim_{z \to a} \left(g(z)(z-a) \right) = 0.$$

$$\int_\gamma g(z)\,dz = 0, \text{ but also } 0 = \int_\gamma g(z)\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz - f(a) \int \frac{dz}{z-a}$$

And $\dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{dz}{z-a} = n(\gamma, a)$, the winding number of γ around a

So
$$\int_\gamma \frac{f(z)}{z-a}\,dz = 2\pi i\, n(\gamma, a)\, f(a) \Rightarrow \boxed{f(a)\, n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz.}$$

Assume for now $n(\gamma, a) = 1$, (e.g. γ is a small circle around a)

Rename variables.
$$\boxed{f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta}$$
Traditional form of CIF.

Next time:
$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta \Longrightarrow f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

So if f is differentiable, it is infinitely so!

Cauchy Integral Formula:

Let D be a closed disk $\subseteq \mathbb{C}$, f holomorphic on D, $\Delta = \mathring{D}$ and $C = \partial D$. Parameterize C by γ.

Easy to see: $n(\gamma, a) = 1$ for all $a \in \Delta$.

Cauchy Integral Formula: For any $z \in \Delta$, $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$

More generally, let γ be piecewise differentiable, ϕ continuous on $im(\gamma)$.
Consider $F(z) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z} \, d\zeta$ for $z \notin im(\gamma)$.

Assume $\gamma : [0,1] \to \mathbb{C}$ piecewise C^1.
$\gamma'(t)$, $\phi(\gamma(t))$ are both bounded.
$im(\gamma)$ is closed, bounded and connected.

If $z \notin im(\gamma)$, then $|\gamma(t) - z|$ bounded away from zero.
$(\exists\, \delta > 0 \quad |\gamma(t) - z| > \delta \ \forall t)$

We show:
(a) F is continuous
(b) F is holomorphic
(c) $F'(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^2} \, d\zeta$

Proof:

(a) Let $z \notin im(\gamma)$. Let $\delta > 0$ s.t. $B(z, \delta) \cap im(\gamma) = \emptyset$, $z_0 \in B(z, \delta/2)$.
Then $|\gamma(t) - z_0| \geq \delta/2$ for all t.

$$F(z) - F(z_0) = \int_\gamma \phi(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) d\zeta$$

$$= \int_\gamma \frac{\phi(\zeta)(z - z_0)}{(\zeta - z)(\zeta - z_0)} \, d\zeta$$

$$= (z - z_0) \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)(\zeta - z_0)} \, d\zeta .$$

bound independent of z_0.

So now let $z_0 \to z$ and observe $F(z_0) \to F(z)$. So F is continuous.

(b) Consider $\dfrac{F(z)-F(z_0)}{z-z_0} = \displaystyle\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)(\zeta-z_0)}\, d\zeta$

Appeal to previous part (a) with ϕ replaced by $\phi^*(\zeta) = \dfrac{\phi(\zeta)}{\zeta - z}$. Set $\dfrac{F(z)-F(z_0)}{z-z_0} \longrightarrow \displaystyle\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^2}\, d\zeta$.

(c) Iterating, we find that F is C^∞ and that

$$F^{(n)}(z) = n! \int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^{n+1}}\, d\zeta \quad . \qquad \blacksquare$$

Returning to Cauchy integral formula:

f is infinitely differentiable, and $f^{(n)}(z) = \dfrac{n!}{2\pi i} \displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, dz$

<u>Liouville's theorem:</u> An entire function on $\mathbb{C}$ is constant if it's bounded.

<u>Proof:</u> Let $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let C_R be a circle of radius R around z. Then

$$f'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{(\zeta-z)^2}\, d\zeta$$

let $\gamma = R\exp(2\pi i t) + z$

$\gamma : [0,1] \to C_R$.

Notice that $f'(z) \longrightarrow 0$ as $R \to \infty$.

So then $f'(z) = 0$ for all $z \in \mathbb{C}$, so f is constant. $\blacksquare$

<u>Next Goal:</u> f holomorphic on $\Omega \subseteq \mathbb{C}$ open. Let $a \in \Omega$, consider the Taylor series $f(a) + (z-a) f^{(1)}(a) + \dfrac{(z-a)^2}{2!} f^{(2)}(a) + \ldots$

It converges to $f(z)$ on any open disk centered on a and contained in Ω.

Refined Cauchy Integral Formula:

$D \subseteq \mathbb{C}$ closed disk, $\Delta = \mathring{D}$, $C = \delta D$ parameterized by γ.

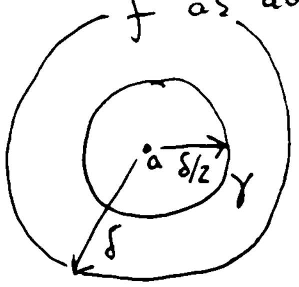
Let $a_1, \dots, a_k \in \Delta$, f holomorphic on $D \setminus \{a_1, \dots, a_k\}$.
For each i, let $\lim_{z \to a_i} f(z)(z - a_i) = 0$.

Then: $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} \, d\zeta$ for $z \in \Delta \setminus \{a_1, \dots, a_k\}$.

Proof: Combine previous proof of CIF with more general Cauchy's theorem. $\blacksquare$

Defn: Let f be holomorphic and defined on an open ball $B(a, \delta) \setminus \{a\}$, $\delta > 0$.
Then f has a removable singularity at a iff $\lim_{z \to a} f(z)(z-a) = 0$.

Theorem: If a is a removable singularity for a function f as above, then f can be extended to f_1 holomorphic on $B(a, \delta)$.



Consider g defined on $B(a, \delta/2)$ by

$$g(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

By general/previous facts, g is holomorphic on $B(a, \delta/2)$ & by CIF, $g(z) = f(z)$ for $z \in B(a, \delta/2) \setminus \{a\}$.
Use g to extend f to f_1 by $f_1(a) = g(a)$, $f_1(z) = f(z)$ for $z \neq a$. $\blacksquare$
Since g is continuous, agrees w/ f, $\lim_{z \to a} f(z) = g(a)$

Taylor Expansion: f holomorphic on Ω, $a \in \Omega$.
$\frac{f(z) - f(a)}{z - a}$ is holomorphic on $\Omega \setminus \{a\}$, removable singularity at a.

So let $f_1(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & z \in \Omega \setminus \{a\} \\ f'(a) & z = a \end{cases}$, repeat.

Taylor Expansions:

Let f be holomorphic on Ω, open. Let $a \in \Omega$. The function $\dfrac{f(z)-f(a)}{z-a}$ is holomorphic on $\Omega \setminus \{a\}$, and has a removable singularity at a. Remove it, thereby defining f_1 such that

$$f_1(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \in \Omega \setminus \{a\} \\ f'(a) & z = a \end{cases} \quad \text{and } f_1 \text{ holomorphic}$$

Inductively define

$$f_n(z) = \cancel{\frac{f_{n-1}(z) - f(}{}} \begin{cases} \dfrac{f_{n-1}(z) - f_{n-1}(a)}{z-a} & z \in \Omega \setminus \{a\} \\ f'_{n-1}(a) & z = a \end{cases}$$

f_n holomorphic on Ω

Easily shown:

$$f(z) = f(a) + f_1(a)(z-a) + \dots + f_{n-1}(a)(z-a)^{n-1} + f_n(z)(z-a)^n$$

$$f(z) = f(a) + f^{(1)}(a)(z-a) + \frac{f^{(2)}(a)}{2!}(z-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \dots$$

Special cases of Cauchy Integral Formula:

Consider a contour C, which is a circle around a

$$\frac{1}{2\pi i} \int_C \frac{dz}{z-a} = n(C, a) = 1$$

$$\frac{1}{2\pi i} \int_C \frac{dz}{(z-a_1)(z-a_2)} = \frac{1}{2\pi i (a_1 - a_2)} \int_C \left(\frac{1}{z-a_1} - \frac{1}{z-a_2} \right) dz$$

$$= 0.$$

Differentiate wrt a_1: $\displaystyle \int_C \frac{dz}{(z-a_1)^n (z-a_2)} = 0$ (*)

Using CIF for $1^{(n)}$, $\displaystyle 0 = \int \frac{dz}{(z-a)^n}$ for $n > 1$, a inside C

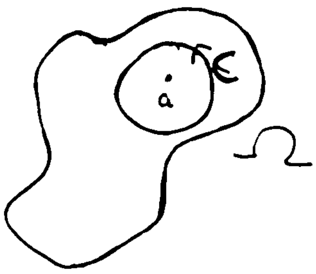
Recall:

$$f(z) = f(a) + f^{(1)}(a)(z-a) + \dots + \frac{f^{(n)}(a)}{n!}(z-a)^n + f_{n+1}(z)(z-a)^{n+1}$$

So $$f_n(z) = \frac{f(z)}{(z-a)^n} - \frac{f(a)}{(z-a)^n} - \dots - \frac{f^{(n-1)}(a)}{(n-1)!\,(z-a)}$$

Use CIF to simplify. Let C be a contour around a, C a circle of radius small s.t. $C \cup (\text{disk} \subseteq C) \subseteq \Omega$.

Assume $(n \gg 1)$



$$f_n(z) = \int_C \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)} = \int \frac{f(\zeta)\,d\zeta}{(\zeta-a)^n(\zeta-z)} + 0$$

other terms drop b/c (*) from previous page.

f holomorphic $\Rightarrow$ f bounded on C since C compact, f cts.

Let D be a closed disk of radius $\frac{1}{2}$(radius of C), center a. Consider only $z \in D$. Then $|\zeta - z| \geq \frac{1}{2}R$ where R is the radius of C. Estimate value of $f_n(z)$ and see that $f_n(z)(z-a)^n \to 0$ on D uniformly as $n \to \infty$. Why is this the case?

$$|f_n(z)(z-a)^n| = |f_n(z)|\,|(z-a)^n| \leq \frac{\sup_{\zeta \in C} |f(\zeta)|}{R^n} |z-a|^n$$

$$\leq \frac{\sup_{\zeta \in C} |f(\zeta)|}{R^n} \left(\tfrac{1}{2}R\right)^n$$

$$\leq \frac{1}{2^n} \sup_{\zeta \in C} |f(\zeta)|$$

Error bound for n^{th} order taylor series for f at point a.

Also this tayl- series converges in a disk around a.

Taylor Series Convergence:

For any $a \in \Omega$ and any δ s.t. $B(a,\delta) \subseteq \Omega$, the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \quad \text{converges to } f(z) \text{ for all } z \in B(a,\delta)$$

Analytic is synonymous with holomorphic.

Topology of $\mathbb{C}$

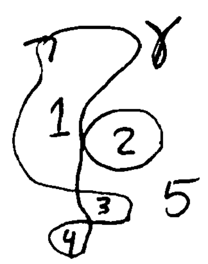
$\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$.

An open set O is connected iff it is path-connected.
iff any 2 points $x, y \in O$, there is a path of finitely many vertical/horizontal line segments from x to y and contained in O.

Defn: A *region* is a connected open subset of $\mathbb{C}$.

Let γ be a closed contour in $\mathbb{C}$, $\gamma : [0,1] \to \mathbb{C}$, piecewise continuously differentiable.

The *regions determined by γ* are the connected components of $\mathbb{C} \setminus \mathrm{im}(\gamma)$.



Easy Fact: $n(\gamma, a)$ is constant as a function of a, on each region determined by γ.

Proof: $n(\gamma, a)$ is a cts function of a which takes integer values. ∎

Proof ctd.

Suppose for contradiction that Ω is a connected component of $\mathbb{C} \setminus im(\gamma)$ and $\xi_1, \xi_2 \in \Omega$. $n(\gamma, \xi_1) < n(\gamma, \xi_2)$

Choose $\alpha \notin \mathbb{Z}$ such that $n(\gamma, \xi_1) < \alpha < n(\gamma, \xi_2)$

$\Omega = \Omega_< \cup \Omega_>$ where $\Omega_> = \{\xi \in \Omega : n(\gamma, \xi) > \alpha\}$

$\Omega_< = \{\xi \in \Omega : n(\gamma, \xi < \alpha\}$

$\divideontimes$ contradicts connectivity of Ω. ■

Inside any region Ω, and any $B(a,\delta) \subseteq \Omega$, the taylor series converges.

Let $a \in \Omega$, and suppose that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$.

There is $\delta > 0$ and $B(a,\delta) \subseteq \Omega$ and $f \restriction B(a,\delta) = 0$

$f^{(n)}(b) = 0$ for all $b \in B(a,\delta)$.

Hence, the collection of $a \in \Omega$ s.t. $f^{(n)}(a) = 0$ is open.
$\uparrow$
Ω_0

Let $a \in \Omega$, $f^{(n)}(a) \neq 0$ for some $n \in \mathbb{N}$.

$f^{(n)}$ is holomorphic, and hence continuous, so there is $\delta > 0$ $B(a,\delta) \subseteq \Omega$ and $f^{(n)}(b) \neq 0$ for all $b \in B(a,\delta)$

Hence, $\{a \in \Omega :$ there is $n \in \mathbb{N}$ s.t. $f^{(n)}(a) \neq 0\}$ is open in Ω
Call it Ω^*

~~Then~~ For any $a \in \Omega$, either $f^{(n)}(a) = 0$ $\forall n$, or $\exists n$ $f^{(n)}(a) \neq 0$

Since Ω is connected, both of Ω_0 and Ω^* are open, then one of ($\Omega_0 = \emptyset$ and $\Omega^* = \Omega$) or ($\Omega_0 = \Omega$ and $\Omega^* = \emptyset$) must be the case.

Assume that f is not identically zero on Ω, so for all $a \in \Omega$ there is n s.t. $f^{(n)}(a) \neq 0$.

Let $f(a) = 0$ and let n be the least such that $f^{(n)}(a) \neq 0$. From the proof of Taylor's theorem, there is analytic g on Ω so

$$f(z) = (z-a)^n g(z), \quad g(a) \neq 0.$$

As g is continuous, there is $\delta > 0$, $B(a,\delta) \subseteq \Omega$ where $g(b) \neq 0$ for all $b \in B(a,\delta)$. So $f(z) \neq 0$ for $B(a,\delta) \setminus \{a\}$.

We say f has a "zero of order n" at a. Small open disk where only a is a zero of f on that disk.

__Corollary:__ If B is a closed, bounded set, $B \subseteq \Omega$ a region, then $\{a \in B : f(a) = 0\}$ is finite.

__Proof:__ Open cover / finite subcover b/c B is compact.

__Corollary:__ If $\gamma : [0,1] \to \mathbb{C}$, then there is a closed/bounded $B \subseteq \Omega$. such that $im(\gamma) \subseteq B$.

__Defn:__ Let f be holomorphic in $B(a,\delta) \setminus \{a\}$. Then f has a __pole__ at a if $\lim_{z \to a} |f(z)| = \infty$

Suppose f has a pole at $a \in \Omega$. Consider $g : B(a,\delta) \setminus \{a\} \to \mathbb{C}$, $g(z) = \frac{1}{f(a)}$. We may assume $f(z) \neq 0$ for $z \in B(a,\delta) \setminus \{a\}$.

g is holomorphic on $B(a,\delta) \setminus \{a\}$. $\lim_{z \to a} g(z) = 0$, so g has a removable singularity at a, so define $g_1(z) = \begin{cases} g(z) & z \neq a \\ 0 & z = a \end{cases}$

$g_1(z)$ is nonzero, holomorphic, has zero at a.

If g_1 has a zero of order n at $z = a$, then $g_1(z) = (z-a)^n h(z)$, $h(a) \neq 0$, h holomorphic on $B(a,\delta)$.

So $f(z) = \dfrac{1}{g_1(z)} = (z-a)^{-n} \underbrace{\dfrac{1}{h(z)}}$ for all $z \in B(a,\delta)\setminus\{a\}$.

$\hookrightarrow$ analytic b/c $h \neq 0$.

"f has a pole of order n at $z=a$".

Extended Complex Plane:

$\mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$.
Also the projective space $\mathbb{P}^1(\mathbb{C})$.

Write $\mathbb{C} \cup \{\infty\} = (\mathbb{C}) \cup (\mathbb{C} \cup \{\infty\} \setminus \{0\})$

$z \in \mathbb{C} \longmapsto \frac{1}{z} \in \mathbb{C} \cup \{\infty\} \setminus \{0\}$ remap coordinates.

$\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.

Local Behavior of Holomorphic functions

f holomorphic on Ω, f not identically zero on Ω.
For simplicity, assume for the moment that f has only finitely many zeroes in Ω, which are $\{a_1, \dots, a_n\}$

We write $f(z) = (z-a_1)(z-a_2)\cdots(z-a_n)\,g(z)$,
g is holomorphic on Ω and $g(z) \neq 0$ for all $z \in \Omega$.

NB: a_i can have repetitions.

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \cdots + \frac{1}{z-a_n} + \frac{g'(z)}{g(z)} \text{ at all } z \in \Omega \setminus \{a_1, \dots, a_n\}$$

Let γ be a closed curve in ~~Ω than~~ $\Omega \setminus \{a_1, \dots, a_n\}$ Then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n(\gamma, a_1) + n(\gamma, a_2) + \cdots + n(\gamma, a_n) + \underbrace{\int_\gamma \frac{g'(z)}{g(z)}\,dz}_{=0 \text{ since } g \text{ has no zeroes in } \Omega}$$

Special case: if γ is a circle in Ω avoiding zeroes of f, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \#\text{ of zeroes of } f \text{ enclosed by } \gamma, \text{ counted up to multiplicity.}$$

Consider the curve $\Gamma = f\circ\gamma$. Since γ avoids zeroes of f, Γ avoids zero. So then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n(\Gamma, 0)$$

Last Time: $f:\Omega\to\mathbb{C}$ holomorphic, not identically zero.

γ closed curve in Ω, γ avoids zeroes of f.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z_j \text{ zero of } f} n(\gamma, z_j) = n(\Gamma, 0) \quad \text{if } \Gamma = f\circ\gamma.$$

Generalization: $f(z)=a \iff z$ is a zero of $f-a$.

If γ is closed and avoids $\{z : f(z)=a\}$, then

$$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$$ where $z_j(a)$ enumerates with multiplicity points where $f(z)=a$.

Key Point: Let γ be a circle, $\Gamma = f\circ\gamma$

$n(\Gamma, a) = \sum_j n(\gamma, z_j(a))$ counts (w/ multiplicity) points inside γ where $f(z)=a$.

As we saw, if a, a' are in the same region determined by Γ, then $n(\Gamma, a) = n(\Gamma, a')$. f assumes values a, a' same number of times inside Γ.

f holomorphic on Ω, $b\in\Omega$, $f(b)=a$, f nonconstant on Ω

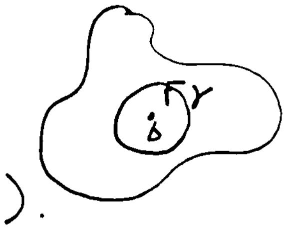
Let b be a zero of order n for $f-a$, that is $f-a=(z-b)^n h$ for a holomorphic function h with $h(a)\neq 0$.

Key Point:

① b is an isolated zero of $f-a$.

② If $f'(b)=0$ (so b is repeated root of $f-a$) then b is also an isolated zero of f'.

Let γ be a circle with center b contained in Ω, along w/ interior of circle. Inside γ, $f-a$ has only b as a zero and f' has only b as a zero, possibly. (i.e. f' vanishes only possibly at b in γ).



For a' sufficiently close to a, $n(\Gamma, a) = n(\Gamma, a')$ = # of times f assumes value a inside $\gamma = n$.
Since $f' \neq 0$ inside γ except possibly at $z=b$, values $a' \neq a$ are assumed n times, each with multiplicity one.

Key consequences: Ω a region, f holomorphic on Ω, nonconstant

(1) f is an open mapping: for all open $U \subseteq \Omega$, $f(U)$ is open

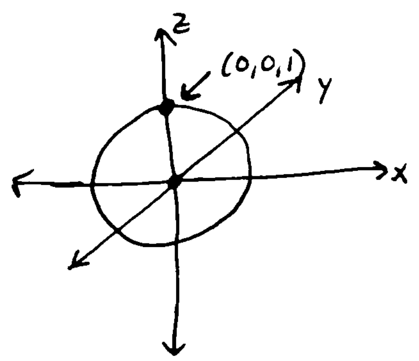
(2) If f is nonconstant on Ω, $|f|$ does not assume a maximum on Ω.

Extended Complex Plane: $\mathbb{C} \cup \{\infty\}$

Riemann Surface

One-point Compactification of $\mathbb{C}$.

Construct the "Riemann Sphere" $x^2+y^2+z^2=1$, identify xy-plane with $\mathbb{C}$.



$S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

For $Q \in S$, $Q \neq P$, associate to Q the unique $x+iy$ s.t. the line PQ meets ~~plane~~ xy-plane at $(x,y,0)$.

Gives Bijection between $S \setminus \{P\}$ and $\mathbb{C}$.
Not an isometry, giving S the Euclidean metric.
Continuous w/ cts inverse, so homeomorphism.

Let P correspond to ∞. S is compact, so then $\mathbb{C} \cup \{\infty\}$ ~~one~~ is compact too.

Extended Complex Plane:

$A \subseteq \mathbb{C} \cup \{\infty\}$ is open iff

for all $a \in A$, there is basic open neighborhood around a but contained in A.

basic open sets are

if $a \in \mathbb{C}$, $B(a,\delta)$

if $a = \infty$, $\{\infty\} \cup \{z \in \mathbb{C},\ |z| > \delta\} = B(\infty, \delta)$.

Let γ be a closed curve, say $\gamma : [0,1] \longrightarrow \mathbb{C}$.

$\mathrm{im}(\gamma)$ is compact, i.e. closed and bounded.

Consider γ as a function $\gamma : [0,1] \longrightarrow \mathbb{C} \cup \{\infty\}$

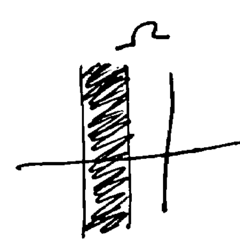
γ divides $\mathbb{C} \cup \{\infty\}$ into regions

There is one region containing ∞, and some others, all bounded in usual metric on $\mathbb{C}$.

Fact: If $a \in \mathbb{C}$ and a is in the unbounded region determined by γ, then $n(\gamma, a) = 0$.

Proof: Find Δ s.t. $\mathrm{im}(\gamma) \subseteq \Delta$, Δ an open disk. Find $a \in \mathbb{C} \setminus \Delta$, a far from Δ. Then $\frac{1}{z-a}$ is holomorphic on Δ, so

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$$



Defn: Let Ω be a region in $\mathbb{C}$. Then Ω is simply connected iff $\mathbb{C} \cup \{\infty\} \setminus \Omega$ is connected.

Theorem: Let Ω be a region. Then TFAE

(1) Ω is simply connected

(2) $n(\gamma, A) = 0$ for all closed γ in Ω, all $A \notin \Omega$.

Proof (1) $\Rightarrow$ (2):

Consider the regions of $\mathbb{C}\cup\{\infty\}$ determined by γ.
$\mathbb{C}\cup\{\infty\}\setminus\Omega$ is contained in <u>one</u> region determined by γ, since $\mathbb{C}\cup\{\infty\}\setminus\Omega$ is connected. Since $\infty\notin\Omega$, ∞ is in the unbounded region.

$n(\gamma,a)=0$ for all a in the unbounded region, since it's constant on each region, and for some $a\notin\Omega$, $\frac{1}{z-a}$ is holomorphic on Ω, so $n(\gamma,a)=0$. ∎

Proof of (2) $\Rightarrow$ (1):

By contrapositive. Ω is <u>not</u> simply connected, so $(\mathbb{C}\cup\{\infty\})\setminus\Omega = A\cup B$, both A, B disjoint, closed, nonempty.

Say $\infty\in B$, so $\infty\notin A \Rightarrow \infty\in A^c$, and A^c is open.

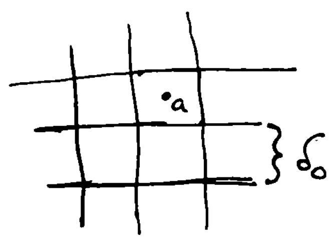
A^c contains an open nbhd of ∞ in $\mathbb{C}\cup\{\infty\}$.

So A is a bounded subset of $\mathbb{C}$, closed $\Rightarrow$ compact.

Also $B\cap\mathbb{C}$ is a closed subset of $\mathbb{C}$.

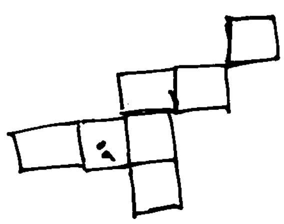
Since A is compact, B closed, $A\cap B=\emptyset$, then there is $\delta>0$ such that $\forall a\in A$ and $\forall b\in B$ $|a-b|\geq\delta$

Fix $a\in A$. Cover $\mathbb{C}$ by squares of size $\delta_0 \ll \delta$ s.t. a is at the center of some such square.



Only finitely many of these squares intersect A, since A is bounded.

Consider the set of the squares which meet A, break it into connected components, and focus on component which contains a.



Proof $(2) \Rightarrow (1)$ Continued:

If a square Q appears in this component and $E \subseteq \partial Q$ is an edge, and there is no other square Q' in the component, $\partial Q' \cap E \neq \emptyset$, then $E \subseteq \Omega$.

Since A meets Q, and $d(A,B) \geq \delta \gg \delta_0 = \text{length } E$, then $E \cap B = \emptyset$.

Proof Aborted.

Goal: (Penultimate Cauchy's theorem):

For any simply connected Ω, holomorphic $f: \Omega \to \mathbb{C}$, and any closed curve γ in Ω, $\oint_\gamma f(z)\,dz = 0$.

Ultimate Cauchy's Theorem:

For any region Ω, any γ s.t. $\forall a \notin \Omega$, $n(\gamma, a) = 0$, and any $f: \Omega \to \mathbb{C}$ holomorphic $\oint_\gamma f(z)\,dz = 0$.

Idea: Cauchy for all functions $\frac{1}{z-a}$, $a \notin \Omega$, $\Longrightarrow$ full Cauchy.

Corollary: If f is holomorphic on simply connected Ω, then f has an antiderivative on Ω.

Salvaging the proof until last time.

Want to show: Ω a region, $\forall a \notin \Omega$, $n(\gamma, a) = 0$ for all closed γ in Ω.
$\Longrightarrow \Omega$ simply connected.

Ω simply connected only when $\mathbb{C} \cup \{\infty\} \setminus \Omega$ is connected.
If $\mathbb{C} \cup \{\infty\} \setminus \Omega$ is not connected, $\mathbb{C} \cup \{\infty\} \setminus \Omega = A \cup B$ nonempty, closed, disjoint. $\infty \in B \Longrightarrow A$ bounded. $B \cap \mathbb{C}$ closed.

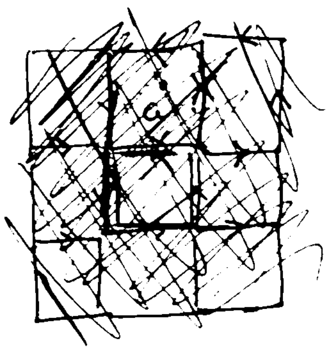
Since A compact, B closed $\exists \delta > 0$ $d(a,b) > \delta$ $\forall a \in A, b \in B$.
Cover plane by closed squares w/ side length $\delta_0 \ll \delta$.

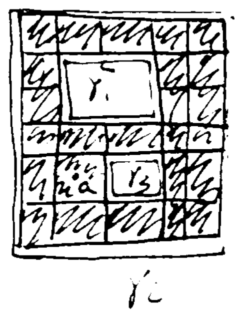
$\longrightarrow$

Arrange some $a \in A$ lies at the center of some square. Consider the finite set of squares meeting A. Start with the one containing a. Inductively add $\square$'s by the rules (a) new $\square$ meets A
(b) has an edge in common w/ previous.

For each square Q meeting A, let $\partial Q = \square$ be a contour.
For each such Q, $n(\partial Q, a) = \begin{cases} 1 & \text{if } Q \text{ is unique square containing } a \\ 0 & \text{otherwise.} \end{cases}$

$$\frac{1}{2\pi i} \oint_{\partial Q} \frac{dz}{z-a} = n(\partial Q, a) = 1 \quad \text{and} \quad \frac{1}{2\pi i} \oint_{\partial Q} \frac{dz}{z-a} = \sum_{Q \text{ meets } a} n(\partial Q, a) = 1 + 0 + 0 + 0.$$





Combining terms, can find finitely many closed $\gamma_1, \dots, \gamma_k$ s.t.

(a) $\displaystyle\sum_{i=1}^{n} n(\gamma_i, a) = 1$ $\ast$

(b) Each γ_i is composed of the boundaries of the squares which occur in exactly one Q meeting A.

<u>Key point:</u> All such edges must avoid B by choice of $\delta_0 \ll \delta$, and avoid A, so are contained in Ω. So we cannot have $n(\gamma_i, a) = 1$ for some γ_i, by assumption, hence $\ast$, and $\mathbb{C} \cup \{\infty\}$ must be connected.

■

Free Abelian Groups:

For any set X, $Fr(X) = \{ f : X \to \mathbb{Z}, \{f(x) \neq 0\} \text{ cofinite} \}$

If H is any abelian group, $f : X \to H$ is any function, f extends to an HM $\phi : Fr(X) \to H$, $\phi\left(\sum_{i=1}^{n} n_i x_i\right) = \sum_{i=1}^{n} n_i f(x_i)$.

Algebraic Topology Stuff

If $\gamma : [a,b] \to \mathbb{C}$, piecewise continuously differentiable, $-\gamma$ is "γ traversed backwards", $(-\gamma)(t) = \gamma(a+b-t)$

$$\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz.$$

If $\phi : [a,b] \to [a',b']$ strictly increasing and C^1, then if $\gamma' = \gamma \circ \phi : [a',b'] \to \mathbb{C}$, then $\int_\gamma f(z)\,dz = \int_{\gamma'} f(z)\,dz$.

Say $\gamma : [a,b] \to \mathbb{C}$ $\quad a = a_0 < a_1 < \cdots < a_n = b$

γ_i is from $[a_i, a_{i+1}]$ to $\mathbb{C}$, $\gamma_i = \gamma \restriction [a_i, a_{i+1}]$

$$\int_\gamma f(z)\,dz = \sum_{i=0}^{n-1} \int_{\gamma_i} f(z)\,dz.$$

Chain Group: Start with $X = \{ \gamma : \gamma \text{ piecewise } C^1 \}$

Form $Fr(X)$, the free group on X.

Let $G \leq Fr(X)$ be the subgroup of $Fr(X)$ generated by terms of the form $\gamma + (-\gamma)$

$\gamma - \gamma \circ \phi$, ϕ increasing, C^1 as above

$\gamma - (\gamma_1 + \gamma_2 + \cdots + \gamma_k)$ where $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k = \gamma$, γ_i come from partition of domain of γ as above.

Chain Group $= \dfrac{Fr(X)}{G}$. Elements $\gamma \in$ Chain Group are called <u>chains</u>.

Chain Group: We will identity γ with its equivalence class $[\gamma] = \gamma + G$ in the chain group, because the integral of a function is invariant on its equivalence class of contours. So the integral is well-defined for $\gamma \in$ Chain Group.

Extend the meaning of $\int_\gamma f(z)dz$ to $\gamma \in \frac{Fr(X)}{G}$ as follows:

$$\int_{\sum n_i \gamma_i} f(z)dz = \sum n_i \int_{\gamma_i} f(z)dz.$$

For all $\gamma \in G$, $\int_\gamma f(z)dz = 0$.

Defn: A chain γ is in Ω iff $\gamma = \sum_{i=1}^{n} n_i \gamma_i + G$; Ω an open subset of $\mathbb{C}$, γ_i are curves in Ω.

Defn: The group of cycles is the subgroup of the chain group generated by cosets of the form $\gamma + G$, γ is a closed curve.

Theorem (revisited): For a region $\Omega \subseteq \mathbb{C}$, TFAE

(1) Ω simply connected

(2) $n(\gamma, a) = 0$ for all cycles γ in Ω, for $a \notin \Omega$

N.B: $n(\gamma, a)$ is well defined for cycles, $= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$

Defn: Let Ω be a region.

(a) A cycle in Ω is homologous to zero iff $n(\gamma, a) = 0$ for all $a \notin \Omega$

(b) γ_1, γ_2 cycles in Ω are homologous iff $\gamma_1 - \gamma_2$ is homologous to zero iff $n(\gamma_1, a) = n(\gamma_2, a)$ for all $a \notin \Omega$

γ_1 homologous to γ_2, we write $\gamma_1 \sim \gamma_2$

Techinically, homologous mod Ω.

Theorem (Ultimate Cauchy):

If f is holomorphic in region Ω, γ a cycle in Ω, and $\gamma \sim 0 \pmod{\Omega}$, then $\int_\gamma f(z)\,dz = 0$.

Corollary: If Ω is simply connected, then $\int_\gamma f(z)\,dz = 0$ for all closed curves (cycles) in Ω.

So f has an antiderivative on Ω.

Corollary: If Ω is simply connected, f holomorphic and nonzero on Ω, then can define a holomorphic function $\log(f)$ on Ω s.t. $\exp(\log(f(z))) = f(z)$ for all $z \in \Omega$, not necessarily unique.

Proof: Since f nonzero on Ω, f' holomorphic on Ω, so $\frac{f'}{f}$ is holomorphic on Ω. Choose an antiderivative F for f'/f.

$$F'(z) = f'(z)/f(z).$$

Look at $g(z) = f(z)\exp(-F(z))$.

$$g'(z) = f'(z)\exp(-F(z)) - f(z)\frac{f'(z)}{f(z)}\exp(-F(z)) = 0.$$

g is a holomorphic function with zero derivative on a connected set, so $g(z)$ is constant.

Fix $z_0 \in \Omega$. Choose ω s.t. $\exp(\omega) = f(z_0)$, possible since $f(z_0) \neq 0$.

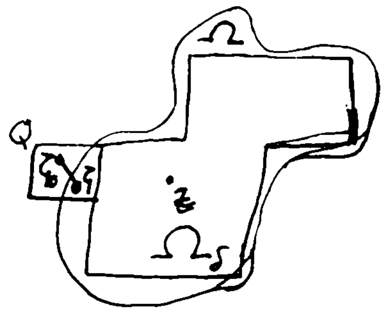
$\exp(F(z) - F(z_0) + \omega) = f(z)$ by calculation, $F(z)$ is $\log(f)$. $\blacksquare$

Corollary: If f is nonzero, holomorphic on Ω a simply connected region, then can choose $\sqrt[n]{f}$ holomorphic on Ω.

Proof: $\sqrt[n]{f} = \exp\left(\frac{\log f}{n}\right)$. $\blacksquare$

Proof of Ultimate Cauchy:

Fix some closed curve γ in Ω. Suffices because cycles are Σ's of closed curves.



Case 1: Ω is bounded.

Cover $\mathbb{C}$ with closed squares, side length δ.
Arrange so at least one square $\subseteq \Omega$. As Ω is bounded, finite, nonempty set of squares meet Ω.

Let $X_1 = \{Q : Q \text{ is square} \subseteq \Omega\}$
$X_2 = \{Q : Q \text{ is square}, Q \cap \Omega \neq \emptyset\}$ $\quad \emptyset \neq X_1 \subseteq X_2$, X_2 finite.

Let $\Omega_\delta = \bigcup_{Q \in X_1} Q$ and arrange that γ is a cycle in Ω_δ.

Form the sum of the boundaries of X_1, get cycle Γ_δ.
Let $\zeta \in \Omega \setminus \Omega_\delta$. Then $\zeta \in Q$ for some $Q \not\subseteq \Omega$. Find $\zeta_0 \in Q \setminus \Omega$.
The line ζ to ζ_0 avoides Ω_δ. So ζ and ζ_0 are in the same region determined by γ, as γ is in Ω_δ. As $\gamma \sim 0 \pmod{\Omega}$, and $\zeta_0 \notin \Omega$, then $n(\gamma, \zeta_0) = 0$. As $n(\gamma, \cdot)$ is constant on regions determined by γ, then $n(\gamma, \zeta) = n(\gamma, \zeta_0) = 0$.

In particular, $n(\gamma, \zeta) = 0$ for all ζ on Γ_δ.

f is holomorphic on Ω, let $z \in \mathring{\Omega}_\delta$, s.t. $z \in \mathring{Q}$, $Q \in X_1$.
Let R be a square, $R \in X_1$, so $R \subseteq \Omega$.

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \begin{cases} f(z) & \text{if } R = Q. \\ 0 & \text{otherwise} \end{cases}$$

Summing, $$\frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z} \, d\zeta = f(z).$$

Special case of Fubini-Torelli Theorem:

If f is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, $[a,b]$, $[c,d]$ closed intervals in $\mathbb{R}$, then

$$\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx.$$

Idea: $$\int_\gamma f(z)\,dz = \int_\gamma \left(\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)}{\zeta - z}\,d\zeta\right)dz$$

$$\text{"="}\frac{1}{2\pi i}\int_{\Gamma_\delta}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,dz\,d\zeta \leftarrow \text{but needs to be justified! } (\bigstar)$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta} f(\zeta)\,(-n(\gamma,\zeta))\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta} f(\zeta)\,(0)\,d\zeta = 0.$$

So now to justify $(\bigstar)$:

Key fact: $\mathrm{im}(\gamma)$, $\mathrm{im}(\Gamma_\delta)$ disjoint, compact, so there is some $\varepsilon > 0$ such that $d(a,b) > \varepsilon$ for all $a \in \mathrm{im}(\gamma)$, $b \in \mathrm{im}(\Gamma_\delta)$.

So $\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)\,d\zeta}{\zeta - z} = f(z)$ for all z on γ.

Furthermore, $|\zeta - z|$ is bounded from below, f holomorphic, so $\frac{f(\zeta)}{\zeta - z}$ is holomorphic on this region. ∎

Case 2: Hence established for bounded region Ω, if Ω unbounded, fix large open disk Δ s.t. γ is a cycle in Ω', $\Omega' = \Omega \cap \Delta$. Then $\gamma \sim 0 \bmod \Omega'$, because if $a \notin \Delta$, $n(\gamma,a) = 0$, b/c γ cycle in Δ. If $a \notin \Omega'$ but $a \in \Delta$, $n(\gamma,a) = 0$ because $\gamma \sim 0 \bmod \Omega$. ∎

Meromorphic Functons:

Defn: Ω a region. A function f is meromorphic function on Ω if and only if for every $a \in \Omega$, there is $\delta > 0$ s.t. $B(a,\delta) \subseteq \Omega$ and either (a) f is holomorphic on $B(a,\delta)$, or (b) f is holomorphic on $B(a,\delta)\setminus\{a\}$, f has a pole at a.

Fact: $\{f : f$ is meromorphic on $\Omega\}$ forms a field, with the normal operations, pointwise, remove removable singularities.

Example: $\Omega = \mathbb{C}$ $\quad f(z) = 1 + \frac{1}{z}$, $g(z) = \frac{-1}{z}$

$(f+g)(z) = 1 + \frac{1}{z} - \frac{1}{z}$, yet undefined at $z = 0$. Hence a removable singularity here, which we can remove to get $f + g = 1$.

Defn:
An region Ω is n-connected if and only if $\mathbb{C} \cup \{\infty\} \setminus \Omega$ has n connected components.

For any given n-connected Ω, let the components be $A_1, A_2, \ldots, A_n$, with $\infty \in A_n$.
We showed that Ω is simply connected (1-connected) $\iff n(\gamma, a) = 0$
$\forall\, \gamma$ cycle in Ω, $a \notin \Omega$.

Choosing a sufficiently fine grid of squares on $\mathbb{C}$, we may construct $a_i \in A_i$ and $\gamma_i \in \Omega$ for $i < n$ such that $n(\gamma_i, a_i) = 1$, $n(\gamma_i, a_j) = 0$ for $j \neq i$.

Claim:
Let γ be an arbitrary cycle in Ω. Let $m_i = n(\gamma, a_i)$. Then

$$\gamma \sim \sum_{i=1}^{n-1} m_i \gamma_i \mod \Omega.$$

Proof: Let $a \notin \Omega$. Sinc $n(\gamma, \cdot)$ is constant on each region defined by γ, $n(\gamma, \cdot)$ constant on each A_i. If $a \in A_i$ for $i < n$, then

$n(\gamma, a) = n(\gamma, a_i) = m_i$ and $n(\sum m_i \gamma_i, a) =$
$n(\sum m_i \gamma_i, a_i) = m_i$

$\longrightarrow$

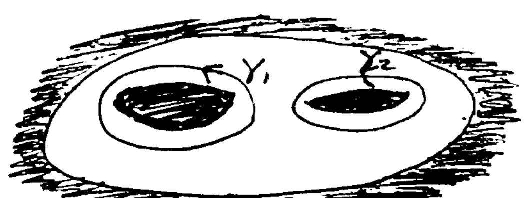
Proof continued:

If $a \in A_n$, then since A_n is unbounded $n(\gamma, a) = 0$ and $n(\gamma_i, a) = 0 \ \forall i$. ■

This claim shows, essentially, that the set of γ_i is a basis for the abelian group of cycles in Ω mod the subgroup of cycles homologous to zero.

Since $\gamma - \sum_{i=1}^{n-1} m_i \gamma_i \sim 0 \pmod{\Omega}$, then

$$\int_{\gamma - \sum m_i \gamma_i} f(z)\,dz = 0 \implies \int_\gamma f(z)\,dz = \sum_{i=1}^{n-1} m_i \int_{\gamma_i} f(z)\,dz .$$



Calculus of Residues:

Let Ω be a region.

Let f be holomorphic in $\Omega' = \Omega \setminus \{a_1, \dots, a_m\}$ a_i distinct points in Ω

For each j, $1 \le j \le m$, find δ_j such that $B(a_j, \delta_j) \subseteq \Omega$, and $a_k \notin B(a_j, \delta_j)$ for $k \neq j$. Let C_j be a circular contour around a_j, radius $\delta_j/2$. Define

$$P_j = \int_{C_j} f(z)\,dz, \quad R_j = P_j / 2\pi i, \text{ and consider } f - \frac{R_j}{z - a_j} .$$

$B(a_j, \delta_j) \setminus \{a\}$ is 2-connected.

$$\oint_\gamma \left(f(z) - \frac{R_j}{z - a_j} \right) dz = 0 \text{ for all cycles } \gamma \text{ in } B(a_j, \delta_j) \setminus \{a_j\}.$$

Note that locally, $f(z) - \frac{R_j}{z - a_j}$ has an antiderivative in this punctured disk $B(a_j, \delta_j) \setminus \{a_j\}$.

Let γ be a cycle in Ω, γ avoids $a_1, a_2, \dots, a_m$.

$\boxed{\text{Assume } \gamma \sim 0. \bmod \Omega.}$?

$$\gamma \sim \sum_{i=1}^{m} n(\gamma, a_i)\, C_i \mod \Omega'$$

f holomorphic in Ω', $\gamma - \sum n(\gamma, a_i) C_i \sim 0 \mod \Omega'$

so
$$\int_\gamma f(z)\,dz = \sum_{k=1}^{m} n(\gamma, a_k) \int_{C_k} f(z)\,dz$$
$$= \sum_{k=1}^{m} n(\gamma, a_k)\, P_k$$

$$\Longrightarrow \frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{i=1}^{m} n(\gamma, a_i)\, R_i$$

R_i is the residue of f at a_i.

Generalization: Suppose that Ω is a region, f holomorphic on $\Omega \setminus \{a_j : j \in J\}$, and the a_j are "isolated singularities",

For all j, there is $\delta_j > 0$ $a_k \notin B(a_j, \delta_j)$ for $k \neq j$.

Key Point: If γ is a cycle in Ω and avoids a_j for each $j \in J$ then $\{j : n(\gamma, a_j) \neq 0\}$ is finite.

Proof: $\{a : n(\gamma, a) = 0,\ a \notin im(\gamma)\}$ is open and there is a closed disk $\bar{\Delta}$ such that γ in $\bar{\Delta}$ and $n(\gamma, a) = 0$ for all $a \notin \bar{\Delta}$.
$\bar{\Delta}$ is compact, so can contain only finitely many a_j.

Residue Theorem: Let Ω be a region, f holomorphic on $\Omega \setminus \{a_j : j \in J\}$, a_j isolated **singularities**. γ is a cycle in Ω, $\gamma \sim 0 \bmod \Omega$.

Then
$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{j \in J} n(\gamma, a_i)\, R_j$$
where R_j is the residue of f at a_j.

Thing: Given a region $\Omega \subseteq \mathbb{C}$, $A \subseteq \Omega$. If f is holomorphic on $\Omega \setminus A$, not defined on A. A is scattered, that is, ~~there is $\delta > 0$~~ no limit points.

γ a cycle in Ω / A, $\gamma \sim 0 \mod \Omega$.

Then (1) $\{a \in A : n(\gamma, a) \neq 0\}$ is finite

(2) $\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in A} n(\gamma, a) \operatorname{Res}_{z=a} f$.

Proof: Find a closed ~~cycle~~ disk $\bar{\Delta}$ s.t. γ is a cycle in $\bar{\Delta}$.

$A^* = \{a \in A : n(\gamma, a) \neq 0\} \subseteq A \cap \bar{\Delta} \subseteq \Omega \cap \bar{\Delta}$.

Suppose for contradiction that A^* is infinite. By compactness, choose a sequence $a_n \in A^*$, converging to a_∞ as $n \to \infty$.

So, $a_\infty \notin A$ because A has no limit points.

$a_\infty \notin \Omega \setminus A$ because $\Omega \setminus A$ is a region, so there would be $B(a_\infty, \delta)$ on which f is defined and holomorphic, but $a_n \to a_\infty$, f is ill-defined on a_i for all i, $\divideontimes$ $a_\infty \notin \Omega \setminus A$.

So $a_\infty \notin \Omega$.

But as $\gamma \sim 0 \mod \Omega$, $n(\gamma, a_\infty) = 0$. As $n(\gamma, x)$ is continuous as a function of x, $n(\gamma, b) = 0$ for all b in open ball around a_∞. But this is a contradiction because $a_n \to a_\infty$, $n(\gamma, a_n) \neq 0$, for all $n \in \mathbb{N}$.

∎

Finish Proof of Residue Theorem:

Argument Principle: Let f be a holomorphic function, nonconstant on Ω. If f has a zero of order n at a, then locally $f = (z-a)^n h$, h holomorphic and $h \neq 0$.

$$\frac{f'}{f} = \frac{n(z-a)^{n-1}h + (z-a)^n h'}{(z-a)^n h} = \frac{n}{(z-a)} + \frac{h'}{h}$$

This has a pole of order 1 at a, ~~and~~ with residue n.
Similarly, if f has a pole of order n at a, f'/f has a simple pole with residue $-n$.

Let γ be a cycle, $\gamma \sim 0 \mod \Omega$, γ avoids zeros and poles of f. Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z \text{ zero of } f} n(\gamma, z) - \sum_{p \text{ pole of } f} n(\gamma, p).$$

Here, note that $\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)}\,dz = n(\Gamma, 0)$ where $\Gamma = f \circ \gamma$.

ROUCHE'S THEOREM: Let $\gamma \sim 0 \mod \Omega$, Ω a region in $\mathbb{C}$.

Assume $n(\gamma, a) \in \{1, 0\}$ for all a not on γ.

"a inside γ" $\longrightarrow n(\gamma, a) = 1$

"a outside γ" $\longrightarrow n(\gamma, a) = 0$

Suppose that f, g holomorphic on Ω, and $|f-g| < |f|$ on γ.
Then "f and g have the same number of zeros inside γ."

Proof of Rouché's Theorem:

Let $h = g/f$. By hypothesis, $|1-h| < 1$ on γ.

Let $\Gamma = h \circ \gamma$. As $0 \notin B(1,1)$, Γ in $B(1,1)$, $n(\Gamma, 0) = 0$.

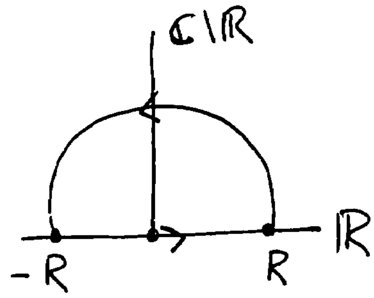
So $\displaystyle\oint_\gamma \frac{h'(z)}{h(z)}\,dz = 0$, so h has the same number of zeros/poles.

Check that this $\Longrightarrow$ f, g have equal number of zeros. ∎

Using residues to compute integrals:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

Consider the contour



$$\oint_{C_R} \frac{dx}{1+x^2} = \int_{C_R^0} \frac{dx}{1+x^2} + \int_{C_R^1} \frac{dx}{1+x^2}$$

$C_R = C_R^0 + C_R^1$

C_R^0: $-R$ to $+R$ on real line

C_R^1: semicircle

Then, since $\frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)}$, the residue of $\frac{1}{1+x^2}$ at $x = i$ is π

So $\displaystyle\oint_{C_R} \frac{dx}{1+x^2} = 2\pi i\left(\frac{1}{2i}\right) = \pi$ constant as $R \to \infty$

$\displaystyle\lim_{R\to\infty} \int_{C_R^0} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ and $\displaystyle\lim_{R\to\infty} \int_{C_R^1} \frac{dx}{1+x^2} = 0$ (look at the size of denominator)

$$\int_0^{2\pi} \frac{d\theta}{2+\sin(\theta)} = \frac{2\pi}{\sqrt{3}}$$

Make a substitution, $\sin(\theta) = \dfrac{\exp(i\theta) - \exp(-i\theta)}{2i}$

So this integral becomes $\displaystyle\int_0^{2\pi} \frac{2ie^{i\theta}\,d\theta}{4ie^{i\theta} + e^{2i\theta} - 1} = \int_{\text{unit circle}} \frac{2\,dz}{4iz + z^2 - 1}$.

Substitute $z = e^{i\theta}$, $\theta \in [0, 2\pi]$

To integrate this, use partial fractions to find poles of $\dfrac{2}{4iz+z^2-1}$, which are at $z=-2i\pm\sqrt{3}\,i = i(-2\pm\sqrt{3})$

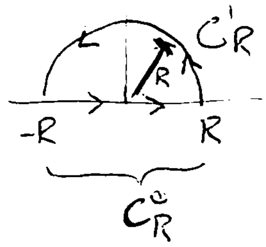
$$\int_{\text{unit circle}} \frac{2}{(z-i(-2+\sqrt{3}))(z-i(-2-\sqrt{3}))}\,dz = \frac{2}{i(\sqrt{3}+-2)-i(-\sqrt{3}-2)} = \frac{1}{\sqrt{3}i}$$

So multiply by $2\pi i$ to get $\dfrac{2\pi}{\sqrt{3}}$

$$\int_0^{2\pi} \frac{\sin\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}.$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx\right)$

Consider $\displaystyle\int_{C_R} \frac{e^{iz}}{z^2+1}\,dz$



$|e^{i(x+iy)}| = |e^{-y}| \leq 1$

As $R\to\infty$ $\displaystyle\int_{C_R'} \frac{e^{iz}}{z^2+1}\,dz \longrightarrow 0$

Residue at $z=i$; $1/2ie$, so $\displaystyle\int_{C_R^0} \frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$

$= \dfrac{2\pi i}{2ie}$ $\boxed{= \pi/e}$.

Series and products.

$f : \Omega \to \mathbb{C}$

Given a sequence of functions (f_n), f_n holomorphic on Ω_n, $\Omega_n \subseteq \Omega_{n+1}$. Let $\Omega = \bigcup_n \Omega_n$

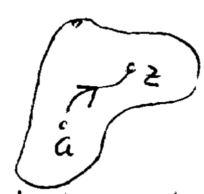
Theorem (Weierstrauss): If $f_n \to f$ uniformly on all compact $K \subseteq \Omega$,
(Remark: As K is compact, $\{\Omega_n\}$ open cover of K, there is m s.t. $K \subseteq \Omega_m$.)
and f_n holomorphic on Ω_n for all n, then (a) f holomorphic on Ω, and
(b) $f_n' \to f'$ uniformly on compact sets.

Before the Proof:

Morera's Theorem: Let Ω be a region, $f : \Omega \to \mathbb{C}$ continuous.
If $\int_\gamma f(z)\,dz = 0$ for all closed γ in Ω. Then f is analytic.

Proof: f has holomorphic antiderivative on Ω,

$F(z) = \int_{\gamma_z} f(\zeta)\,d\zeta$ where γ_z is:



"by magic" derivative of holomorphic function is holomorphic. ∎

Proof of Weierstrauss: Let $a \in \Omega = \bigcup_m \Omega_m$, so $a \in \Omega_m$ $\exists m$,
Ω_m open $\to$ find $\delta > 0$, $\overline{B(a,\delta)} \subseteq \Omega_m \subseteq \Omega_n$ $\forall n \geq m$.
By hypothesis, $f_n \rightrightarrows f$ on $\overline{B(a,\delta)}$.
As $\overline{B(a,\delta)}$ is simply connected, $\int_\gamma f_n(z)\,dz = 0$ for all closed γ in $B(a,\delta)$, all $n \geq m$.
$\int_\gamma f(z)\,dz = 0$ for all closed γ in $B(a,\delta)$
$\implies$ f holomorphic on $B(a,\delta)$ by Morera.

∎ (a)

Proof of Weierstrauss (b):

$a \in \Omega_m$, $\overline{B(a,\delta)} \subseteq \Omega_m$

For all $z \in B(a,\delta)$ $\quad f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad n \geq m.$

$\implies f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$ by uniform convergence.

As f is continuous on γ, f is holomorphic in $B(a,\delta)$.

$$f_n'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

As $n \to \infty$, RHS $\longrightarrow \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = f'(z)$.

Only shows pointwise convergence, not uniform.

Argue that $f_n' \rightrightarrows f'$ on $\overline{B(a,\delta/2)}$ by ε's and δ's.

Given compact $K \subseteq \Omega$, cover w/ suitable finite of small, closed disks.

Corollary: Let f_n holomorphic on Ω for $n \in \mathbb{N}$. If $\left(\sum_{n=1}^{N} f_n\right)$ conveges on compact sets, then the sum is holomorphic and we can differentiate term by term.

Laurent Series: $\sum_{n \in \mathbb{Z}} a_n z^n$

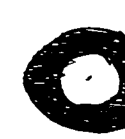
To analyze: split into

$$a_0 + a_1 z + a_2 z^2 + \dots = \sum_{n=0}^{\infty} a_n z^n$$

$$a_{-1} z^{-1} + a_{-2} z^{-2} + \dots = \sum_{m=1}^{\infty} a_{-m} w^m$$

let $w = z^{-1}$

Radius of convergence of $\sum_n a_n z^n$ is R_1, of $\sum_m a_{-m} w^m$ is S_2

$\{z : |z| < R_1 \text{ and } |1/z| < S_2\}$, series converges $\longrightarrow$ 

Recall: If f is meromorphic, f has a pole at b. In a nbhd of b, ~~$f(z) = (z-b$~~

$$f(z) = (z-b)^{-n} g(z), \quad g \neq 0, \ g \text{ holomorphic.}$$

From taylor series of g, get $f = \overbrace{P\left(\frac{1}{z-b}\right)}^{\text{Singular part}} + h(z)$, h holomorphic in a nbhd of b, P a polynomial with no constant term, $\deg(P) = n$.

Problem: Construct meromorphic function f on $\mathbb{C}$ with specified poles and singular parts.

Let b_ν, $\nu \in \mathbb{N}$ be distinct, $b_\nu \longrightarrow \infty$ as $\nu \longrightarrow \infty$.

For each ν, P_ν is nonzero polynomial w/ zero constant term.

Goal: Construct f w/ poles b_ν and singular part $P_\nu\left(\frac{1}{z-b_\nu}\right)$ at b_ν.

Fix: Find polynomials $p_\nu(z)$ such that

$$\sum_\nu \left(P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z) \right)$$

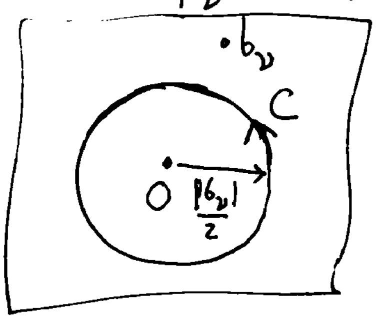
converges for $z \notin \{b_\nu : \nu \in \mathbb{N}\}$.

p_ν will be an initial segment of the taylor series for $P_\nu\left(\frac{1}{z-b_\nu}\right)$ in powers of z.

Let $g_\nu(z) = P_\nu\left(\frac{1}{z-b_\nu}\right)$, holomorphic on $B(0, |b_\nu|)$.

g_ν has a taylor series expansion valid in this disk. Estimate:

Let p_ν be the first $n_\nu + 1$ terms of the Taylor Series (up to the x^{n_ν} term)



Let C be a circular contour w/ radius $\frac{|b_\nu|}{2}$

Estimate error: (using integral form of Taylor Series remainder, $z \leq \frac{|b_\nu|}{4}$.)

$|g_\nu(z) - p_\nu(z)|$

$$g_\nu(z) = p_\nu(z) + \frac{z^{n_\nu+1}}{2\pi i} \oint_C \frac{g_\nu(\zeta)\, d\zeta}{\zeta^{n_\nu+1}(\zeta - z)} \qquad \text{since } |z| \leq |b_\nu|/4, \quad |\zeta - z| \geq \frac{|b_\nu|}{4}.$$

$$\Longrightarrow |g_\nu(z) - p_\nu(z)| \leq \frac{|z|^{n_\nu+1}}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi} \cancel{|b_\nu|}}{2} \frac{M_\nu \cdot 2^{n_\nu+1} \cdot 4}{|b_\nu|^{n_\nu+1} \cdot \cancel{|b_\nu|}}, \text{ where } M_\nu = \sup_{\zeta \in \operatorname{im} C} |g_\nu(\zeta)|.$$

$$= 2M_\nu \left(\frac{2|z|}{|b_\nu|} \right)^{n_\nu+1} \leq 2M_\nu \left(\frac{1}{2} \right)^{n_\nu+1} = \frac{M_\nu}{2^{n_\nu}}$$

Choose n_ν s.t. $2^{-n_\nu} M_{n_\nu} \leq 2^{-\nu}$.

Fix $z \notin \{b_\nu : \nu \in \mathbb{N}\}$. For all but finitely many ν, $|z| \leq \frac{|b_\nu|}{4}$ since $b_\nu \xrightarrow[\nu \to \infty]{} \infty$

So break up $\sum_\nu (g_\nu(z) - p_\nu(z))$ into several parts.

$$\sum_{\nu : |b_\nu| < 4|z|} \left(g_\nu(z) - p_\nu(z)\right) + \sum_{\nu : |b_\nu| \geq 4|z|} \left(g_\nu(z) - p_\nu(z)\right)$$

This shows that for each δ, $\sum_\nu (g_\nu - p_\nu) = T_1 + T_2$

T_1 meromorphic

T_2 uniformly convergent on $\overline{B(0,\delta)}$, $(|b_\nu| \geq 4\delta)$.

By Weierstrauss theorem, $T_2 = \sum_{\nu : |b_\nu| > 4\delta} g_\nu - p_\nu$ holomorphic on $\overline{B(0,\delta)}$.

If $f = \sum_\nu g_\nu - p_\nu$, f is as required. ∎

Is this function unique? Yes!

Let h be any meromorphic function with poles b_ν and singular parts $P(\frac{1}{z-b_\nu})$. Construct f as above, consider $h-f$. Has removable singularities at each b_ν. So we can find an entire function g such that $h = f + g$.

<u>Example:</u> $\dfrac{\pi^2}{\sin^2(\pi z)}$, where $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

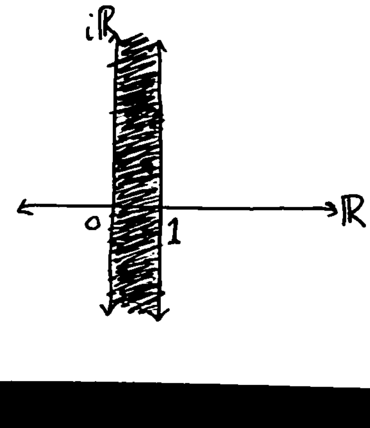
Has poles at all integers $n \in \mathbb{Z}$, with singular part $\dfrac{1}{(z-n)^2}$.

$\sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$ absolutely convergent for $z \notin \mathbb{Z}$.

uniformly convergent on $\overline{B(0,\delta)}$ if we exclude terms $|n| \leq \delta$

Then $\dfrac{\pi^2}{\sin^2(\pi z)} - \sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$ is entire.

both have period 1, so estimate on $\{x+iy : 0 \leq x \leq 1\}$



Estimate

$$\sin(\pi(x+iy)) = \frac{e^{i\pi(x+iy)} - e^{-i(x+iy)\pi}}{2}$$ for $|y|$ large?

$$|\sin(\pi(x+iy))| = \frac{|e^{-\pi y}| + |e^{\pi y}|}{2}$$

$\left.\begin{array}{l} \text{as } y \to \infty \\ \text{as } y \to -\infty \end{array}\right\} \longrightarrow \infty$

Let $H = \dfrac{\pi^2}{\sin^2(\pi z)} - \sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$, H is entire, $H \to 0$ as $z \to \infty$

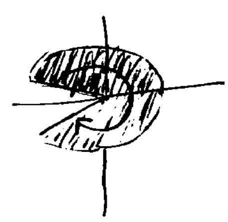
Find δ s.t. $|H| \leq 1$ for z w/ $|z| > \delta$

H bounded on $\overline{B(0,\delta)}$, so H bounded. Hence, by Liouville, H is constant.

Since it tends to zero, $H=0$, so:

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n^2)}$$

<u>Infinite Products</u>:



Branch cut: $\Omega = \mathbb{C} \setminus \{r \in \mathbb{R} : r \leq 0\}$

Define $\mathrm{Log}(\omega) = \displaystyle\int_\gamma \frac{dz}{z}$ where γ is some curve from 1 to ω.

$$\underset{r>0,\ \theta \in (-\pi,\pi)}{\mathrm{Log}(re^{i\theta})} = \ln(r) + i\theta, \quad \theta = \mathrm{Arg}(re^{i\theta})$$

$\prod_{n=0}^{\infty} b_n$, $b_n \in \mathbb{C}$

This product <u>converges</u> iff (1) $\{n : b_n = 0\}$ is finite ← a matter of "logical hygene"

(2) "partial products" of nonzero entries converge

~~If $\prod_{n=0}^{\infty} b_n$ converges, then $b_n \to 1$ as $n \to \infty$.~~ Actually if $\prod_{n=0}^{\infty} b_n$ converges <u>to a nonzero value</u>, then $b_n \to 1$.

Let $b_n = 1 + a_n$, then $a_n \to 0$

As $b_n \to 1$, $\mathrm{Log}(b_n)$ exists for all large n.

Analyze $\prod_n (1+a_n)$, $a_n \longrightarrow 0$, $1+a_n \in \Omega$ for all n.

Consider $\sum_n \mathrm{Log}(1+a_n)$.

<u>Theorem:</u> $\prod_n (1+a_n)$ converges to a nonzero value $\Longleftrightarrow \sum_n \mathrm{Log}(1+a_n)$ converges.

<u>Proof ($\Longleftarrow$):</u> If $\sum_n \mathrm{Log}(1+a_n) = S$ and $S_n = \sum_{i=0}^{n} \mathrm{Log}(1+a_i)$

$\exp(S_n) = P_n = \prod_{i=0}^{n}(1+a_i)$. Exp is continuous, so $P_n \to \exp(S) = P$.

<u>($\Longrightarrow$):</u> $P_n \longrightarrow P$, and $P_n/P \longrightarrow 1$.

Thus $\mathrm{Log}(P_n/P) \Rightarrow \mathrm{Log}(1) = 0$.

$$\exp(\mathrm{Log}(P_n/P) - S_n + \mathrm{Log}(P))$$
$$= \frac{P_n}{P} \cdot \frac{1}{\exp(S_n)} P = 1$$

So then $\mathrm{Log}(P_n/P) - S_n + \mathrm{Log}(P) = h_n 2\pi i$, $h_n \in \mathbb{Z}$

$$(h_{n+1} - h_n)(2\pi i) = \mathrm{Log}\left(\frac{P_{n+1}}{P}\right) - \mathrm{Log}\left(\frac{P_n}{P}\right) - \mathrm{Log}(1+a_{n+1})$$

As RHS $\longrightarrow 0$ with n, then $h_{n+1} - h_n \longrightarrow 0$ with n as well.

Since $h_n \in \mathbb{Z}$ for all n, then $h_n = h$ is constant for large n.

$$\mathrm{Log}\left(\frac{P_n}{P}\right) - S_n + \mathrm{Log}(P) = h 2\pi i, \quad h \in \mathbb{Z}.$$

$n \longrightarrow \infty \Longrightarrow \mathrm{Log}(P_n/P) \longrightarrow 0$, $S = \mathrm{Log}(P) - 2\pi i h$ ∮ $h \in \mathbb{Z}$.

<u>Defn:</u> $\prod_n (1+a_n) \neq 0$, $a_n \to 0$, $1+a_n \in \Omega$ with $\Omega = \mathbb{C} \setminus \{r \in \mathbb{R}, r \leq 0\}$

This product is <u>absolutely</u> <u>convergent</u> $\Longleftrightarrow \sum_n \mathrm{Log}(1+a_n)$ is also convergent absolutely.

Taylor series for $\mathrm{Log}(1+z)$ around zero, has radius of convergence $= 1$.

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

$\lim_{z \to 0} \frac{\log(1+z)}{z} = 1$. If $a_n \to 0$, then for all $\varepsilon > 0$ for all large n,

$$(1-\varepsilon)|a_n| \leq |\mathrm{Log}(1+a_n)| \leq (1+\varepsilon)|a_n|.$$

Hence, we conclude that

$$\prod_n (1+a_n) \text{ absolutely convergent} \iff \sum_n \log(1+a_n) \text{ absolutely convergent}$$
$$\iff \sum_n a_n \text{ absolutely convergent.}$$

An analysis of entire functions:

Easy: if g is entire, then $\exp(g(z))$ is also entire and has no zeroes.

Fact: If h is entire and has no zeroes, then $h=\exp(g(z))$ for some entire g.

Proof: Use an old result to choose g as a holomorphic logarithm of h, defined on whole of $\mathbb{C}$.

Let h be entire, h has finitely many zeroes. Assume $h(0)=0$ with order $m \geq 0$, and zeroes at $a_1, \dots, a_N$, including multiplicities.

Let $g(z) = \dfrac{h(z)}{z^m \prod_{i=1}^{N} (1-\frac{z}{a_i})}$. The denominator is a polynomial in z, and after removing removable singularities, g is an entire function with no zeroes.

So $g=\exp(f(z))$ for some entire f, hence

$$h = z^m \prod_{i=1}^{N} (a_i - z) \exp(f(z)).$$

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Recall: $\prod(1+a_n)$ converges absolutely $\iff \sum a_n$ converges absolutely.

Entire functions:

Recall: if f is an entire function with no zeros, then there is an entire g with $f(z)=\exp(g(z))$

Suppose f is entire, f has a zero of order m at $z=0$, and zeros $a_1, \dots, a_n, (a_i \neq 0)$.

Consider $g = \dfrac{f}{z^m \prod_{i=1}^{n} (1-\frac{z}{a_i})}$. Removing singularities at zeroes of f, get entire function with no zeros. $g=\exp(h(z))$ for some entire h.

$$f = z^m \prod_i (1-z/a_i) \exp(h(z)).$$

What if our function has infinitely many zeros?

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Let $(a_i : i \in \mathbb{N})$ be a sequence of nonzero complex numbers, and let $a_i \to \infty$.

Want to make an entire function with zeros at a_i.

<u>FAILED ATTEMPT</u>: $\prod \left(1 - \frac{z}{a_i}\right)$

This converges absolutely & uniformly on every closed disk $\overline{B(0,R)}$ $\Leftrightarrow$ $\sum \frac{1}{|a_i|}$ converges absolutely, which it ~~will~~ may not

What about

$\operatorname{Log}\left(1 - \frac{z}{a_i}\right)$ for $z \in B(0, |a_i|)$?

Taylor series is $\quad -\frac{z}{a_i} - \frac{1}{2}\left(\frac{z}{a_i}\right)^2 - \frac{1}{3}\left(\frac{z}{a_i}\right)^3 - \cdots$

Let m_i be a natural number, and let $p_i(z) = \frac{z}{a_i} + \frac{1}{2}\left(\frac{z}{a_i}\right)^2 + \cdots + \frac{1}{m_i}\left(\frac{z}{a_i}\right)^{m_i}$.

$$\left|\operatorname{Log}\left(1 - \frac{z}{a_i}\right) + p_i(z)\right| = \left|\frac{1}{m_i+1}\left(\frac{z}{a_i}\right)^{m_i+1} + \frac{1}{m_i+2}\left(\frac{z}{a_i}\right)^{m_i+2} + \cdots\right| \qquad \text{with } |z| < |a_i|$$

$$\le \frac{1}{m_i+1} \sum_{n=m_i+1}^{\infty} \frac{|z|^n}{|a_i|^n}$$

$$= \frac{1}{m_i+1} \frac{|z|^{m_i+1}}{|a_i|^{m_i+1}} \frac{1}{1 - \frac{|z|}{|a_i|}} \qquad ★$$

Fix R, n s.t. $|a_i| \ge 2R$ for $i \ge n$.

For $z \in \overline{B(0,R)}$ and $i \ge n$, choose $m_i = i$, so ★ decays exponentially by comparison test using sum of logs, above bound ★

So $\quad \sum_{i=n}^{\infty} \left(\log\left(1 - \frac{z}{a_i}\right) + p_i(z)\right)$ converges absolutely, uniformly.

Hence, $\prod_{i=0}^{\infty} \cancel{\log}\left(1 - \frac{z}{a_i}\right) e^{p_i(z)}$ converges absolutely and uniformly.

We may conclude $\prod_{i=0}^{\infty} \left(1 - \frac{z}{a_i}\right) e^{p_i(z)}$ is entire, has zeros exactly at $\{a_i : i \in \mathbb{N}\}$.

Recall: $\dfrac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \dfrac{1}{(z-n)^2}$

Consider the entire function $\sin(\pi z) = \dfrac{e^{i\pi z} - e^{-i\pi z}}{2i}$.

$$\sin(\pi z) = z \prod_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \left(1-\frac{z}{n}\right) e^{z/n} \xrightarrow[\text{to}]{\text{Corresponds}} \log\left(1-\frac{z}{n}\right) + \frac{z}{n} = \frac{1}{2}\left(\frac{z}{n}\right)^2 + \dots$$

Converges absolutely to on $B(0,R)$ for all R.

Then $\dfrac{\sin(\pi z)}{z \prod_{n\neq 0}(1-\frac{z}{n})e^{z/n}}$ is entire, has no zeros, hence $= g(z)$ for some entire $g(z)$

$$\sin(\pi z) = \exp(g(z))\, z \prod_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \left(1-\frac{z}{n}\right) e^{z/n}$$

What is the logarithmic derivative of $\sin(\pi z)$?

$$\frac{d}{dz}\log(\sin(\pi z)) = \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \pi\cot(\pi z).$$

To find $g(z)$, use the logarithmic derivative.

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$$\pi\cot(\pi z) = \frac{1}{z} + g'(z) + \sum_{n\neq 0}\left(\frac{-1/n}{1-z/n} + \frac{1}{n}\right) = \frac{1}{z} + g'(z) + \sum_{n\neq 0} \frac{z}{n(z-n)}$$

Let $H = 1/z + \sum_{n\neq 0} \dfrac{z}{n(z-n)}$ $\quad H'(z) = -1/z^2 + \sum_{n\neq 0} -1/(z-n)^2 = \dfrac{-\pi^2}{\sin^2(\pi z)}$ ← from before.

$$\frac{d}{dz}\pi\cot(\pi z) = \frac{-\pi^2}{\sin^2(\pi z)} \qquad \text{Hence } \frac{d}{dz}\left(H - \pi\cot(\pi z)\right) = 0$$

$H - \pi\cot(\pi z) = K$ $\exists$ constant K.

Note that the LHS is an odd function.
Therefore, K is an odd function, but constant, so $K=0$.

Hence, $g'(z)=0$. So $g(z)=\ell$ for some constant ℓ.

$$\sin(\pi z) = z e^{\ell} \prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}$$

divide both sides, take limit as $z\to 0$, to get $e^{\ell} = \pi$.

Detour into Functional Analysis

Schwarz Lemma: Let f be holomorphic on $B(0,1)$. If $|f(z)| \leq 1$ for all z and $f(0)=0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Also: if equality holds (either $|f(z)|=|z|$ $\forall z$ or $|f'(0)|=1$) then $f(z)=cz$ for some c with $|c|=1$.

Proof: Let $r<1$ and consider the behavior of $f(z)/z$ on $\overline{B(0,r)}$. By compactness, $f(z)/z$ has a maximum in the disk. By the maximum principle, finds max on boundary $\{z : |z|=r\}$ of $\overline{B(0,r)}$. So for $|z| \leq r$, so $\frac{|f(z)|}{|z|} \leq \frac{1}{r} \Rightarrow |f(z)| \leq \frac{|z|}{r}$. Let $r \to 1$, and conclude $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. $\longleftarrow$ from difference quotient.

If f also attains max on interior, then f is constant. So equality holding means that f is just constant anyway. ∎