Detour into Functional Analysis

Schwarz Lemma: Let f be holomorphic on $B(0,1)$. If $|f(z)| \leq 1$ for all z and $f(0)=0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Also: if equality holds (either $|f(z)| = |z|$ for z ($z \neq 0$) or $|f'(0)| = 1$) then $f(z) = cz$ for some c with $|c| = 1$.

Proof: Let $r < 1$ and consider the behavior of $f(z)/z$ on $\overline{B(0,r)}$. By compactness, $f(z)/z$ has a maximum in the disk. By the maximum principle, finds max on boundary $\{z : |z| = r\}$ of $\overline{B(0,r)}$. So for $|z| \leq r$, so $\frac{|f(z)|}{|z|} \leq \frac{1}{r} \implies |f(z)| \leq \frac{|z|}{r}$. Let $r \to 1$, and conclude $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. $\longleftarrow$ from difference quotient.

If f also attains max on interior, then f is constant. So equality holding means that f is just constant anyway. ∎

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Recall:

Theorem (Weierstrauss): If (f_n) is a sequence of functions, each holomorphic on Ω, and for all compact $E \subseteq \Omega$ $(f_n \restriction E)$ converges uniformly then $f_n \to f$ where f is holomorphic

Defn: Let $\Omega \subseteq \mathbb{C}$ be a region. Let $\mathcal{F}$ be a family of continuous functions from Ω to Y. Then $\mathcal{F}$ is normal $\iff$ for every (f_n), $f_n \in \mathcal{F}$ there is a subsequence which is uniformly convergent on all compact $E \subseteq \Omega$.

Y is some metric space.

Recall: TFAE for a metric space X

(1) X is compact.

(2) X is sequentially compact.

(3) X is complete and totally bounded.

Defn: Let $A \subseteq X$, X a metric space. Then X is pre-compact $\iff \overline{A}$ is compact.

($\iff$ covered by some compact set)

Easy Fact: A is pre-compact $\iff$ every sequence from A has convergent subsequence.

Claim: If Ω is a region, then there exists an increasing sequence of compact sets $(E_k)_{k\in\mathbb{N}}$ such that

(1) $\bigcup_{k\in\mathbb{N}} E_k = \Omega$

(2) Every compact $E \subseteq \Omega$ is a subset of E_k for some k.

Proof: Let $E_k = \{z \in \Omega \mid |z| \le k \text{ and } B(z, 1/k) \subseteq \Omega\}$

Clearly $E_k \subseteq E_{k+1}$, and clearly E_k is bounded, so enough to show E_k closed. Show E_k closed by noting that it is $\overline{B(0,k)}$ minus a lot of open sets which are together not bad enough to keep it from being closed.

Hence E_k is compact, and note that $\bigcup_{k\in\mathbb{N}} E_k \subseteq \Omega$, and for any $z \in \Omega$, since Ω open, there is k s.t. $B(z, 1/k) \subseteq \Omega$, and $|z|$ finite, so $z \in$ some E_k.

Hence $\Omega \subseteq \bigcup_{k\in\mathbb{N}} E_k$, and therefore we find (1). (2) easily follows.

[Cover E by open sets, each in some E_k, use compactness w/ open covers]

Simplifying Assumption Y is a metric space of diameter 1, $d(x,y) \le 1 \ \forall x,y \in Y$.

Introduce metric ρ on $C(\Omega, Y) := \{f : \Omega \to Y \mid f \text{ continuous}\}$, as follows:

Let $f, g \in C(\Omega, Y)$. For each k, let $\delta_k(f,g) = \max\{d_Y(f(x), g(x)) : x \in E_k\}$

$$\rho(f,g) = \sum_{k=0}^{\infty} \frac{\delta_k(f,g)}{2^k}.$$

Fact: $f_n \to f$ in ρ metric $\iff$ $f_n \to f$ uniformly on compact $E \subseteq \Omega$.

Proof ($\Longrightarrow$): Let $f_n \xrightarrow{\rho} f$. Let $E \subseteq \Omega$ compact. Let $\varepsilon > 0$.

Choose k such that $E \subseteq E_k$. Find N such that $\rho(f_n, f) < \varepsilon/2^k$ for $n \ge N$.

$2^{-k}\delta_k(f_n, f) < 2^{-k}\varepsilon$ for all $n \ge N \implies \max\{d(f_n(x), f(x)) : x \in E_k\} \le \varepsilon \ \ \forall n \ge N$.

continued $\longrightarrow$

converge uniformly

Proof: ($\Longleftarrow$) Let $f_n \rightrightarrows f$ on compact $E \subseteq \Omega$. Fix $\varepsilon > 0$.

Let K be so large that $\sum_{k=K+1}^{\infty} 2^{-k} < \varepsilon/2$. Then $f_n \rightrightarrows f$ on E_K (uniformly),

So for all $k \le K$, $\delta_k(f_n, f) \le \delta_K(f_n, f)$. Hence,

$$\rho(f_n, f) = \sum_{k=0}^{\infty} \frac{\delta_k(f_n,f)}{2^k} = \sum_{k=0}^{K} \frac{\delta_k(f_n,f)}{2^k} + \sum_{k=K+1}^{\infty} \frac{\delta_k(f_n,f)}{2^k} \le \sum_{k=0}^{K} \frac{\delta_K(f_n,f)}{2^k} + \sum_{k=K+1}^{\infty} 1/2^k$$

$$\le \delta_K(f_n, f) \frac{(2^{-K-1} - 1)}{(2^{-1} - 1)} + \varepsilon/2.$$

Choose N so large that $\delta_K(f_n, f) < \dfrac{\varepsilon/2}{\sum_{k=0}^{K} 2^{-k}}$, for all $n \ge N$. ■

Payoff: Normality = Precompactness in ρ-metric.

Defn: A family $\mathcal{G}$ of functions $f : X \to Y$ is (uniformly) equicontinuous $\Longleftrightarrow$ for all $\varepsilon > 0$ there is $\delta > 0$ s.t. for all $g \in \mathcal{G}$, $x, x' \in X$

$$d_X(x, x') < \delta \Longrightarrow d_Y(g(x), g(x')) < \varepsilon$$

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<u>Theorem</u> (Arzela-Ascoli): The following are equivalent for a family $\mathcal{F}$ of continuous functions $\Omega \to Y$, $\Omega \subseteq \mathbb{C}$ a region, Y a metric space:

(1) $\mathcal{F}$ is normal

(2) $\mathcal{F}$ is equicontinuous on compact sets, and for all $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is pre-compact.

<u>Proofs</u>

<u>Last time</u>: $\mathcal{F}$ normal $\Longrightarrow$ F equicontinuous on compact sets. To show that $\{f(z) : f \in \mathcal{F}\}$ is precompact, it is enough to show any sequence from this set has a convergent subsequence. Given $(f_n(z))$ with $f_n \in \mathcal{F}$, there is an infinite subsequence which converges uniformly on compact sets. $\{z\}$ is compact, so (1) $\Rightarrow$ (2).

<u>(2) $\Longrightarrow$ (1)</u>: Let $(f_n)_{n \in \mathbb{N}}$, $f_n \in \mathcal{F}$. Let $\Omega_0 = \{a+bi : a+bi \in \Omega,\ a, b \in \mathbb{Q}\} = \Omega \cap \mathbb{Q}(i)$

Ω_0 is countable and dense in Ω. As Ω_0 is countable, $\{f(z) : f \in \mathcal{F}\}$ is precompact, an easy diagonal argument lets us find a subsequence $(f_{n_i})_{i \in \mathbb{N}}$ s.t. $(g_i(z))_{i \in \mathbb{N}}$ converges at each $z \in \Omega_0$, where $f_{n_i} = g_i$.

Let $E \subseteq \Omega$ be compact, and let $\varepsilon > 0$. Fix $\delta > 0$ s.t. $\forall z, z' \in E$, $|z - z'| < \delta$ $\Longrightarrow$ $\forall f \in \mathcal{F}$, $d(f(z), f(z')) < \varepsilon/3$. Cover E by a finite set of open disks $D_1, \dots, D_k$ such that radius$(D_i) < \delta/2$ and $\overline{D_i} \subseteq \Omega$. Choose for each i some $z_i \in \Omega_0 \cap D_i$. $(g_n(z_i))_{n \in \mathbb{N}}$ is Cauchy for each i, $1 \leq i \leq k$.

So there is $N \in \mathbb{N}$ s.t. for all i, for all $m, n \geq N$, $d(g_m(z_i), g_n(z_i)) < \varepsilon/3$.

Let $z \in E$ and find i s.t. $z \in D_i$. As $z, z_i \in D_i$, and radius of $D_i < \delta/2$, $|z - z_i| < \delta$.
For $m, n \geq N$

$$d(g_m(z), g_n(z)) \leq d(g_m(z), g_m(z_i)) + d(g_m(z_i), g_n(z_i)) + d(g_n(z_i), g_n(z))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So $(g_m(z))_{m \in \mathbb{N}}$ is Cauchy. $\longrightarrow$

$(g_m(z))_{m\in\mathbb{N}}$ is Cauchy inside a compact set (complete + totally bounded)

so $g_m(z) \longrightarrow g(z)$ for some g.

AND, $d(g(z), g_N(z)) \leq \varepsilon$ for all $z \in E$. ■

Consider the family of functions $\mathcal{F}$, each of which is continuous from Ω to $\mathbb{C}$.
$\mathcal{F}$ is normal, $\overset{\text{iff}}{\Longleftrightarrow}$ (by Arzela-Ascoli), $\mathcal{F}$ is equicontinuous on compact sets, furthermore for all $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is precompact $\Rightarrow$ bounded.

<u>Claim</u>: $\mathcal{F}$ normal $\Longleftrightarrow$ $\mathcal{F}$ is both equicontinuous and bounded on all compact $E \subseteq \Omega$.

<u>Proof</u>: ($\Longrightarrow$) Let $\mathcal{F}$ be Normal (equicontinuous and bounded at each point)

Fix $E \subseteq \Omega$ compact. Fix δ such that if $z, z' \in E$ and $|z - z'| < \delta \Longrightarrow |f(z) - f(z')| < 1$ for all $f \in \mathcal{F}$. Cover E w/ finitely many open disks $B(z_i, \delta/2)$, $1 \leq i \leq k$, and insist $\overline{B(z_i, \delta/2)} \subseteq \Omega$. By hypothesis, for each i, there is M_i s.t. $|f(z_i)| \leq M_i$ for all $f \in \mathcal{F}$. Let $M = \max_i M_i + 1$, then easy to see $|f(z)| \leq M$ for all $z \in E$, $f \in \mathcal{F}$.

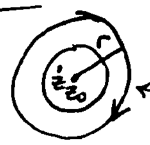
($\Longleftarrow$) Apply Arzela Ascoli, bounded on all compact $E \subseteq \Omega \Longrightarrow$ bounded at each point $\{z\}$. ■

<u>Theorem</u>: If $\mathcal{F}$ is a family of holomorphic functions from Ω to $\mathbb{C}$, then $\mathcal{F}$ is normal $\Longleftrightarrow$ $\mathcal{F}$ is bounded on each $E \subseteq \Omega$.

<u>Proof</u>: Let $E \subseteq \Omega$ compact, $\varepsilon > 0$. Choose M so that $|f(z)| \leq M$ for all $f \in \mathcal{F}$, $z \in E$. (Unfinished)

<u>Goal</u>: For $\mathcal{F}$ a family of holomorphic functions $f:\Omega\to\mathbb{C}$, uniform boundedness on compact $\Omega \subseteq E \Longrightarrow$ equicontinuous on compact $E \subseteq \Omega$.

<u>Proof</u>: $M = \max_{z \text{ in } C_2} |f(z)|$



$$|f(z) - f(z_0)| \le 4M\frac{|z-z_0|}{r} \quad (*)$$

Fix $\varepsilon > 0$, and finitely many ζ_k, r_k, M_k s.t. $B(\zeta_k, r_k/4)$ cover E, $\overline{B(\zeta_k, r_k)} \subseteq \Omega$, $\max\limits_{z:|z-\zeta_k|=r_k} |f(z)| = M_k$ (for <u>all</u> $f \in \mathcal{F}$)

Let $r = \min r_k > 0$, $M = \max M_k$

Let $\delta < r/4, \frac{\varepsilon r}{M}$, with $\delta > 0$. Let $z, z_0 \in E$ s.t. $|z - z_0| < \delta$. Fix k such that $|z - \zeta_k| < \frac{r_k}{4}$, so $|z_0 - \zeta_k| \le |z_0 - z| + |z - \zeta_k| < r_k/4 + r_k/2 = \frac{r_k}{2}$

Thus, $z, z_0 \in B(\zeta_k, r_k/2)$. Use the estimate $(*)$:

$$|f(z) - f(z_0)| \le 4M_k \frac{|z-z_0|}{r_k} \le \frac{4M\, \varepsilon r/4M}{r} = \varepsilon. \quad (\underline{\text{All of the above at} \le \text{sign}})$$

■

<u>Hurwitz Theorem</u>. Let Ω be a region and $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions, and (f_n) converges uniformly on compact $E \subseteq \Omega$, say $f_n \to f$. Suppose that each f_n is nonzero throughout Ω. Then <u>either</u> $f = 0$ on Ω or f is never zero in Ω.

<u>Proof</u>: If f is not identically zero on Ω, then zeroes of f are isolated. Let $z \in \Omega$, $f(z) = 0$. Let $r > 0$ be small enough so that z is the only zero of f in $\overline{B(z,r)}$. Let C be $\{z : |\zeta - z| = r\} = \delta B(z,r)$. f is nonzero on C, and C compact, so $\exists M > 0$, $|f(z)| \ge M$ on $z \in C$. Easy to see $f_n' \to f'$ uniformly on compact sets by Weierstrauss. Also, $1/f_n \to 1/f$ uniformly on C. So:

in particular, C

$$\underbrace{\frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}\,dz}_{\#\text{zeros of } f_n = 0,\ \text{by assumption}} \longrightarrow \underbrace{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz}_{\#\text{ zeros of } f} \Longrightarrow f \text{ has no zeros on } \Omega.$$

■

Exercise: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of injective, holomorphic functions, ~~from the unit circle~~ uniformly convergent on compact sets to a function f.

Prove: f is injective or constant.

03/31/14

Given a family $\mathcal{F}$ of holomorphic (meromorphic?) functions $f:\Omega \to \mathbb{C}$, can view them as functions to $\mathbb{C}\cup\{\infty\}$, with metric on the riemann sphere. We reserve the word "normal" for families normal in this sense.

Theorem: TFAE: (1) $\mathcal{F}$ is normal

(2) For every $(f_n)_{n\in\mathbb{N}}$, $f_n\in\mathcal{F}$, there is a subsequence which either (a) converges uniformly on compact sets to some holomorphic $f:\Omega\to\mathbb{C}$ or (b) converges uniformly to infinity on compact sets (that is for all compact $E\subseteq\Omega$ $\forall M$ there is N, s.t. $|f_n(z)|>M$ for all $n\geq N$, $z\in E$).

converges to constant function $g(z)\equiv\infty$.

Proof: ~~[illegible]~~

Find a subsequence converging uniformly on compact sets w.r.t. the spherical metric. If (a), (b) both fail, converges to meromorphic functions. Let the limit be $f:\Omega\to\mathbb{C}\cup\{\infty\}$. Let $f(a)=\infty$. As f is not constant w/ value ∞, there is $\delta>0$ s.t. a is the only $z\in B(a,\delta)$ $f(z)=\infty$ and we may assume $|f(z)|\geq 1$ for all $z\in B(a,\delta)\setminus\{a\}$.

By uniform convergence, we may find N s.t. for $n\geq N$, $z\in B(a,\delta)$ we have $|f_n(z)|\geq 1/2$. Consider $1/f_n \to 1/f$ uniformly on compact subsets of $B(a,\delta)$. Since f_n ~~is~~ bounded away from zero, not $=\infty$ at $z\neq a$, $1/f_n$ has no zeroes in $B(a,\delta)$. Contradicts Hurwitz Theorem. ∎

Riemann Mapping Theorem: Let Ω be simply connected subset of $\mathbb{C}$.
Find a bijection, holomorphic, $f: \Omega \to B(0,1)$, w/ holomorphic inverse.

We'll fix $z_0 \in \Omega$ and find f such that $f(z_0) = 0$, $f'(z_0)$ is positive real.
It turns out there is exactly one such f.

Define a class of functions $\mathcal{F} = \{ f: \Omega \to \mathbb{C} \mid f$ holomorphic, injective, $|f(z)| \le 1 \ \forall z \in \Omega$ and $f(z) = 0$, $f'(z_0) \in \mathbb{R}$, positive$\}$.

Plan:

(1) $\mathcal{F} \neq \emptyset$
(2) There is $f \in \mathcal{F}$ which maximizes $f'(z_0)$
(3) This f works.

Proof of (1): Let $a \in \mathbb{C}$, $a \notin \Omega$. Since Ω is simply connected, $z - a \neq 0$ for all $z \in \Omega$, there is $h: \Omega \to \mathbb{C}$ such that $h(z)^2 = z - a$, for all $z \in \Omega$. (Choose a square root)

Note that $h(z_1) = h(z_2)$ or $h(z_1) = -h(z_2) \implies z_1 - a = z_2 - a \implies z_1 = z_2$
Injective. Also, not constant. Also if $h(z) = x$, then $h(z) \neq -x$ for any $z \in \Omega$.

So $h(\Omega)$ contains an open ball $B(h(z_0), \delta)$, $\delta > 0$. Hence $h(\Omega) \cap B(-h(z_0), \delta) = \emptyset$.
That is, for all $z \in \Omega$, $|h(z) + h(z_0)| \ge \delta$. In particular, $2|h(z_0)| \ge \delta$.

We will produce a nonzero $K \in \mathbb{C}$ such that $f(z) = K \dfrac{h(z) - h(z_0)}{h(z) + h(z_0)}$ is in $\mathcal{F}$.

Since h holomorphic, $|h(z) + h(z_0)| \neq 0$, f holomorphic.
Since h injective, f injective.
Also, f vanishes at z_0.

Need to show $|f(z)| \le 1$ for all $z \in \Omega$.

$$|f(z)| = |K|\,|h(z_0)| \cdot \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \le |K|\,|h(z_0)|\, 4/\delta$$

Choosing K small satisfies $|f(z)| \le 1$.

So now we need $f'(z_0) \in \mathbb{R}$ and > 0.

$\longrightarrow$

Proof of (1) continued:

$$f'(z) = K\left(\frac{h'(z)(h(z)+h(z_0)) - h'(z)(h(z)-h(z_0))}{(h(z)+h(z_0))^2}\right)_{\text{at } z=z_0}$$

$$= \frac{K\,2h'(z_0)h(z_0)}{4\,h(z_0)^2} = K\frac{h'(z_0)}{2h(z_0)}$$

Since $h(z)^2 = z-a$, $2h(z)h'(z)=1$, so $h'(z)\neq 0$ throughout Ω.

Can choose K to rotate $\frac{h'(z_0)}{2h(z_0)}$ around in complex plane to lie on $\mathbb{R}^+$.

Hence $f\in\mathcal{F}$. ■

Let ~~$B=\sup\{f'(z_0): f\in\mathcal{F}$~~

Proof of (2):

Let $B=\sup\{f'(z_0): f\in\mathcal{F}\}$.

Choose (f_n) such that (a) $f_n'(z_0)\longrightarrow B$

(b) f_n converges uniformly on compact subsets of Ω.

Say $f_n\longrightarrow f$, $|f(z)|\leq 1$ on all $z\in\Omega$.

Also $f_n'\longrightarrow f'$ on compact sets, and in particular $f'(z_0)=B<\infty$.

Also f is injective (by Hurwitz theorem), so $f\in\mathcal{F}$. ■

Proof of (3): By maximum principle, $|f(z)|<1$ for all $z\in\Omega$; else, it assumes maximum on boundary of Ω, so it is constant throughout Ω, yet f is injective, so this cannot be.

Recall from HW 1:

If $|a| < 1$ $(a \in B(0,1))$ then the map $S_a : z \longmapsto \dfrac{z-a}{1-\bar{a}z}$ is an invertible holomorphic map from $B(0,1)$ to $\Delta = B(0,1)$, with inverse $S_a^{-1} = ?$ ✓ who cares.

Then $S_a' = \dfrac{1-|a|^2}{(1-\bar{a}z)^2} = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$ $\quad S_a'(a) = \dfrac{1}{1-|a|^2}$.

Riemann Mapping Theorem; Proof part (3):

Let $f \in \mathcal{F}$ with $|f'(z_0)| = B$ maximal among $f \in \mathcal{F} = \{f$ injective + holomorphic, $f(z_0) = 0$, $|f(z)| \leq 1$ $\forall z$, $f'(z_0)$ real, positive$\}$.

Suppose for contradiction that $\exists a \in \Delta$, $f(z) \neq a$ for all $z \in \Omega$.

Notice that $S_a \circ f$ nonzero on Ω, where $S_a \circ f(z) = \dfrac{f(z) - a}{1 - \bar{a} f(z)}$

We may find G holomorphic on Ω such that $G(z)^2 = S_a \circ f$. Note that

(*) $G(z_0)^2 = S_a(f(z_0)) = S_a(0) = -a$. Now differentiate:

(**) $2\,G(z)\,G'(z) = f'(z)\,S_a'(f(z)) = \dfrac{f'(z)(1-|a|^2)}{(1-\bar{a}f(z))^2} \Longrightarrow 2\,G(z_0)\,G'(z_0) = \overset{f'(z_0)}{B}(1-|a|^2)$

f is an injection $\Omega \longleftrightarrow \Delta$, $S_a : \Delta \xrightarrow{\sim} \Delta$, so $S_a \circ f$ injective and therefore G is injective as well.

Let $G(z_0) = w_0$, and let $F = S_{w_0} \circ G$. F is holomorphic, injective, $F(z_0) = 0$ and $|F(z)| \leq 1$ b/c of defn of S_{w_0}, G.

So compute $F'(z) = G'(z)\dfrac{1-|w_0|^2}{(1-\bar{w}\,G(z))^2}$

At $z = z_0$, $F'(z_0) = \dfrac{G'(z_0)}{1-|w_0|^2}$

By (*) and (**), we see that $2w_0\,G'(z_0) = B(1-|a|^2)$

$w_0^2 = -a \Longrightarrow |w_0|^2 = |a|$.

$\longrightarrow$

proof continued:

$$F'(z_0) = \frac{G'(z_0)}{1-|w_0|^2} = \frac{1}{1-|a|} \; \frac{B(1-|a|^2)}{2w_0} = \frac{B(1+|a|)}{2w_0}$$

$$|F'(z_0)| = \frac{B(1+|w_0|^2)}{2|w_0|} > B$$

Finally forming λF for suitable λ with $|\lambda| = 1$, we obtain $\lambda F \in \mathcal{F}$, $\lambda F'(z_0) > B$, contradicts value of B as max value for derivatives of $f \in \mathcal{F}$. So the chosen f must work. ∎

Fact: There is a unique biholomorphic $f: \Omega \to \Delta$ such that $f(z_0) = 0$, $f'(z_0)$ real, positive.

Proof: Let f_1, f_2 be such functions and consider $h = f_2 \circ f_1^{-1}: \Delta \to \Delta$.
h is a holomorphism (isomorphism, biholomorphic) $\Delta \to \Delta$, $h(0) = 0$.
Applying the Schwarz lemma to both h and h^{-1}, as there must be $a \in \Delta$ $|h(a)| = |a|$, there is λ, $|\lambda| = 1$ s.t. $h(z) = \lambda z$ for all $z \in \Delta$.
This implies uniqueness using $f'(z_0)$ real, positive. ∎

Example: Let $H = \{z = a + bi \in \mathbb{C},\ b > 0\}$

Let $g(z) = \frac{z-i}{z+i}$. Then g is an holomorphism from H to Δ.

04/04/14

Periodic Functions:

Let $\omega \in \mathbb{C}$, $\omega \neq 0$. f is <u>periodic</u> with period ω iff $f(z) = f(z+\omega)$ for all z.

E.g. $\exp(2\pi i z/\omega)$

Let f be meromorphic on Ω, where $z \in \Omega \Rightarrow z+\omega, z-\omega \in \Omega$.

Let $\Omega' = \{\cancel{\exp(z)}\ \exp(2\pi i z/\omega) : z \in \Omega\}$, Ω' also a region.

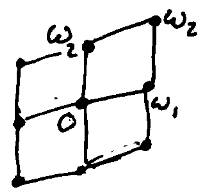
Every f which is meromorphic on Ω which is meromorphic and has period ω is of the form $g \circ \exp(2\pi i z/\omega)$, g meromorphic on Ω'.

Elliptic Functions:

Let $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. Consider f such that $f(z) = f(z+\omega_1) = f(z+\omega_2)$, f meromorphic on $\mathbb{C}$.

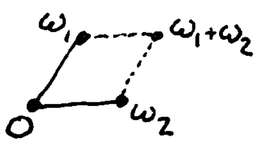
<u>Fact 1</u>: if f is holomorphic, then f is constant.

Let $\Lambda = \{m\omega_1 + n\omega_2 : m,n \in \mathbb{Z}\}$. Algebraically, $\Lambda \cong \mathbb{Z}^2$ as an additive group.
Geometrically, Λ is a lattice.



For all $z \in \mathbb{C}$, $\lambda \in \Lambda$, $f(z) = f(z+\lambda)$.

<u>Proof of Fact 1:</u> For all $z \in \mathbb{C}$, there are $s,t \in \mathbb{R}$ s.t. $s,t \in [0,1]$, $z - s\omega_1 - t\omega_2 \in \Lambda$.
The elements of $\mathbb{C}/\Lambda$ (as a group) can be represented by an element in



Let P be the closed parallelogram $\{r\omega_1 + s\omega_2 : 0 \le r,s \le 1\}$.
P compact $\Rightarrow f$ bounded on $P \Rightarrow f$ bounded on $\mathbb{C} \xRightarrow{\text{Liouville.}} f$ constant. ■

Weierstrass $\wp$-function:

Let $\omega_1, \omega_2, \Lambda$ be as above. The Weierstrass $\wp$-function is:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

As on the homework, $\wp$ is meromorphic, poles at $z \in \Lambda$.

Fact: $\wp$ is periodic.

pf: $\wp'(z) = -2\sum_{\lambda\in\Lambda}\frac{1}{(z-\lambda)^3}$. $\wp$ is even and $\wp'$ is odd, by symmetry of Λ.

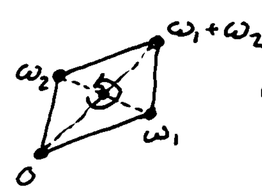
$\wp'$ is periodic with periods ω_1 and ω_2.

Consider $\left(\wp(z+\omega_1)-\wp(z)\right)' = \wp'(z+\omega_1)-\wp'(z) = 0 \Longrightarrow \wp(z+\omega_1)-\wp(z)$ constant.

$\underset{\text{even}}{\wp(\omega_1/2) = \wp(-\omega_1/2)} = \wp(\omega_1-\omega_1/2) \Longrightarrow$ constant $=0$.

Some reasoning shows $\wp$ is ω_2-periodic. ■

$\wp(z) = \wp(\omega_1+\omega_2-z)$ by periodicity $\Longrightarrow$



rotationally symmetric around the center of the parallelogram.

Facts:

The set $\{f : f$ meromorphic over $\mathbb{C}$ and Λ periodic$\}$ is a field, and generated over $\mathbb{C}$ by $\wp$ and $\wp'$; that is, $\mathbb{C}(\wp,\wp')$ is the field of Λ-periodic meromorphic functions.

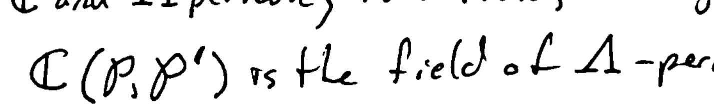
In analysing poles + zeros in the parallelogram, identify such poles and zeros if they differ by an element of Λ.

To count zeros + poles, integrate around boundary of parallelogram.

Fact 1: For any f as above, sum of residues at poles of f in Ω is zero.

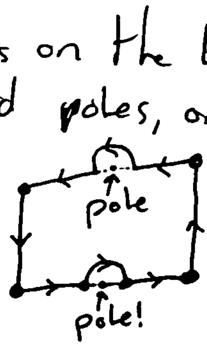
Proof: If f has no poles on boundary of Ω = parallelogram, integrate

$\int f$. By periodicity, $\int f + \int f = 0$ and $\int f + \int f = 0$.



Then use Residue Theorem.

If there are poles on the boundary of Ω, either shift whole parallelogram a tiny bit to avoid poles, or integrate avoiding poles by tiny ~~circles~~ half-poles.



■

Fact 2: Counting w/ multiplicity and identifying poles and zeros if congruent mod Λ, f has same number of poles and zeros in P.

Proof: Apply Fact 1 to f'/f.

Example: For all $c \in \mathbb{C}$, $\wp(z) - c$ has one pole of order 2 in the parallelogram, so $\wp$ assumes value c twice in parallelogram.

04/07/14

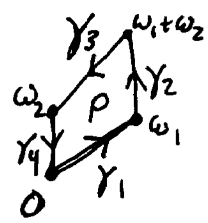
Let $E_\Lambda = \{f : f \text{ meromorphic on } \mathbb{C} \text{ and } f(z) = f(z+\lambda)\ \forall \lambda \in \Lambda\}$. Let P be the fundamental parallelogram.

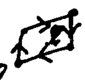
Recall: if $f \in E_\Lambda$, then f has the same number of zeros as poles in the fundamental parallelogram.

Fact: If $f \in E_\Lambda$, then $\displaystyle\sum_{p \text{ pole of } f} p - \sum_{z \text{ zero of } f} z \in \Lambda$ (disregarding multiplicity, counting mod Λ)

Proof: Assume for simplicity there are no poles/zeros on boundary. Consider

$$\frac{1}{2\pi i}\int z \frac{f'(z)}{f(z)}\, dz$$



$P \rightarrow$ 

$$\frac{1}{2\pi i}\left(\int_{\gamma_1} z\frac{f'(z)}{f(z)}\,dz + \int_{\gamma_3} z\frac{f'(z)}{f(z)}\,dz\right) = \frac{-\omega_2}{2\pi i}\int_{\gamma_1}\frac{f'(z)}{f(z)} = -\omega_2\underbrace{(n(\Gamma_1, 0))}_{\text{integer}} \qquad \Gamma_1 = f\circ\gamma_1$$

Similarly

$$\frac{1}{2\pi i}\left(\int_{\gamma_2} z\frac{f'(z)}{f(z)}\,dz + \int_{\gamma_4} z\frac{f'(z)}{f(z)}\,dz\right) = \frac{-\omega_1}{2\pi i}\int_{\gamma_2}\frac{f'(z)}{f(z)} = -\omega_1\underbrace{(n(\Gamma_2, 0))}_{\text{integer}} \qquad \Gamma_2 = f\circ\gamma_2$$

$$\Longrightarrow \quad \frac{1}{2\pi i}\int z\frac{f'(z)}{f(z)}\,dz = -\omega_2(\text{integer}) - \omega_1(\text{integer}) \in \Lambda. \quad \blacksquare$$

Recall: $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$ $\qquad \wp'(z) = \sum_{\lambda\in\Lambda} \frac{-2}{(z-\lambda)^3}$.

For each c, $\wp - c$ has two zeros in P, and $\wp' - c$ has three zeros in P.

$\wp'(z) = -\wp'(-z) = \underbrace{\cancel{\quad}}_{-\wp'(\lambda - z)} \implies \wp'$ has zero at $\omega_1/2$, $\frac{\omega_2}{2}$, $\frac{\omega_1+\omega_2}{2}$.

Since $\wp'$ has at most 3 zeros in P, then these are the only zeros mod Λ. Each has order 1 as a zero.

Define: $\quad e_1 = \wp(\frac{\omega_1}{2})$, $e_2 = \wp(\frac{\omega_2}{2})$, $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$.

$\wp$ assumes each value e_i twice (i.e. $\wp - e_i$ has zeros of order two at the relevant point).

Consider $f = \frac{(\wp')^2}{(\wp-e_1)(\wp-e_2)(\wp-e_3)}$. Numerator and denominator have same poles, namely points of Λ. Poles have same order (six).

Moreover, top+bottom have same zeros $\left(\Lambda + \{\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}\right)$ with same order (two) at each point.

Hence f is holomorphic, but also doubly periodic $\implies f$ is constant.

Which constant?

We will show that $(\wp')^2 = 4(\wp-e_1)(\wp-e_2)(\wp-e_3)$ $\xrightarrow{\text{so}}$ $f=4$. looks like elliptic curve equation!

N.B: Since $\wp$ assumes each value only twice, e_1, e_2, e_3 distinct.

$\wp - \frac{1}{z^2}$ is holomorphic in a nbhd of 0, so it has a Taylor series:

$$\wp(z) - \frac{1}{z^2} = \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \longrightarrow \wp = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \dots$$

where $G_k = \sum_{\lambda \in \Lambda \setminus \{0\}} 1/\lambda^k$

~~An~~ And therefore $\wp' = \frac{-2}{z^3} + 6G_4 z + 20 G_6 z^3 + \dots$

So
$$\left.\begin{aligned} (\wp')^2 &= \frac{4}{z^6} + \frac{-24 G_6}{z^2} - 80 G_6 + \dots \\ (\wp)^3 &= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15 G_6 + \dots \end{aligned}\right\} (\wp')^2 - 4(\wp)^3 = \frac{-60 G_4}{z^2} - 140 G_6 + \dots$$

$$\underbrace{(\wp')^2 - 4\wp^3 + 60 G_4 \wp + 140 G_6 = \sum_{i=1}^{\infty} a_{2i} z^{2i}}_{\text{as the LHS is periodic (doubly) and RHS is holomorphic}, =0.}$$

where we don't care about the a_i, but only that there are only positive powers of z.

Hence $(\wp')^2 = 4\wp^3 - g_2 \wp - g_3$ where $g_2 = 60 G_4$ and $g_3 = 140 G_6$.

<u>Projective Geometry</u>:

Let K be a field. Define $\mathbb{P}^n(K)$, projective n-space over K, as the set of equivalence classes of $K^{n+1} \setminus \{0\}$ under $x \sim y \iff x = \lambda y \ \exists\ \lambda \neq 0, \lambda \in K$.

Compare to affine space $\mathbb{A}^n(K) = K^n$. $\mathbb{P}^n(K)$ contains a copy of $\mathbb{A}^n(K)$ as $\{(x_1, \dots, x_n, 1) : x_i \in K\}$, but also "points at infinity" of the form $(x_1, \dots, x_n, 0)$.

<u>Examples</u>:

$\mathbb{P}^1(\mathbb{C}) \longleftarrow$ Riemann Sphere

$\mathbb{P}^2(\mathbb{R}) \longleftarrow$ lines through origin.

04/09/14

Algebraic Geometry

A <u>variety</u> in $\mathbb{A}^n(K)$ is the set of common zeroes of a family of polynomials in $K[x_1, \dots, x_n]$.

In $\mathbb{P}^{n+1}(K)$, define <u>projective varieties</u> using families of homogeneous polynomials.

<u>Notes</u> $f \in K[x_1, \dots, x_{n+1}]$ homogeneous, then $f(\lambda \vec{a}) = \lambda^t f(\vec{a})$ where $\deg(f) = t$.

$\mathbb{P}^{n}(K)$ contains a copy of $\mathbb{A}^n(K)$, given by $\{(a_1, \dots, a_n, 1) : (a_1, \dots, a_n) \in K^n\}$.
The other points of $\mathbb{P}^n(K)$, of the form $(a_1, \dots, a_n, 0)$ are "points at ∞".

If $f \in K[x_1, \dots, x_n]$, with degree d, then associate to f a homogeneous polynomial $x_{n+1}^d f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right) \in K[x_1, \dots, x_{n+1}]$. The <u>homogenization</u> of f.

The variety in $\mathbb{P}^n(K)$ determined by the homogenization of f intersected with copy of $\mathbb{A}^n(K)$ is the copy of the variety determined in $\mathbb{A}^n(K)$ by f.

<u>Example</u>: <u>Recall</u>: $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ where e_1, e_2, e_3 are values of $\wp$ at half-lattice points. Consider $y^2 - 4(x - e_1)(x - e_2)(x - e_3) \in \mathbb{C}[x, y]$ determines a curve in $\mathbb{A}^2(\mathbb{C})$.

<u>Recall</u>: $\wp$ assumes each complex value twice in the fundamental parallelogram (i.e. twice in $\mathbb{C}/\Lambda$). For any $x \in \mathbb{C}$, we may find $z \in \mathbb{C}$ s.t. $\wp(z) = x$.
If $\wp(z) = x$, $\wp'(z) = y$, then $\wp(-z) = x$ and $\wp'(-z) = -y$.

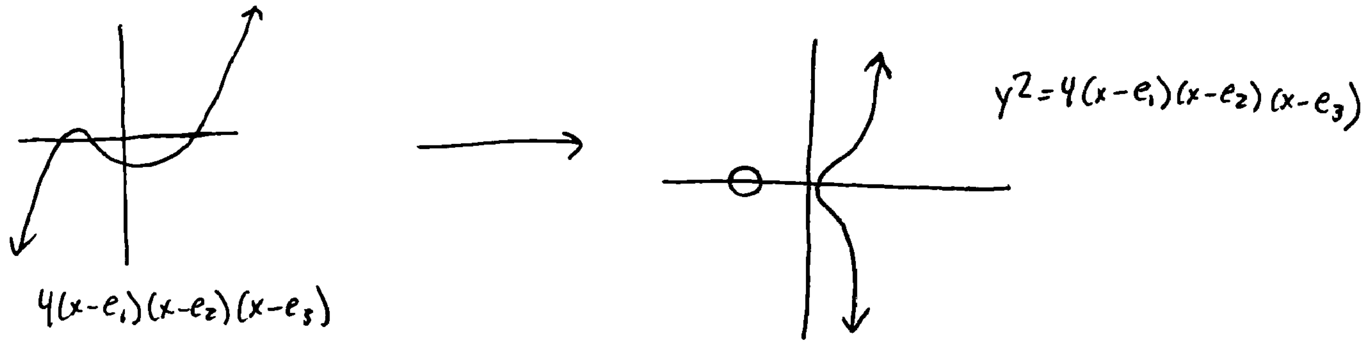
Hence, all points on this curve have their reflection also on the curve.

<u>Homogenizing</u> gives

$$y^2 t - 4(x - e_1 t)(x - e_2 t)(x - e_3 t)$$

Now includes a unique point at ∞ with coordinates $(0,1,0)$.
Corresponds to the point $z = 0$ under $z \mapsto (\wp(z), \wp'(z))$.

If e_1, e_2, e_3 are all real, and $4(x-e_1)(x-e_2)(x-e_3)$ looks like



$z \mapsto (\wp(z), \wp'(z))$ sets up a bijection between the projective curve

$$y^2 t = 4(x-e_1 t)(x-e_2 t)(x-e_3 t)$$

and the torus $\mathbb{C}/\Lambda$. We may think of $\mathbb{C}/\Lambda$ as a topological group.

<u>Idea:</u> Copy the group operation on $\mathbb{C}/\Lambda$ to get the group law on the curve $y^2 t = 4(x-e_1 t)(x-e_2 t)(x-e_3 t)$ in $\mathbb{P}^2(\mathbb{C})$.

But first, <u>lines in projective space:</u> $ax+by=c \xrightarrow{\text{homogenize}} ax+by=ct$. ← plane through origin in affine 3-space.

two of always meet in one point

To add two points P, Q on our curve E.

(1) Draw a line through P and Q.

(2) Has another point of intersection, R ← because of Bézout's Theorem.

(3) Reverse the y-coordinate of R, the result is $P+Q$.

<u>Fact:</u> [redacted] "$(\wp(z), \wp'(z)) + (\wp(w), \wp'(w)) = (\wp(z+w), \wp'(z+w))$" ← should be projective

The same group law given by the parameterization by $\wp$ and $\wp'$.

In fact, this group is isomorphic to $(\mathbb{C}/\Lambda, +) \cong S^1 \times S^1$.

04/14/14

Recall: ω_1, ω_2 linearly independent over $\mathbb{R}$, $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$

$E_\Lambda = \{f \text{ meromorphic} : f(z) = f(z+\lambda) \text{ for all } \lambda \in \Lambda\}$.

$\wp, \wp'$ satisfy $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$

$z \mapsto (\wp(z), \wp'(z))$ sets up a bijection between $\mathbb{C}/\Lambda$ and the projective curve $y^2 t = 4(x - e_1 t)(x - e_2 t)(x - e_3 t)$.

Claim: This map $z \mapsto (\wp(z), \wp'(z))$ is actually an isomorphism between $(\mathbb{C}/\Lambda, +)$ and the geometric group of the elliptic curve.

"Proof:"

Want $(\wp(z), \wp'(z)) + (\wp(w), \wp'(w)) = (\wp(z+w), \wp'(z+w))$.

Assuming (1) $z, w \notin \Lambda$
(2) $z \neq w \mod \Lambda$
(3) $z \neq -w \mod \Lambda$
$\Big\}$ consciously ignore corner cases.

These assumptions imply that the line joining $(\wp(z), \wp'(z)) = P$ and $Q = (\wp(w), \wp'(w))$ has equation $y = mx + b$ $\exists m, b$. Consider the function in E_Λ given by $\wp' - m\wp - b$. Both z and w are zeros of this function.

The function $\wp' - m\wp - b$ has poles of order 1 at all lattice points, and zeros at z, w (b/c line through z, w). Let the third zero be u, let $R = (\wp(u), \wp'(u))$

$P + Q = (\wp(u), -\wp'(u))$, by group law on elliptic curve.

As $g = \wp' - m\wp - b \in E_\Lambda$, sum of zeros − sum of poles $\in \Lambda$ as before. — counting only those zeros and poles in fundamental parallelogram.

Hence, $z + w + u - 0 - 0 - 0 \in \Lambda \implies z + w = -u \mod \Lambda$.

Therefore $\wp(z+w) = \wp(-u) = \wp(u)$

$\wp'(z+w) = \wp'(-u) = -\wp'(u)$. ∎

Fact: Every $g \in E_\Lambda$ can be expressed as an element of $\mathbb{C}(\wp, \wp')$.

Proof: We will show that (a) every even $g \in E_\Lambda$ is just a rational function of $\wp$.
and (b) every $g \in E_\Lambda$ can be written as $g_0 + \wp' g_1$ for $g_0, g_1 \in E_\Lambda$ and are even.

First, ~~[illegible]~~

(a) $\Rightarrow$ (b)

Proof:
$$g = \underbrace{\frac{g(z)+g(-z)}{2}}_{\text{even}} + \underbrace{\wp'\left(\frac{g(z) \mp g(-z)}{2\wp'}\right)}_{\text{even}}$$

Proof of (a): Let $g \in E_\Lambda$, g even. We will produce g^* a rational function of $\wp$ with the same poles and zeros ~~[illegible]~~ as g.

$\frac{g}{g^*}$ is holomorphic, elliptic $\Longrightarrow$ constant, so g is constant times rational.

It remains to find g^*. Enumerate nonzero zeros of g in the fundamental parallelogram, up to congruence mod Λ.

Divide these zeros into two classes:

Type I: $\{a \mid g(a)=0 \text{ and } a \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2} \bmod \Lambda\}$

Type II: $\{a \mid g(a)=0 \text{ and } a = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2} \bmod \Lambda\}$

- If a is type I, $\omega_1+\omega_2-a$ is a distinct type I zero. they come in pairs, list each pair only once: one per pair.
- If a is type II zero, it is of even order (by evenness and periodicity), Enumerate a exactly $\frac{m}{2}$ times, where m is its order.

So let $a_1, \dots, a_t$ enumerate the zeros of g as above.
Similarly, enumerate poles by $b_1, \dots, b_s$ in the same way.

$\longrightarrow$

Proof of (a) continued:

Define $g^* = \dfrac{\prod_{i=1}^{t} (\wp(z) - \wp(a_i))}{\prod_{j=1}^{s} (\wp(z) - \wp(b_j))}$. Check that the zeros and poles of g and g^* match, using the ~~harmonic~~ elliptic properties of $\wp, \wp'$.

As $g \in E_\Lambda$, g has the same number of poles and zeros, up to multiplicity, in the fundamental parallelogram. This means we also match zeros and poles on the corners of the parallelogram. ■

Example: $\wp(2z)$ has poles at half lattice points, in addition to regular lattice points.

04/16/14

Let $SL_2(\mathbb{Z})$ act on $\Lambda = \langle \omega_1, \omega_2 \rangle \subseteq \mathbb{C}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$. act on generators of the lattice as a group.

Defn: Lattices Λ_1, Λ_2 are similar iff there is nonzero $c \in \mathbb{C}$ s.t. $\Lambda_2 = c\Lambda_1$. (scale + rotate one to another)

If Λ_1, Λ_2 are similar lattices, the theory of E_{Λ_1} is the same as that for E_{Λ_2}.

Recall: $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$.

We can see that every lattice is similar to one of the form $\langle 1, \tau \rangle$ where $\tau \in \mathbb{H}$, but not uniquely.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, so $\langle 1, \tau \rangle = \langle a\tau + b, c\tau + d \rangle$ which is similar to $\left\langle 1, \dfrac{a\tau + b}{c\tau + d} \right\rangle$.

Recall: If we build $\wp$ from the lattice Λ,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3 \qquad g_2 = 60G_4 \quad g_3 = 140G_6 \qquad G_{2k} = \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}}.$$

If $\Lambda = \langle 1, \tau \rangle$, view G_4 and G_6 as functions of τ.

$$G_{2k}(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2k}}, \quad k \geq 2.$$

Since τ is in $\mathbb{H}$, $G_{2k}(\tau)$ is holomorphic on $\mathbb{H}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = G_{2k}\left(\left\langle 1, \frac{a\tau+b}{c\tau+d}\right\rangle\right) = G_{2k}\left(\frac{1}{c\tau+d}\langle 1, a\tau+b\rangle\right) = (c\tau+d)^{2k} G_{2k}(\tau)$$

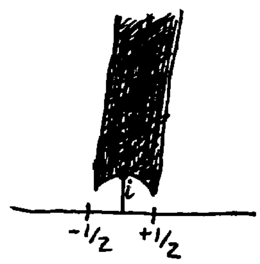
Notice: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, action on a lattice $\langle 1, \tau \rangle$ is $\langle 1, \tau \rangle \longmapsto \langle 1, \tau+1 \rangle$

Hence, $G_{2k}(\tau) = G_{2k}(\tau+1)$.

Recall: Defining for $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathbb{H}$, $\Gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ gives a group action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$.

We want to understand the orbits of this action.

Defn: The fundamental Region is $\{z \in \mathbb{H} : |z| \geq 1, \, Re(z) \in (-1/2, 1/2)\}$



We will see that for any $z \in \mathbb{H}$, there is $\Gamma \in SL_2(\mathbb{Z})$ such that $\Gamma \cdot z$ is in the fundamental region. Additionally, if w, z are distinct points in the interior of the fundamental region, there is no $\Gamma \in SL_2(\mathbb{Z})$ such that $\Gamma w = z$.

Fact: $SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$

$\tau \mapsto \tau+1$ $\qquad$ $\tau \mapsto -1/\tau$.

N.B. Since $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are both the identity with this action, then this action is not faithful.

Proof: Let $G \leq SL_2(\mathbb{Z})$ be the subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\tau \in \mathbb{H}$.

If $\Gamma \in SL_2(\mathbb{Z})$, $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Im(\Gamma \cdot \tau) = \dfrac{\Gamma \cdot \tau - \overline{\Gamma \cdot \tau}}{2i} = \dfrac{\frac{a\tau+b}{c\tau+d} - \frac{a\bar{\tau}+b}{c\bar{\tau}+d}}{2i}$

$= \dfrac{(ad-bc)(\tau-\bar{\tau})}{2i|c\tau+d|^2} = \dfrac{Im(\tau)}{|c\tau+d|^2}$. Fix $\tau \in \mathbb{H}$. Since $\tau \in \mathbb{H}$, if either c or d is large, then $|c\tau+d|$ is also large.

Let $\Gamma \cdot \tau$ be such that $\mathrm{Im}(\Gamma\cdot\tau)$ be maximal among $\{\mathrm{Im}(\delta\cdot\tau) : \delta \in G\}$
$(\Gamma \in G)$

Let O be the orbit of τ under action by G: $\{\Gamma\cdot\tau \mid \Gamma \in G\}$.
Then ~~$\{\tau' : \tau' \in O\}$~~ $\{\mathrm{Im}(\tau') : \tau' \in O\}$ has a maximum value, say at r.
Let $\tau' = \gamma\cdot\tau \in O$ be such that $\mathrm{Im}(\tau') = r$. Then $|\tau'| \geq 1$ because else if $|\tau'| < 1$, acting on it with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ get τ'', $\mathrm{Im}(\tau'') = \dfrac{\mathrm{Im}(\tau')}{|\tau'|^2} > \mathrm{Im}(\tau$
Act on τ' by some power of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we can find τ'' in O such that $\mathrm{Re}(\tau'') \in (-1/2, 1/2)$ and $\mathrm{Im}(\tau'') = \mathrm{Im}(\tau') = r \geq 1$.

Claim: if $\tau_0, \tau_1 \in \mathring{F}$, and $\tau_1 = \gamma\tau_0$ for some $\gamma \in G$, $\tau_0 = \tau_1$.

Proof: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma^{-1} = \begin{pmatrix} d & b \\ -c & -a \end{pmatrix}$

$$\mathrm{Im}(\tau_1) = \frac{\mathrm{Im}(\tau_0)}{|c\tau_0 + d|^2} \qquad \mathrm{Im}(\tau_0) = \frac{\mathrm{Im}(\tau_1)}{|-c\tau_1 + a|^2}$$

For points in the interior, $\mathrm{Im}(\tau_1) \geq 1$ and $\mathrm{Im}(\tau_0) \geq 1$.
If $\mathrm{Im}(\tau_1) = \mathrm{Im}(\tau_0)$, then they differ by a real translate, but the real translate is ≥ 1, so they are equal.
Else, WLOG $\mathrm{Im}(\tau_1) > \mathrm{Im}(\tau_0)$ so the two equations above have $|c\tau_0 + d|^2 > 1$ and $|-c\tau_1 + a^2| > 1$, so we get a contradiction by combining them. ∎

Claim: A point in $\mathring{F}$ is not in the same orbit as a point on the boundary. (Follows from previous claim)

Claim: $G = SL_2(\mathbb{Z})$

Proof: Let $\gamma \in SL_2(\mathbb{Z})$ be arbitrary. Let $\tau = \gamma\cdot(2i)$. Arguing as above there is $\delta \in G$ s.t. $\delta\cdot\tau = \delta\gamma\cdot 2i \in F$. So in fact $(\delta\gamma)\cdot 2i = 2i$. Let $\delta\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\mathrm{Im}(2i) = \dfrac{\mathrm{Im}(2i)}{|2ic + d|^2} \implies c = 0,\ d = \pm 1,\ a = d = \pm 1,\ b = 0$ ← matrix translates $2i$ to $2i$ ∎

Recall: $\Lambda = \{m\omega_1 + n\omega_2 : n, m \in \mathbb{Z}\}$.

From Λ we defined $\wp$. $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$

As Λ is similar to $\Lambda_\tau = \langle 1, \tau \rangle$, if we view G_{2k} as a function on τ, then G_{2k} is holomorphic on H and $G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau)$.

G_{2k} is periodic with period 1: choose $a = b = d = 1$, $c = 0$. So we may view G_{2k} as a function of $q = e^{2\pi i \tau}$. This change of variables maps the upper half plane $\mathbb{H}$ to the punctured unit disk, $B(0,1) \setminus \{0\}$

Fact: As a function of q, G_{2k} is meromorphic at 0.

Recall: $e_1 = \wp(\omega_1/2)$ $e_2 = \wp(\omega_2/2)$ $e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$, all three points are distinct

Let $\lambda = \dfrac{e_3 - e_2}{e_1 - e_2}$. $\lambda \in \mathbb{C}$, $\lambda \neq 0, \lambda \neq 1$.

Let $\Lambda = \langle 1, \tau \rangle$ and view λ as a function of τ, where $\tau = \frac{\omega_2}{\omega_1}$.

So $\lambda\left(\frac{a\tau + b}{c\tau + d}\right) \stackrel{?}{=} \ldots$

Defn: $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$ is the "congruence subgroup"

Can also consider as subgroup of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

04/21/14

(Little) Picard's Theorm: Let $f: \mathbb{C} \to \mathbb{C}$ be nonconstant, holomorphic and entire, then for each $z \in \mathbb{C} \setminus \{z_0\}$, there is w s.t. $f(w) = z$, for some z_0.

"f omits at most one value"

Let $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, from Λ, define $\wp$. $e_1 = \wp(\frac{\omega_1}{2})$ $e_2 = \wp(\frac{\omega_2}{2})$ $e_3 = \wp(\frac{\omega_1 + \omega_2}{2})$

$\lambda = \dfrac{e_3 - e_2}{e_1 - e_2}$, λ depends on choice of generators for lattice. By an easy homogeneity argument, λ depends only on $\tau = \frac{\omega_2}{\omega_1}$.

Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\left.\begin{array}{l} \omega_1' = a\omega_1 + b\omega_2 \\ \omega_2' = c\omega_1 + d\omega_2 \end{array}\right\} \Longrightarrow$ generate same lattice as ω_1, ω_2 (same $\wp$-function).

In general, $e_j' \neq e_j$. However, if $\begin{array}{l} a \equiv d \equiv 1 \mod 2 \\ b \equiv c \equiv 0 \mod 2 \end{array}$ then $e_j = e_j' \Longrightarrow \lambda$ unchanged.

Recall:

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}$$

$\phi : SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/2\mathbb{Z})$ natural HM, $\Gamma(2) = \ker(\phi)$

If λ is a function of $\tau = \frac{\omega_2}{\omega_1}$, then λ has the property $\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau)$ for matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$.

As e_1, e_2, e_3 are distinct for $\tau \in \mathbb{H}$, λ omits values $0, 1$ considered as a function holomorphic on $\mathbb{H}$.

The group of transformations of $\mathbb{H}$ of the form $\tau \longmapsto \frac{a\tau + b}{c\tau + d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ is generated by transformations $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Question: How do they affect the half-lattice points? (mod Λ)

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $\quad \frac{\omega_1}{2} \longmapsto \frac{\omega_1}{2} \quad \frac{\omega_2}{2} \longmapsto \frac{\omega_1 + \omega_2}{2} \quad \frac{\omega_1 + \omega_1}{2} \longmapsto \frac{\omega_2}{2} \Longrightarrow$ e_1 fixed, e_2, e_3 switch

$$\lambda = \frac{e_3 - e_2}{e_1 - e_2} \longmapsto \frac{\lambda}{\lambda - 1} = \frac{e_2 - e_3}{e_1 - e_3}$$

Therefore, $\lambda(\tau + 1) = \dfrac{\lambda(\tau)}{\lambda(\tau) - 1}$

Similarly, $\lambda(-1/\tau) = 1 - \lambda(\tau)$.

Note that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma(2)$, so easily $\lambda(\tau) = \lambda(\tau + 2)$. Let $q = e^{\pi i \tau}$, consider λ as a function of q. As $e^{\pi i \tau}$ ~~has a maps~~ maps $\mathbb{H}$ to $B(0,1) \setminus \{0\}$, then as function of q, λ is defined on $B(0,1) \setminus \{0\}$, but has a removable singularity at $q = 0$, with value 0. "$\lambda(i\infty) = 0$".

Recall: $$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{m=-\infty}^{\infty} \frac{1}{(z-m)^2}.$$

Recall also:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Let $\Lambda = \langle 1, \tau \rangle$, so $e_1 = \wp(1/2)$ $e_2 = \wp(\tau/2)$, $e_3 = \wp(\frac{1+\tau}{2})$. Hence

$$e_3 - e_2 = \wp\left(\tfrac{1+\tau}{2}\right) - \wp(\tau/2) = \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(\lambda - 1/2 - \tau/2)^2} - \frac{1}{(\lambda - \tau/2)^2} \right), \quad \Lambda = \mathbb{Z}[\tau].$$

$$= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left(\frac{1}{(m - 1/2 + (n-1/2)\tau)^2} - \frac{1}{(m + (n-1/2)\tau)^2} \right)$$

$$= \pi^2 \sum_{n \in \mathbb{Z}} \left(\frac{1}{\cos^2((n-1/2)\pi\tau)} - \frac{1}{\sin^2((n-1/2)\pi\tau)} \right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{m \in \mathbb{Z}} \frac{1}{(z-m)^2}$$

Similarly, $e_1 - e_2 = \pi^2 \sum_{n \in \mathbb{Z}} \left(\frac{1}{\cos^2(n\pi\tau)} - \frac{1}{\sin^2((n-1/2)\pi\tau)} \right)$.

as $\tau \to i\infty$ then

Now substitute $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, then we may see that as $\tau \to i\infty$, $e_3 - e_2 \to 0$, and $e_1 - e_2 \to \pi^2$

So $\lambda \to 0$ as $\tau \to i\infty$.

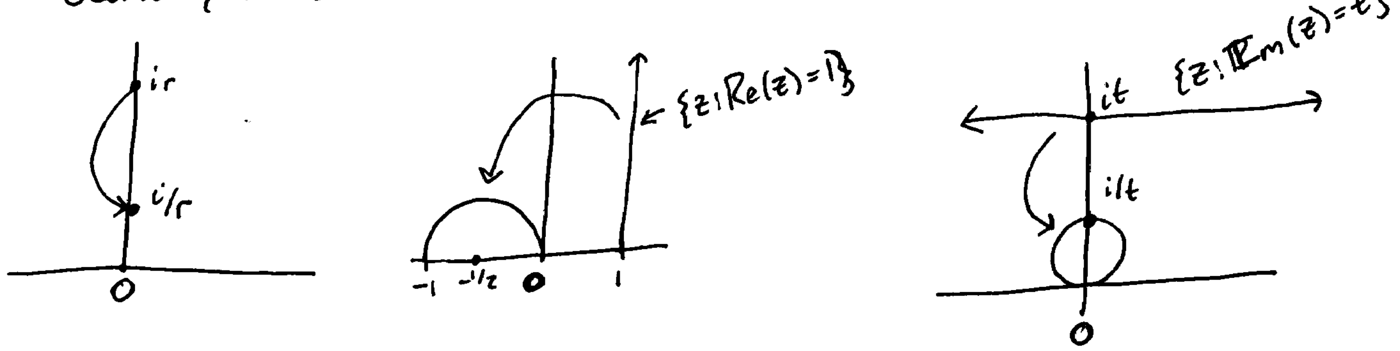
b/c of the $n=0$ term, $\frac{1}{\cos^2(0)} = 1$.

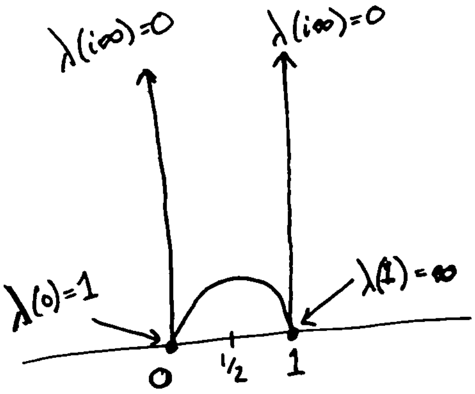
04/23/14

Note that if τ is pure imaginary (i.e. $\mathrm{Re}(\tau) = 0$) then $\lambda(\tau) \in \mathbb{R}$.

Recall: $\lambda(\tau+1) = \frac{\lambda(\tau)}{1 - \lambda(\tau)}$ and $\lambda(1/-\tau) = 1 - \lambda(\tau)$ and $\lambda(\gamma \cdot \tau) = \gamma \cdot \tau$ if $\gamma \in \Gamma(2) \leq SL_2(\mathbb{Z})$

Geometry of $\tau \mapsto -1/\tau$:





On the boundaries of the fundamental region, $\lambda(\tau)$ is real. As $\tau \longrightarrow i\infty$, $\lambda(\tau) \longrightarrow 0$.

$\lambda(-1/\tau) = 1-\lambda(\tau) \Longrightarrow$ as $\tau \longrightarrow 0$ along the imaginary axis, $\lambda(\tau) \longrightarrow 1$.

If we know λ is real along the imaginary axis, then $\lambda(\tau+1) = \dfrac{\lambda(\tau)}{\lambda(\tau)-1} \Longrightarrow \lambda$ real along the line $\{Re(z)=1\}$. Combining this with the other formula gives $\lambda(1-1/\tau) = \dfrac{1-\lambda(\tau)}{-\lambda(\tau)} = 1 - 1/\lambda(\tau)$, so λ is real on the semicircle w/ center $1/2$ and radius 1. This equation also gives $\lambda(1)=\infty$ by letting $\tau \longrightarrow i\infty$.

Recall: since $\lambda(\tau)=\lambda(\tau+2)$, we may write λ as a function of $q=e^{i\pi\tau}$ the "nome"
In this change of coordinates, λ is defined on $B(0,1)\setminus\{0\}$.
Singularity at $q=0$ is removable, $\lambda(0)=0$.

We want a taylor series for λ in terms of q to get an idea of its behavior. In terms of $q=e^{i\pi\tau}$,

$$\frac{1}{\cos^2(x)} = \frac{4}{(e^{ix}+e^{-ix})^2} = \frac{4}{(q^{n-1/2}+q^{1/2-n})^2} = \frac{4q}{(q^n+q^{1-n})^2}$$

$x = \pi(n-1/2)\tau$

$e^{ix} = e^{i\pi(n-1/2)\tau} = q^{n-1/2}$

$$\frac{-1}{\sin^2(x)} = \frac{4q}{(q^n-q^{1-n})^2}$$

$$e_3 - e_2 = \pi^2 \sum_{n\in\mathbb{Z}} \left(\frac{4q}{(q^n+q^{1-n})^2} + \frac{4q}{(q^n-q^{1-n})^2} \right)$$

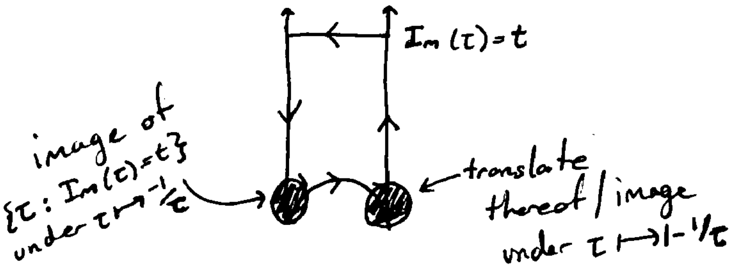
But for the Taylor series, only the $n=0, n=1$ terms contribute, so look at $2\left(\dfrac{4q}{(1+q)^2} + \dfrac{4q}{(1-q)^2}\right)$. In the Taylor series for λ, coefficient of q^1 term is $16q$ $\quad \lambda(\tau)e^{-i\pi\tau} \longrightarrow 16$ as $Im(\tau) \longrightarrow \infty$.

Next goal: λ assumes each value in the upper half plane exactly once in the interior of Ω, where Ω is the fundamental region, and assumes each value in the lower half plane exactly once in the interior of Ω'.

Let $w_0 \in \mathbb{H}$. How many times does λ wind around it? Gives how many times λ hits w_0 by argument principle.

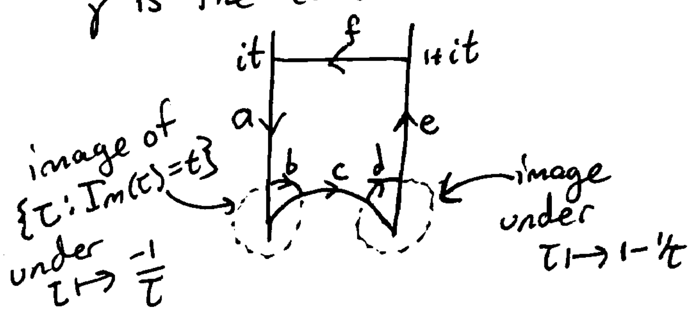
Let γ be the contour

Key points: $\lambda(-1/\tau) = 1 - \lambda(\tau)$
$\lambda(1 - 1/\tau) = 1 - 1/\lambda(\tau)$



04/28/14

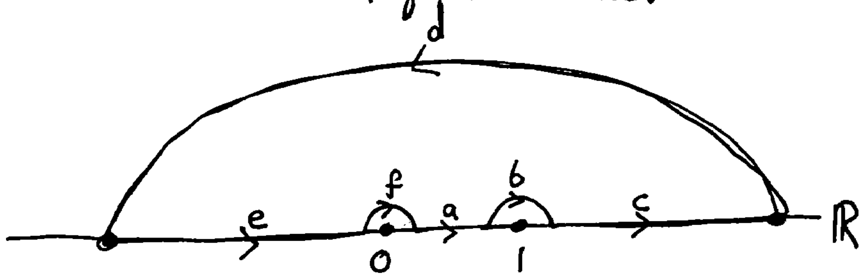
γ is the contour



Let $\Gamma = \lambda \circ \gamma$.

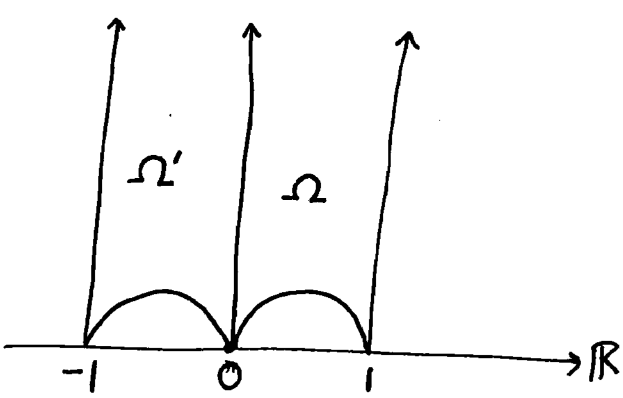
We find that $n(\Gamma, a) = 1$ for each $a \in \mathbb{H} \setminus \{0,1\}$. (see email)

The contor $\Gamma = \lambda \circ \gamma$ looks like:



Where the letters labelling the arrows correspond to labels in previous picture.

Letting $t \to \infty$, the contour $\Gamma = \lambda \circ \gamma$ eventually encloses ~~any a∈H~~ any $a \in \mathbb{H} \setminus \{0,1\}$. To pick up lower half plane as well, let $\Omega' = \Omega - 1 = \{z - 1 : z \in \Omega\}$. By symmetry properties of λ, Ω' is also a fondamental region and λ assumes each value in lower half-plane exactly once in $\mathring{\Omega}'$.



Fact: $\lambda' \neq 0$ at all points on $\partial\Omega \cap \mathbb{H}$.

Proof: Write taylor series for λ. If $\lambda' = 0$, λ can't map points on one side of boundary to lower half plane and points on other side to upper half plane.

because $\lambda' \neq 0$ on $\partial\Omega$, then $\lambda(\tau)$ moves monotonically around Γ as τ moves around γ.

Recall: $\Gamma(2) = \ker\left(SL_2(\mathbb{Z}) \xrightarrow{\text{no hook!}} SL_2(\mathbb{Z}/2\mathbb{Z})\right)$.

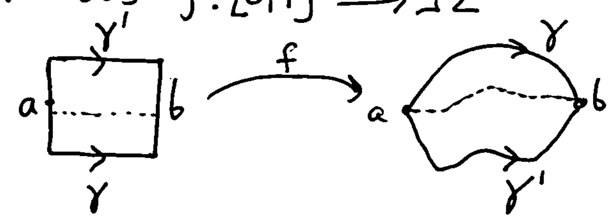
Fact: Every $z \in \mathbb{H}$ can be moved to a point of $\Omega \cup \Omega'$ by some $\gamma \in \Gamma(2)$

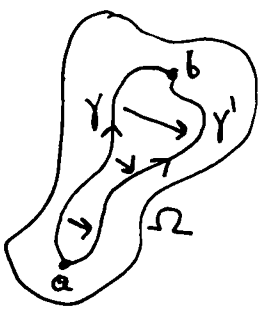
Algebraic Topology Stuff.

Let $\gamma, \gamma' : [0,1] \to \Omega$ be continuous, $\Omega \subseteq \mathbb{C}$ connected, open, and $\gamma(0) = \gamma'(0) = a$ and $\gamma(1) = \gamma'(1) =$

Defn: γ, γ' are homotopic iff there is a continuous $f : [0,1]^2 \to \Omega$ such that

$$f(t,0) = \gamma(t) \qquad f(0,s) = a$$
$$f(t,1) = \gamma'(t) \qquad f(1,s) = b$$





Fact: homotopy is an equivalence relation on $\{\gamma : \gamma(0) = a, \gamma(1) = b\}$.

Defn: γ as above is a loop if $\gamma(0) = \gamma(1) = a$.

Defn: The fundamental group $\pi_1(\Omega, a)$ has elements which are equivalence classes of loops based at a, with operation $\gamma * \delta = \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \delta(2t-1) & t \in [1/2, 1] \end{cases}$. Perform δ after γ.

Fact: If Ω is simply connected, then $\pi_1(\Omega, a) = 1$.

04/30/13 Analytic Continuations

Defn: The pair (f, Ω) is a function element iff Ω is a region and $f : \Omega \to \mathbb{C}$ holomorphic

Fact: If (f_1, Ω_1) is a function element, Ω_2 is a region with $\Omega_1 \cap \Omega_2 \neq \emptyset$, then there is at most one (f_2, Ω_2) s.t. $f_2 \restriction \Omega_1 \cap \Omega_2 = f_1 \restriction \Omega_1 \cap \Omega_2$.

Proof: If there were two candidates (f_2, Ω_2) and (f_2^*, Ω_2), then $f_2 - f_2^*$ is identically zero on $\Omega_1 \cap \Omega_2$, so zero on all of Ω_2. Hence $f_2 = f_2^*$. ∎

Define an equivalence relation on function elements by

$(f,\Omega) \sim (g,\Omega') \iff$ there is a sequence of function elements $((f_n,\Omega_n))_{0\le n\le K}$ such that $(f_0,\Omega_0) = (f,\Omega)$, $(f_n,\Omega_n) = (g,\Omega')$

$$\Omega_i \cap \Omega_{i+1} \neq \phi$$
$$f_i \restriction \Omega_i \cap \Omega_{i+1} = f_{i+1} \restriction \Omega_i \cap \Omega_{i+1}.$$

Defn: An equivalence class is called a global analytic function.

Analytic continuation along a path:

$\gamma : [a,b] \longrightarrow \mathbb{C}$ $\gamma(a)=A$, $\gamma(b)=B$, piecewise differentiable.

(f,Ω) is a function element, $A \in \Omega$.

Formally, this is a continuation along the path as $\{(f_t,\Omega_t) : t\in[a,b]\}$

$\gamma(t) \in \Omega_t$, $\Omega_s \cap \Omega_t \neq \phi \Longrightarrow f_s, f_t$ agree on $\Omega_s \cap \Omega_t$

$f \restriction \Omega \cap \Omega_a = f_a \restriction \Omega \cap \Omega_a$.

Monodromy Theorem: Let Ω be a simply connected region, $A \in \Omega$, (f,Σ) a function element with $A \in \Sigma$, $\Sigma \subseteq \Omega$. Assume that for any path $\gamma : [0,1] \longrightarrow \Omega$ with $\gamma(0)=A$, (f,Σ) can be continued along γ. Then: there is a function element (g,Ω) such that $g \restriction \Sigma = f$.

Key point of proof: As Ω is simply connected, γ is homotopic to any ~~point~~ curve which ends at the same path. So as γ transforms into γ', we can analytically continue f along γ', and this gives a global function g on Ω.

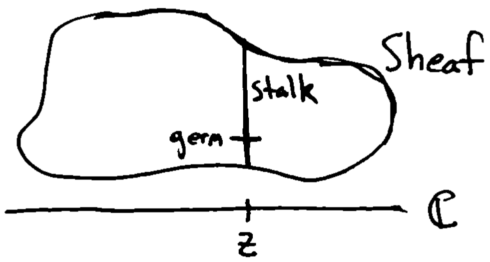
Let $z \in \mathbb{C}$. Introduce an equivalence relation on $\{(f,\Omega) : z \in \Omega\}$, by

$(f,\Omega) \simeq (g,\Sigma) \iff$ there is a region Γ containing z, $\Gamma \subseteq \Omega \cap \Sigma$ such that $f \restriction \Gamma = g \restriction \Gamma$.

The equivalence classes are called germs at z. $\longrightarrow$

The collections of all germs at z is called the <u>stalk</u> at z.
The collections of all germs for all $z \in \mathbb{C}$ is called a <u>sheaf</u>.

Generic useless picture in the spirit of every algebraic geometry book ever.



The stalk over z is a ring.

05/02/14

<u>Little Picard's Theorem</u>: If f is entire and non-constant, then it omits at most one complex value.

<u>Recall</u>: λ omits $0,1$ as a function $\mathbb{H} \to \mathbb{C}$.

<u>Proof</u>: Suppose that f is entire, ~~non-constant~~, and omits two distinct points $a,b \in \mathbb{C}$. Replacing f by $\dfrac{f-a}{f-b}$, we may assume that f omits the points 0 and 1.

We will find $h: \mathbb{C} \to \mathbb{H}$ such that $f(z) = \lambda(h(z))$ for all $z \in \mathbb{C}$, such that h is entire. Composing h with some biholomorphic $\alpha: \mathbb{H} \cong B(0,1)$, $\alpha \circ h$ is a bounded entire function, hence constant. So h is constant, and thus f is constant.

It remains to find h. By the analysis of λ, if $\tau, \tau' \in \mathbb{H}$, $\lambda(\tau) = \lambda(\tau')$, then there is $\gamma \in \Gamma(2)$ s.t. $\gamma \cdot \tau = \tau'$

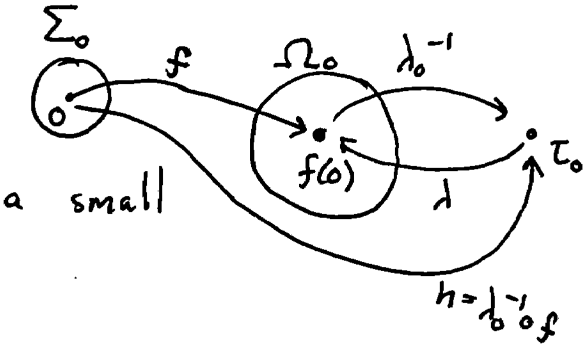
<u>Idea</u>: construct a small piece of h, show it can be extended along any path in $\mathbb{C}$ (which is simply connected), and by Monodromy theorem define a global h.

Choose $\tau_0 \in \mathbb{H}$ s.t. $f(0) = \lambda(\tau_0)$. Possible b/c f omits $0,1$ and λ assumes $\longrightarrow$

proof continued.

every value except $0, 1$. We also know $\lambda' \neq 0$ throughout $\mathbb{H}$.
There is a small open neighborhood Ω_0 of $f(0)$ in which there exists an inverse λ^{-1}_0, and $\lambda_0^{-1}(f(0)) = \tau_0$, by inverse function theorem.

inverse for λ around $f(0)$



Composing, define h to be $\lambda_0^{-1} \circ f$ in a small neighborhood of 0.
Σ_0

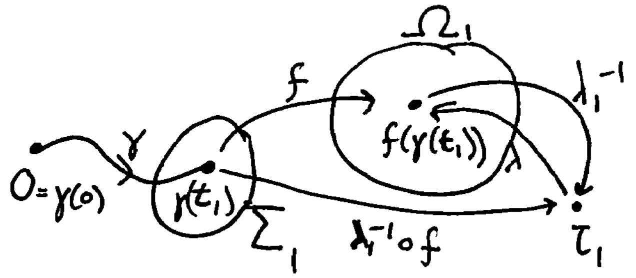
<u>Claim:</u> The function elements $(\lambda_0^{-1} \circ f, \Sigma_0)$ can be extended along any path $\gamma : [0,1] \longrightarrow \mathbb{C}$ such that $\gamma(0) = 0$.

<u>Proof:</u> If not, then there is a path $\gamma : [0, t_1] \longrightarrow \mathbb{C}$ such that

(a) we may extend along $\gamma \restriction [0,s]$ for all $s < t_1$

(b) we cannot extend along γ_1 <u>OR</u> $\mathrm{Im}(h \text{ extended along } \gamma(s)) \xrightarrow[s \to t_1]{} 0$.

Consider $f(\gamma(t_1))$, and find τ_1 such that $\lambda(\tau_1) = f(\gamma(t_1))$. This is okay again, as before.



Again find λ_1^{-1} an inverse of λ defined on $\Omega_1 \ni f(\gamma(t_1))$, $\lambda_1^{-1}(f(\gamma(t_1))) = \tau_1$.

Find a neighborhood Σ_1 of $\gamma(t_1)$, so that we may define $\lambda_1^{-1} \circ f$ on Σ_1.

Choose $t_2 \in (0, t_1)$ so that $\gamma(t_2) \in \Sigma_1$.

Let $\rho = (\text{continuation of } \lambda_0^{-1} \text{ along } \gamma \restriction [0, t_2])(t_2)$

$f(\gamma(t_2)) = \lambda(\rho)$.

Let ~~$\rho^* = (\lambda_1^{-1} \circ f)(t_2)$~~ $\quad \lambda(\rho^*) = f(\gamma(t_2)) = \lambda(\rho)$.

$\rho^* = (\lambda_1^{-1} \circ f)(\gamma(t_2))$

$\longrightarrow$

Since $\lambda(p) = f(\gamma(t_2)) = \lambda(p)$, there is $\delta \in \Gamma(2)$ such that $\delta \cdot p^* = p$.

Define a function element $\left(z \longmapsto \delta \cdot \left(\lambda_1^{-1}(f(z))\right), \Sigma_1\right)$.

Verify that we have extended along γ up to and including t_2, and at t_2 choose preimage for $f(\gamma(t_2))$ lying in $\mathbb{H}$. ■

Riemann Surfaces: (Complex 1D manifolds)

Examples: $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$, $\mathbb{H} \simeq B(0,1)$, $\mathbb{C}/\Lambda$

Uniformization Theorem: A simply connected Riemann surface is isomorphic to one of $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$, $B(0,1)$, and any Riemann surface is isomorphic to a quotient of one of the three simply connected ones by a "discrete subgroup" of its automorphism group.

Eg. $\mathbb{C}/\Lambda$ is the quotient of $\mathbb{C}$ by linearly independent (over $\mathbb{R}$) translations $z \longmapsto \omega_1 + z$, $z \longmapsto \omega_2 + z$.