Differential Geometry and Lie Groups

Classical mechanics governed by $\vec{F} = m\vec{a}$.

Where does this equation come from?

Principle of Least Action $\Longrightarrow$ Newton's Laws

$$\underbrace{S}_{\text{action}} = \int \underbrace{T}_{\substack{\text{kinetic}\\ \text{energy}}} - \underbrace{U}_{\substack{\text{potential}\\ \text{energy}}} \, dt$$

In many systems, very hard to find action

e.g. relativistic $v \approx c$

Fix: we can determine the action up to a constant by symmetries, and the number of such constants (mass, charge, etc) depends on accuracy needed.

A symmetry is a transformation which leaves action invariant.

Consider a single particle moving in a potential field $V(\vec{x})$

$$S = \int \tfrac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \, dt$$

Suppose $V(\vec{x}) = V(|\vec{x}|)$. Then get rotational invariance.

Particle Action $S = \int L(x_i, \dot{x}_i)\, dt$

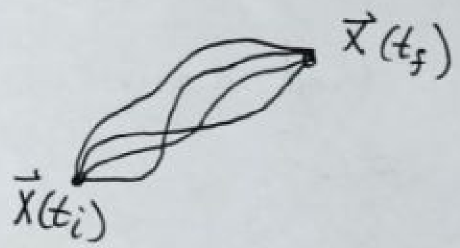
Field Action $S = \int L\left(\phi(\vec{x},t), \frac{\partial}{\partial x_i}\phi(\vec{x},t), \frac{d}{dt}\phi(\vec{x},t)\right) d^n\vec{x}\, dt$

L is the <u>Lagrangian</u>

<u>Equations of Motion:</u>

$S = T - V$

$\vec{x}_i(t) : \mathbb{R} \to \mathbb{R}^3$ parameterizes path of single particle through space.



Hold endpoints fixed & vary the path the particle takes.

$$\delta S = \int_{t_i}^{t_f} \frac{\partial}{\partial t} L(\vec{x}_i(t), \dot{\vec{x}}_i(t))\, dt = 0$$

infinitesimal change in action

principle of least action

$$= \int_{t_i}^{t_f} \frac{\partial L}{\partial \vec{x}_i}\frac{d\vec{x}_i}{dt} + \frac{\partial L}{\partial \dot{\vec{x}}_i}\frac{d\dot{\vec{x}}_i}{dt}\, dt = 0$$

reverse chain rule.

$$= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x_i}\frac{dx_i}{dt} + \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\frac{dx}{dt}\right)}_{\star} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right)\frac{dx_i}{dt} \right) dt = 0$$

Note now that we can integrate $\star$.

$$\int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\frac{dx_i}{dt}\right) dt = \left[\frac{\partial L}{\partial \dot{x}_i}\frac{dx_i}{dt}\right]_{t_i}^{t_f} = 0$$

endpoints are unchanging in time

$$\frac{dx}{dt}(t_f) = \frac{dx}{dt}(t_i) = 0.$$

Therefore, we are left with

$$\int_{t_i}^{t_f} \frac{\partial L}{\partial x_i}\frac{dx_i}{dt} - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\right)\frac{dx_i}{dt}\, dt = 0$$

This must hold for all possible changes to the path. That is, for <u>any</u> $\frac{dx_i}{dt}$. Hence, we must have

$$\boxed{\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}} \leftarrow \text{Euler-Lagrange Equation.}$$

If $L = \frac{1}{2}m\dot{x}^2$, then $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}(m\dot{x}) = 0$

Gives <u>conservation of momentum</u>.

This is a version of Noether's Theorem:

"Continuous symmetries imply conservation laws."

Noether's Theorem

invariances $\Longrightarrow$ conservation laws

translation invariance $\Longrightarrow$ momentum conservation

$\frac{\partial L}{\partial x} = 0$ is translation invariance

So we get $0 = \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)$

$$\Longrightarrow 0 = \frac{d}{dt}(m\dot{x})$$

Exactly the statement of Newton's 2nd Law.

rotational invariance $\Longrightarrow$ angular momentum.

Groups

Define groups by the things they leave invariant.

For example, $O(3)$ leaves magnitude fixed.

What must group elements look like?

$x' = Rx$, $|x'| = |x|$

$$|x'|^2 = x'^T x' = x^T R^T R x = x^T x \Longrightarrow R^T R = I$$

$$\Longrightarrow R^T = R^{-1}.$$

Orthogonal Matrices.

DIFFERENTIAL GEOMETRY

09/03/14

Lie Algebra determines the group locally but not uniquely. Examples: (of Lie groups)

$SO(3)$ has manifold structure $\mathbb{P}^2(\mathbb{R})$

There is another group with the same Lie Algebra but a different manifold structure.

$SU(2)$ has a Lie Algebra isomorphic to that of $SO(3)$. $su(2) \cong so(3)$

Consider $SO(3) = \{ T : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} \mid x \mapsto x' \text{ and } \|x\| = \|x'\| \}$

Let $\vec{x} \in \mathbb{R}^3$

Basis for 2×2 traceless hermitian Matrices is given by Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$$

$$x_i \sigma^i = \begin{bmatrix} x_3 & x_1 + i x_2 \\ x_1 - i x_2 & -x_3 \end{bmatrix} \qquad \det(x_i \sigma^i) = -x_3^2 - x_2^2 - x_1^2 = -\|\vec{x}\|$$

Also, for h ~~traceless~~ hermitian, $\|\vec{x}\| = \det(h(x_i \sigma^i) h^{-1})$

$$\vec{\sigma} \cdot \vec{x} = (h \vec{\sigma} h^{-1}) \cdot \vec{x} = R \vec{x}$$

So there are 2 h's for each R: ($\pm h$ do same thing).

What are h? Well, require $h \sigma h^{-1} \in \{\text{traceless hermitian}\}$

$\mathrm{tr}(h \sigma h^{-1}) = \mathrm{tr}(\sigma) = 0 \qquad (h \sigma h^{-1})^{\dagger} = h \sigma h^{-1}$

$(h\sigma h^{-1})^\dagger = h\sigma h^{-1} \Rightarrow (h^{-1})^\dagger \sigma^\dagger h^\dagger = h\sigma h^{-1} \Rightarrow (h^{-1})^\dagger = h$

So h is unitary. $\det(h) \in U(1)$.

Shows any element in $U(2)$ can be written as $e^{i\theta} SU(2)$. Hence $U(2) \cong U(1) \times SU(2)$

This gives a 2 to 1 map $SU(2) \longrightarrow SO(3)$.
"double cover"

Elements of $SU(2)$ have the form

$$A = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \quad \text{such that } |a|^2 + |b|^2 = 1.$$

if $a = a_1 + a_2$, $b = a_3 + a_4$

$$\det A = 1 \Rightarrow a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$$

gives $SU(2)$ the manifold structure of S^3.

$SO(3) \cong SU(2)/(\mathbb{Z}/(2))$ has manifold structure $\mathbb{P}^2(\mathbb{R})$.

$\mathbb{P}^2(\mathbb{R})$ is B^3 (the 2-sphere and interior, ball in $\mathbb{R}^3$) with antipodal points identified.

Not simply connected:
The path is not contractible

closed path b/c poles are identified

However $SU(2)$ w/ topology of S^2 is simply connected.
"You can't lasso an orange."

n-Manifolds

Given a Hausdorff, Second-countable topological space M, it is a smooth manifold if

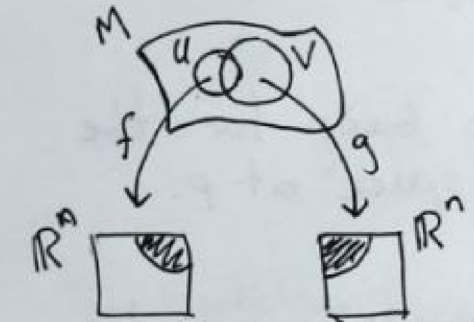
(1) Locally Euclidean:

For each $p \in M$, $\exists\, \mathcal{U} \ni p$ open, $\mathcal{U}$ is homeomorphic to an open subset of $\mathbb{R}^n$ by φ

charts $(\mathcal{U}_\alpha, \varphi_\alpha)$ cover M.

(2) Transition maps between charts are smooth.

$$g \circ f^{-1} : \mathbb{R}^n \to \mathbb{R}^n \text{ is } C^\infty.$$



Example: S^2 must have more than one chart. S^2 is compact, $\mathbb{R}^n$ is not, so a single coordinate system won't do.

09/05/14

ϕ = Phi = φ

Polar coordinates for a sphere don't cover the poles, so cannot do with only one chart here.

Tangent Spaces:

For each $p \in M$, there is a tangent space with all the properties of a vector space.

$$T_pM = \{ \text{linear maps } D : C^\infty(M) \to \mathbb{R} \mid D(fg) = D(f)g(p) + f(p)D(g) \}$$

Vectors on a Manifold:

A curve on a manifold is $\gamma : \mathbb{R} \to M$ that is injective.

The tangent vector to γ at p is $\vec{V}_p = \left.\frac{d\gamma}{dt}\right|_{t=0}$

$\vec{V}_p(f) =$

Choose coordinates so that

$$\gamma_1 = (t, 0 \ldots 0) \qquad \vec{e}_i = \left.\frac{d\gamma_i}{dt}\right|_{t=0}$$
$$\gamma_2 = (0, t, 0 \ldots 0)$$
$$\vdots$$
$$\gamma_n = (0, \ldots, 0, t)$$

Chose a basis for the tangent space at p.

Most abstractly, $\vec{V}_p = \frac{d}{dt}$. After choosing coordinates,

$$\frac{d}{dt} = \frac{dy^i}{dt} \frac{\partial}{\partial y^i} \quad \text{(chain rule)}$$

$$= a^i e_i$$

component basis vector

What about a coordinate transformation $x^i \to y^j$?

$$\frac{dx^i}{dt} = \frac{dy^i}{dt}\frac{dx^i}{dy^j} \qquad \frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

$$\frac{dx^i}{dt}\frac{\partial}{\partial x^i} = \frac{\partial x^i}{\partial y^j}\frac{dy^j}{dt}\frac{\partial y^k}{\partial x^i}\frac{d}{dy^k} = \frac{dy^j}{dt}\frac{\partial}{\partial y^i} \qquad \frac{\partial x^i}{\partial y^j}\frac{\partial y^k}{\partial x^i} = \delta^k_j.$$

Length on Manifolds

At each point p, introduce a symmetric bilinear map to $\mathbb{R}$, called the metric g.

$$g: M^2 \longrightarrow \mathbb{R} \qquad g(\vec{u},\vec{v}) = g(\vec{v},\vec{u}),\ g \text{ bilinear}$$

If $g(\vec{u},\vec{v}) = 0$ for all $\vec{u}$ and fixed $\vec{v}$, then $\vec{v} = 0$.

g is a "tensor" on M, define $g_{ij} = g(e_i, e_j)$

$$g(a^i e_i, b^j e_j) = a^i b^j g_{ij} \longleftarrow \text{"inner product" } \vec{a}\cdot\vec{b}$$

$$dS^2 = g_{ij}\, dx^i dx^j = g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} \longleftarrow \text{components of vector}$$

In Euclidean Space, $dS^2 = \sum_{i=1}^{3} (dx^i)^2 \qquad g_{ij} = \delta_{ij}$

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Vector is a differential on a Manifold.

$$\frac{d}{d\lambda} = A^i \partial_i := A^i e_i \qquad X = \frac{d}{d\lambda} = \frac{\partial x^i}{\partial \lambda}\frac{d}{dx^i} := A^i e_i$$

components ↗ ↖ basis

$$Y = \frac{d}{d\cancel{\mu}} = \frac{dy^i}{d\cancel{\mu}}\frac{d}{dy^i} = A^{i'} e_i'$$

Contravariant (Indices up) $\quad A^{i'} = \frac{dy^i}{d\lambda} = \frac{dy^i}{dx^j}\frac{dx^j}{d\lambda} \longrightarrow A^{i'} = \frac{dy^i}{dx^j} A^j$

Covariant (Indices down) $\quad e_i' = \frac{d}{dy^i} = \frac{dx^j}{dy^i}\frac{d}{dx^j} \Longrightarrow \frac{dx^j}{dy^i} e^j$

Metric: $g(d/d\lambda, d/d\mu) \in \mathbb{R}$

Components: $g_{ij}(e_i, e_j) = g_{ij}$

Coordinate Invariant: $g(A, B) = g(A', B')$

$$g_{ij}(e_i, e_j) = g_{ij}$$

$$g'_{ij} = g_{ij}\left(\frac{\partial x^i}{\partial y^k} e_k, \frac{\partial x^j}{\partial y^\ell} e_\ell\right) = \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial y^\ell} g_{ij}$$

When there is a nontrivial metric, up/down notation for summation:

$$A^i B_i := A^i B^j g_{ij}$$

g_{ij} lowers an index

Example: Metric in Euclidean space is $g_{ij} = \delta_{ij}$

Matrix given by $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\delta_{ij}\delta^{ij}$ is the trace of the identity.

$$\varepsilon_{ijk}\varepsilon_{iab} = A\,\delta_{ja}\delta_{kb} + B\,\delta_{jb}\delta_{ka} + C\,\delta_{jk}\delta_{ab}$$

But $C=0$ because the quantity has to be antisymmetric in j and k.

$$\varepsilon_{ijk}\varepsilon_{iab} = A\delta_{ja}\delta_{kb} + B\delta_{jb}\delta_{ka}$$

Contract with $\delta_{ja}\delta_{kb}$

$$\delta_{ja}\delta_{ja}\delta_{kb}\delta_{kb} = (tr\, I)^2$$

$$\varepsilon_{iab}\varepsilon_{iab} = A\times 3\times 3 + B\times 3$$

$$\delta_{ja}\delta_{jb}\delta_{ka}\delta_{kb} = \delta_{ab}\delta_{ab} = tr\, I$$

$$6 = 9A + 3B \quad ①$$

$\varepsilon_{iab}\varepsilon_{iab}$ = number of permutations of 3 elements

Contract with $\delta_{jb}\delta_{ka}$

$$\varepsilon_{iba}\varepsilon_{iab} = A\times 3 + B\times 3\times 3$$

flip permutation gives a sign

$$-6 = 3A + 9B \quad ②$$

Add ① and ② to get $A = -B$ and $A = 1$.

$$\varepsilon_{ijk}\varepsilon_{iab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}$$

<u>Example:</u> What is $\nabla\times(\nabla\times\vec{A})$?

$$\nabla\times\vec{A} = \varepsilon_{ijk}\partial_j A_k$$

So consider $\nabla\times(\nabla\times\vec{A})_b$ ← b^{th} component.

$$= \partial_a(\varepsilon_{ijk}\partial_j A_k)\varepsilon_{aib} = (\partial_a\partial_j A_k)\varepsilon_{ijk}\varepsilon_{aib}$$

$$= -[\delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}](\partial_a\partial_j A_k) = -\partial_a\partial_a A_b + \partial_k\partial_b A_k$$

$$= \nabla(\nabla\cdot A)_b - \Delta A_b$$

$\mathbb{R}^4$ space-time manifold

At each point we have an "event"

Particles follow world-lines in $\mathbb{R}^4$

$X^\mu(\lambda) \leftarrow$ world line parameterized by λ

Coordinates on this manifold $(\underset{\substack{\uparrow \\ \text{time}}}{x^0}, \underbrace{x^1, x^2, x^3}_{\text{space}})$

This manifold is covered by one chart, trivially.

Inertial frames are those in which a free particles transverse a straight line.

$X^i = V^i t + X_0^i \leftarrow$ constant initial position K_0^i

Use time (X^0) to parameterize curves: $\lambda = X^0$

$$\begin{bmatrix} X^0 \\ X^i \end{bmatrix} = \begin{bmatrix} 1 \\ V^i \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ K^i \end{bmatrix}$$ in coordinate chart $(\phi_1, \mathbb{R}^4)$

In another chart, $(\phi_2, \mathbb{R}^4)$, give particles y-coordinates

Translation between charts is smooth.

"Poincaré Transformation" $\longrightarrow$ $y^\mu = \Lambda^\mu{}_\nu X^\nu + K^\nu$ — Transformation between inertal frames preserves straight lines, hence linear.

Assumptions: (1) Speed of light is the same in all frames. that are inertial.

(2) All inertial frames are indistinguishable.

Then, Poincaré transformations leave

$$(\Delta x)^2 := c^2(\Delta x^0)^2 - \sum_i (\Delta x^i)^2 \quad \text{invariant}$$

Work in units where the speed of light is 1.

<u>Poincaré group</u> is set of transformations leaving $(\Delta x)^2$ invariant.

<u>Lorenz Group</u> is the set of Lorenz boosts, that is, rotations Λ in $y^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + b^{\alpha}$.

<u>Theorem:</u> All poincare transformations leave the inner product $(\Delta t)^2 - (\Delta x)^2$ constant.

Define metric $g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $x^{\mu} y^{\nu} g_{\mu\nu} = \underset{\substack{\uparrow \\ \text{time}}}{x^0 y^0} - \underset{\substack{\uparrow \\ \text{distance.}}}{\vec{x} \cdot \vec{y}}$

The Lorenz group can now be realized as as the set of all transformations which preserve the metric g.

$SO(1,3)$ "proper orthochronous transformations"

doesn't reverse flow of time $\Longrightarrow$ determinant 1.

$\Lambda^0{}_0 > 0$

determinant 1 matrices A such that $A^T \left(\begin{array}{c|c} +1 & 0 \\ \hline 0 & -I_{3\times 3} \end{array}\right) A = I$.

$\cancel{g^{\mu\nu} = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} g^{\rho\sigma} = g^{\mu\nu}}$

* Orthochronous transformations preserve sense of time; and preserve metric.

$$g'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}$$

$\mathbb{R}^4$ endowed with metric $g_{\mu\nu} (:= \eta_{\mu\nu})$ is called "Minkowski Spacetime"

Now we want to find the Laws of motion
- Write down action
- Consistent with symmetries $\qquad \vec{x} = \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$
- Make physical sense

$$S = \int d\lambda \, L(\dot{\vec{x}}, \vec{x})$$

$d\lambda$ parameterizes path through spacetime.

Should be invariant under Poincaré transformations.

Cannot be a function of $\vec{x}$ b/c translation invariant.

Should be invariant under Lorenz transformations.

$\dot{x}^\mu \longrightarrow \Lambda^\mu{}_\nu \dot{x}^\nu$ leaves us with $\int d\lambda \, L(\dot{x}^2)$

What is the simplest Lagrangian now? $L = \dot{x}^2 := x^\mu x^\nu g_{\mu\nu}$

$\hookrightarrow \dot{x}^T g \dot{x}$

guess $\quad S = \int d\lambda \, \dot{x}^2 = \int d\lambda \, \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}$

But this doesn't work, not invariant under change of coordinates:

$$\int \frac{d\lambda}{d\lambda'} d\lambda' \left(\frac{dx^\mu}{d\lambda'} \frac{d\lambda'}{d\lambda} \right)^2 = \int \left(\frac{d\lambda}{d\lambda'} \right) \left(\frac{d\lambda'}{d\lambda} \right)^2 \dot{x}^2 d\lambda' \neq S.$$

But an action which is invariant under change of coordinates is

$$S = m\int \sqrt{\dot{x}^2}\, d\lambda$$

Apply Euler-Lagrange

$$\boxed{\frac{d}{d\lambda}\frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}} = 0}$$

What are the conserved quantities?

Use Noether's theorem.

- translations in t, x, y, z $\longrightarrow$ 4 generators
- Rotate around x, y, z axes $\longrightarrow$ 3 generators
- Lorenz boosts for x, y, z $\longrightarrow$ 3 generators

~~Lorenz~~ Poincaré group should have 10 generators.

For translations,

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu\right) = 0$$ and δx^μ is constant for translation

$\hbar = c = 1$ $[\phi] = 1$ $[\varphi] = 2$ $\Rightarrow [\sqrt{\varphi}] = [\phi]$

So $$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{d}{d\lambda}\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}} = 0$$ call $\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}} := p^\mu$

"four momentum"

In these units, $\hbar = c = 1$.

So S is unitless, $[\text{time}] = [\text{energy}]$

$p^0 =$ "energy"

$p^i =$ "momentum"

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Choose a frame comoving with the particle through spacetime.

$$x^0 = \tau \leftarrow \text{proper time}$$
$$\vec{x} = 0 \leftarrow \text{moving with particle.}$$

Consider the interval between two space-time events. All observers agree on the invariant interval $(\Delta x^0)^2 - (\Delta \vec{x})^2$.

For observer in comoving frame, $(\Delta \cancel{x})^2 = (\Delta \tau)^2$

The action (using the chain rule)
$$S = -m \int \frac{d\lambda}{d\tau} d\tau \sqrt{\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \left(\frac{d\tau}{d\lambda}\right)^2 \eta_{\mu\nu}}$$
$$= -m \int d\tau \sqrt{\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu}}$$

But in this frame, $\frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau} \eta_{\mu,\nu} = \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} \eta_{00} = 1$

$$S = -m \int d\tau \qquad \vec{x} = \begin{pmatrix} \tau \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Consider a lab observer with coordinates $\vec{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$

$$(\Delta \tau)^2 = (\Delta t)^2 - (\Delta \vec{x})^2 \implies \left(\frac{\Delta \tau}{\Delta t}\right)^2 = 1 - \left(\frac{\Delta \vec{x}}{\Delta t}\right)^2$$

So $\frac{d\tau}{dt} = \sqrt{1 - \vec{v}^2}$ $\qquad \vec{v}$ = velocity of particle as seen by lab observer

$$d\tau = \sqrt{1 - \vec{v}^2}\, dt \qquad S_{lab} = -m \int \sqrt{1 - \vec{v}^2}\, dt$$

time dialation.

So the equation of motion we derived earlier from the Euler Lagrange equations is

$$\frac{d}{d\lambda}\left(\frac{m\,\frac{dx^\mu}{d\lambda}}{\sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\eta_{\mu\nu}}}\right)=0$$

In the comoving frame, this becomes

$$\frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right)=0$$

But $\frac{dx^\mu}{d\tau}=\frac{dx^\mu}{dt}\frac{dt}{d\tau}=\frac{dx^\mu/dt}{\sqrt{1-\vec{v}^2}}$ by chain rule

So $\frac{d}{d\tau}=\frac{dt}{d\tau}\frac{d}{dt} \longrightarrow \frac{d}{dt}\left(m\frac{dx^\mu}{dt}\right)=0$

$$\Longrightarrow \frac{d}{dt}\left(\frac{m\,dx^\mu/dt}{\sqrt{1-\vec{v}^2}}\right)=0$$

$$\frac{d}{dt}\Bigg(\underbrace{\frac{mv^i}{\sqrt{1-\vec{v}^2}}}_{p^i}\Bigg)=0$$ ← equations of ~~momentum~~ motion ⟹ conservation of momentum.

In comoving frame $E=p^0=\frac{m}{\sqrt{1-\vec{v}^2}}$

but $\vec{v}=0$ here, so $E=m$

$c=1$ so $E=mc^2$.

More math

$$\frac{d}{d\lambda} = \underbrace{\frac{dx^i}{d\lambda}}_{\text{components}} \underbrace{\frac{d}{dx^i}}_{\text{basis vectors}} = V^i e_i \qquad g(\cdot, w): T_p \longrightarrow \mathbb{R}$$

$$(g_{ij} V^i)\, w^j \in \mathbb{R}$$

A covector is an element of the co-tangent space

$$\underbrace{T_p^*}_{\text{forms a vector space}} = \{ f: T_p \xrightarrow[\text{linear}]{} \mathbb{R} \}. \quad \text{(also called a 1-form)}$$

Given a basis $e_i \overset{\partial / \partial x^i}{=}$ for T_p, there is a dual basis $\tilde{d}x^i$ such that $\tilde{d}x^i(e_j) = \delta^i_j$

Under coordinate transformation,

$$\tilde{d}x^i \longrightarrow \frac{\partial x^i}{\partial y^j} \tilde{d}y^j$$

$$e_i = \frac{\partial}{\partial x^i} \longrightarrow \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

Define $f: M \to \mathbb{R}$ as a scalar function. Then

$\tilde{d}f := \frac{\partial f}{\partial x^i} \tilde{d}x^i$ is the "1-form field" of f.

Like a gradient. This is a functional on TM, so

$$\tilde{d}f(d/d\lambda) = \frac{\partial f}{\partial x^i} \tilde{d}x^i(d/d\lambda) = \frac{\partial f}{\partial x^i} \tilde{d}x^i\left(\frac{dx^j}{d\lambda}\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i} \frac{dx^j}{d\lambda} \tilde{d}x^i\left(\frac{\partial}{\partial x^j}\right)$$

$$= \frac{\partial f}{\partial x^i} \frac{dx^j}{d\lambda} \tilde{d}x^i(\partial/\partial x^j) = \frac{\partial f}{\partial x^i} \frac{dx^j}{d\lambda} \delta^i_j = \frac{df}{d\lambda}.$$

Tensors

$T_{a_1,\dots,a_m}{}^{b_1,\dots,b_m}$ is a map that takes m 1-forms and n vectors and puts out a number.

e.g. metric tensor g_{ij} takes two vectors and gives a real number as output.

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Define 1-forms as maps $\omega : T_pM \longrightarrow \mathbb{R}$.

A general "tensor" is the tensor product of n-vectors and m 1-forms.

$$T_{i_1,\dots,i_n}{}^{j_1,\dots,j_m} = \omega_{i_1} \otimes \omega_{i_2} \otimes \dots \otimes \omega_{i_n} \otimes V^{j_1} \otimes \dots \otimes V^{j_m}$$

A type $T(n,m)$ tensor. Vectors are $T(1,0)$ tensors, and 1-forms are $T(0,1)$ tensors.

$T_{i_1 \dots i_n}{}^{j_1 \dots j_m}$ is a map from $\underbrace{T_p^*M \otimes T_p^*M \otimes \dots \otimes T_p^*M}_{n\text{- copies.}} \otimes \underbrace{T_pM \otimes \dots \otimes T_pM}_{m \text{ copies}}$

Maps from manifolds to manifolds

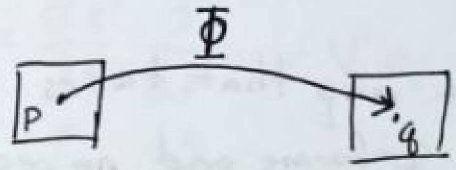
$\phi : M \rightarrow N$

Two ways of thinking about these

(1) Passive view $\longrightarrow$ change of coordinates
(2) Active view $\longrightarrow$ move points around

} essentially the same thing

Let $\Phi : M \to N$ and $\Phi(p) = q$



We can use Φ to define maps from T_pM to T_qN and from T_p^*M to T_q^*N.

Given $f : N \to \mathbb{R}$, the ~~push forward~~ pullback of f by Φ is $\Phi^*(f) : M \to \mathbb{R}$ defined by $\Phi^*(f) = f \circ \Phi$.

$$\Phi^*(f)(p) = f(\phi(p))$$

If $g : M \to \mathbb{R}$ and Φ^{-1} exists, the pushforward of g is the pullback along Φ^{-1}!

When both Φ and Φ^{-1} exist, and Φ, Φ^{-1} smooth, then Φ is called a diffeomorphism.

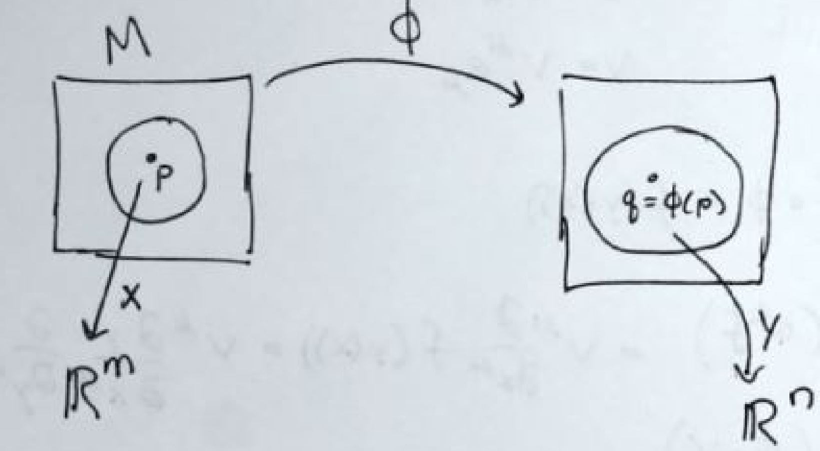
By having a map from a manifold to itself, we can put tangent vectors over the same point and compare them. This allows us to do calculus on manifolds.

Given a vector $v \in T_pM$, define the push-forward of v by ϕ as $(\phi_* v)(f) = v(\phi^* f)$

$$(\phi_* V)^\alpha \frac{\partial f}{\partial x^\alpha} = V^\mu \frac{\partial (f \circ \phi)}{\partial x^\mu} = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial (f \circ \phi)}{\partial y^\alpha}$$

y^α local coordinates of N around q

x^α local coordinates of M around p



$$\phi_* : T_p M \longrightarrow T_{\phi(p)} N$$

$V \in T_p M$, $V : C^\infty M \longrightarrow \mathbb{R}$

$\phi_* V \in T_{\phi(p)} N$, $\phi_*(V) : C^\infty N \longrightarrow \mathbb{R}$.

Given a 1-form ω, it can be pulled back but not pushed forward. $\phi^*(\omega)(V) = \omega(\phi_*(V))$

$$(\phi^* \omega)_\mu = (\phi^*)^\alpha_\mu \, \omega_\alpha$$

<u>Recall</u>:

$$\phi : M \to N$$

$$\begin{tikzcd} \boxed{\cdot p} \arrow[r, bend left, "\phi"] \arrow[d, "x"] & \boxed{\cdot q} \arrow[d, "y"] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

$$\omega : T_pM \to \mathbb{R}$$

$$V \in T_{\phi(p)}N \qquad f \in C^\infty(N)$$

$$\omega = \omega_\alpha \, dx^\alpha$$

$$V = V^\mu e_\mu$$

$$\phi^*(f) = f \circ \phi = f(y(x))$$

$$(\phi_* V)(f) = V(\phi^* f) = V^\mu \frac{\partial}{\partial x^\mu} f(y(x)) = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial y^\alpha} f(y)$$

$$(\phi^*\omega)(V) = \omega(\phi_* V)$$

$$= \omega\left(V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}\right) = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \omega\left(\frac{\partial}{\partial y^\alpha}\right)$$

$$= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha \longrightarrow (\phi^*\omega)_\mu$$

For an arbitrary $(0,\ell)$ tensor, can pull back:

$$\phi^* T(V^{(1)}, \ldots, V^{(\ell)}) = T(\phi_* V^{(1)}, \ldots, \phi_* V^{(\ell)})$$

For an $(\ell, 0)$ tensor, can push-forward

$$\phi_* T(\omega^{(1)}, \ldots, \omega^{(\ell)}) = T(\phi^* \omega^{(1)}, \ldots, \phi^* \omega^{(\ell)})$$

$$(\phi_* T)^{\mu_1, \ldots, \mu_\ell} = \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\mu_\ell}}{\partial x^{\alpha_\ell}} T^{\alpha_1, \ldots, \alpha_\ell}$$

Example:

$$M := S^2 \quad \text{and} \quad N := \mathbb{R}^3$$

coordinates $x = (\theta, \phi)$ coordinates $y = (x, y, z)$

$$\phi : M \longrightarrow N$$

$$(\theta, \phi) \longmapsto (\sin\theta\cos\phi, \sin\theta, \sin\phi, \cos\theta)$$

metric $g_{\mu\nu}$ on N by $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\phi^*(g) = g(\phi_* v_1, \phi_* v_2) \qquad (\phi_* v)^2 = \frac{\partial y^\alpha}{\partial x^\mu} v^\mu$$

Jacobian $$\frac{\partial \vec{y}}{\partial \vec{x}} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{bmatrix}$$

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}$$

$$(\phi^* g)_{11} = \frac{\partial y^1}{\partial x^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial y^2}{\partial x^1}\frac{\partial y^2}{\partial x^1} + \frac{\partial y^3}{\partial x^1}\frac{\partial y^3}{\partial x^1}$$

$$= \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = 1$$

$$(\phi^* g)_{21} = (\phi^* g)_{12} = \frac{\partial y^\alpha}{\partial x^1}\frac{\partial y_\alpha}{\partial x^2} = 0$$

$$(\phi^* g)_{22} = \sin^2\theta$$

So the pullback is $(\phi^* g) = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$. Induced metric on S^2 or pullback of metric onto sphere.

Diffeomorphism: $\phi : M \to N$ is a diffeomorphism if ϕ and ϕ^{-1} are both C^∞.

pushforward of an arbitrary (k,l) tensor on M

$$(\phi_* T)(\omega^{(1)},\dots,\omega^{(k)}; V^{(1)},\dots,V^{(l)})$$

$$= T(\phi^*\omega^{(1)},\dots,\phi^*\omega^{(k)}; (\phi^{-1})_* V^{(1)},\dots,(\phi^{-1})_* V^{(l)})$$

$$= T\left(\frac{\partial y^{\mu_1}}{\partial y^{\alpha_1}}\omega^{(1)}_{\mu_1},\dots,\frac{\partial y^{\mu_k}}{\partial x^{\alpha_k}}\omega^{(k)}_{\mu_k}; \cancel{\frac{\partial x^{\nu_1}}{\partial y^{\beta_1}} V^{(1)}_{\nu_1},\dots,\frac{\partial x^{\nu_l}}{\partial y^{\beta_l}} V^{(l)}_{\nu_l}}\ \frac{\partial x^{\beta_1}}{\partial y^{\nu_1}} V^{(1)\nu_1},\dots,\frac{\partial x^{\beta_l}}{\partial y^{\nu_l}} V^{(l)\nu_l}\right)$$

$$(\phi_* T)^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l} = \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{\mu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial y^{\nu_l}} T^{\alpha_1\dots\alpha_k}{}_{\beta_1\dots\beta_l}.$$

Consider a family of diffeomorphisms $M \to M$ parameterized by $t \in \mathbb{R}$, $\{\phi_t : t \in \mathbb{R}\}$ such that

$$\phi_s \circ \phi_t = \phi_{s+t} \quad \text{and} \quad \phi_0 = 1. \qquad \text{(Everything smooth)}$$

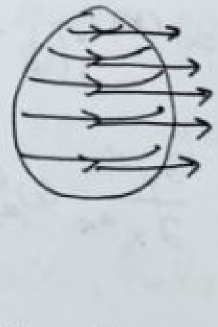
for fixed p, $\phi_t(p)$ is an integral curve on M.

Since $\phi_t(p)$ gives a curve for all p, and ϕ_t is 1 to 1, this "foliates" the manifold.

The one-parameter family of curves ϕ_t gives a vector field $V(\vec{x})$ for $\vec{x} \in M$.

V is tangent to $\phi_t(p)$ at p

<u>Examples</u> on S^2, $\phi_t(\theta, \phi) = (\theta, \phi + t)$



$V^\mu = \dfrac{dx^\mu}{dt}$. Call V^μ the "generators of diffeomorphisms."

So far, have only talked about tangents at single points. This allows us to talk about the tangent space on the whole manifold.

How do tensors change along curves?

$$\Delta T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_s}(p) = \phi_t^*\left(T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_\ell}\right)\left((p)\right) - T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_\ell}(p)$$

Define the Lie Derivative:

$$\mathcal{L}_V T = \lim_{t\to 0}\left(\Delta T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_k}\right)/t$$

Obeys the Leibniz rule $\mathcal{L}_V(T \otimes S) = (\mathcal{L}_V T) \otimes S + T \otimes (\mathcal{L}_V S)$

Example: Lie derivative of a function:

$$\mathcal{L}_V(f) = \lim_{t\to 0} \frac{\phi_t^* f(\phi_t(p)) - f(p)}{t}$$

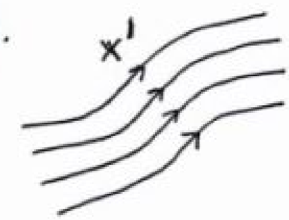
$$= \lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t}$$

$$= \lim_{t\to 0} \frac{f(x(t)) - f(x(0))}{t} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial t} = V^\mu \partial_\mu f.$$

Action of Lie Derivative on another vector field. 09/19/14

Choose a coordinate system such that $V^i = x^i$

x^1 is the parameter along the integral curve.



Hence $V = (1, 0, 0, \ldots, 0)$

$$\begin{tikzcd} M \arrow[r, "\phi_t"] \arrow[d, "x"] & M \arrow[d, "y"] \\ \mathbb{R}^m & \mathbb{R}^m \end{tikzcd}$$

If $x = (x^1, x^2, \ldots)$, then under ϕ_t, the y coordinates are given by

$y = (x^1 + t, x^2, \ldots)$

$$\frac{\partial y^\alpha}{\partial x^\beta} = (\phi_t^*)^\alpha{}_\beta = \delta^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial y^\beta} = \phi^{*-1}$$

So $$\mathcal{L}_V U = \lim_{t\to 0} \frac{U^\mu(x_1 + t, 0, \ldots, 0) - U^\mu(x_1, 0, \ldots, 0)}{t} = \frac{\partial U^\mu}{\partial x^1} = V^\nu \partial_\nu U^\mu.$$

Now consider the commutator of two vector fields:

$$[V,U]^\nu = \frac{d}{d\lambda}\frac{d}{d\mu} - \frac{d}{d\mu}\frac{d}{d\lambda} = \text{another vector field}$$

$V = d/d\lambda$

$U = d/d\mu$

$$[V^\alpha \partial_\alpha, U^\beta \partial_\beta] = V^\alpha(\partial_\alpha U^\beta)\partial_\beta + \cancel{V^\alpha U^\beta \partial_\alpha \partial_\beta} - U^\alpha \partial_\alpha V^\beta \partial_\beta - \cancel{V^\beta U^\alpha \partial_\beta \partial_\alpha}$$

$$= V^\alpha(\partial_\alpha U^\beta)\partial_\beta - U^\alpha(\partial_\alpha V^\beta)\partial_\beta$$

$$= V^1(\partial_1 U^\beta)\partial_\beta - U^\alpha(\cancel{\partial_\alpha V^1}^{\,0})\partial_1 = V^1(\partial_1 U^\beta)\partial_\beta$$

by our choice of coordinate systems

$V = (1,0,0\ldots 0)$

Therefore $[V,U]^\nu = (V\cdot dU)^\nu$

and so finally, conclude that

$$\mathcal{L}_V(U) = V^\nu \partial_\nu U^\mu = [V,U]$$

Therefore $\mathcal{L}_V(U) = -\mathcal{L}_U(V)$.

<u>Consider:</u> $\mathcal{L}_V(\omega_\alpha U^\alpha) = V^\beta \partial_\beta(\omega_\alpha U^\alpha) = (\mathcal{L}_V\omega)_\alpha U^\alpha + \omega_\alpha(\mathcal{L}_V U)^\alpha$

$$= V^\beta(\partial_\beta \omega_\alpha)U^\alpha + V^\beta \omega_\alpha \partial_\beta U^\alpha = \underbrace{(\mathcal{L}_V\omega)_\alpha U^\alpha}_{\star} + \omega_\alpha(\mathcal{L}_V U)^{\alpha}$$

$[V,U] = U^\alpha \partial_\alpha U^\beta - U^\alpha \partial_\alpha V^\beta$

$$\star\ (\mathcal{L}_V\omega)_\alpha U^\alpha = V^\beta(\partial_\beta\omega_\alpha)U^\alpha + \cancel{V^\beta \omega_\alpha \partial_\beta U^\alpha} - \cancel{V^\rho(\partial_\rho U^\alpha)\omega_\alpha} + U^\rho \partial_\rho V^\alpha \omega_\alpha$$

$$= (V^\beta \partial_\beta \omega_\alpha)U^\alpha + (U^\beta \partial_\beta V^\alpha)\omega_\alpha$$

$$\Longrightarrow (\mathcal{L}_V(\omega))_\alpha = V^\beta \partial_\beta \omega_\alpha + (\partial_\alpha V^\beta)\omega_\beta .$$

How does the Lie derivative work on a general (n,m) tensor?

$$\mathcal{L}_V T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m} = V^\sigma \partial_\sigma T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m} - (\partial_\lambda V^{\mu_1}) T^{\lambda,\mu_2,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m}$$

$$- (\partial_\lambda V^{\mu_2}) T^{\mu_1,\lambda,\mu_2,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m} -$$

$$\vdots$$

$$- (\partial_\lambda V^{\mu_n}) T^{\mu_1,\mu_2,\dots,\mu_{n-1},\lambda}{}_{\nu_1,\dots,\nu_m}$$

$$+ (\partial_{\nu_1} V^{\lambda}) T^{\mu_1,\dots,\mu_n}{}_{\lambda,\nu_1,\dots,\nu_m} +$$

$$\vdots$$

$$+ (\partial_{\nu_m} V^{\lambda_m}) T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_{m-1},\lambda}$$

08/22/14

Recall:

$$\mathcal{L}_V T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m} = V^\sigma \partial_\sigma T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_m} - \sum_{i=1}^{n} (\partial_\lambda V^{\mu_i}) T^{\mu_1,\dots,\mu_{i-1},\lambda,\mu_{i+1},\dots,\mu_n}{}_{\nu_1,\dots,\nu_n}$$

$$+ \sum_{j=1}^{m} (\partial_{\nu_j} V^{\lambda}) T^{\mu_1,\dots,\mu_n}{}_{\nu_1,\dots,\nu_{j-1},\lambda,\nu_{j+1},\dots,\nu_m}.$$

$$g_{ij}(v_1^i, v_2^j) = g_{ij}v_1^i v_2^j$$

<u>Example</u>: $\mathcal{L}_V g_{ij} = ?$ g_{ij} is the metric.

$$\mathcal{L}_V g_{ij}(v_1^i v_2^j) = V^\mu \partial_\mu (g_{ij} v_1^i v_2^j)$$

$$= V^\mu \left((\partial_\mu g_{ij}) v_1^i v_2^j + g_{ij} (\partial_\mu v_1^i) v_2^j + g_{ij} v_1^i (\partial_\mu v_2^j) \right)$$

$$= \left(\mathcal{L}_V g_{ij} \right) v_1^i v_2^j + g_{ij} \left(\mathcal{L}_V v_1^i \right) v_2^j + g_{ij} v_1^i \left(\mathcal{L}_V v_2^j \right)$$

given a vector field V on M and a metric, if $\mathcal{L}_V g = 0$, V is called a "killing vector" and V generates an isometry: metric doesn't change.

<u>Example</u>: in $\mathbb{R}^3$, $g_{ij} = \delta_{ij}$

$$\underbrace{\begin{matrix} V_1 = (1,0,0) & \partial/\partial x \\ V_2 = (0,1,0) & \partial/\partial y \\ V_3 = (0,0,1) & \partial/\partial z \end{matrix}}_{\text{killing vectors}}$$

There are actually three more killing vectors on $\mathbb{R}^3$. Go to polar coordinates:

$$x = r\cos\phi \sin\theta$$
$$y = r\sin\phi \sin\theta \qquad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$z = r\cos\theta$$

$\frac{\partial}{\partial \phi}$ is a killing vector b/c no ϕ in ds^2

$$\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad V_\phi = (-y, x, 0)$$

generates rotations, also an isometry.

$$L_z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$L_z f = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$ rotation around z-axis.

$$\left.\begin{aligned} L_x &= -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y} \\ L_y &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \end{aligned}\right\}$$ two more killing vectors.

So in $\mathbb{R}^3$ we have at least 6 killing vectors

A space (Like $\mathbb{R}^3$) is maximally symmetric and can only have at most $\frac{\dim(\dim+1)}{2}$ killing vectors.

Let V be a vector field defined by ϕ_t $t \in \mathbb{R}$.

<u>Claim:</u> $\phi_t^* = e^{t\mathcal{L}_V}$ ← call it $e^{t\partial/\partial\lambda}$ ← defined by power series.

"exponential map"

$$e^{t\partial/\partial\lambda} f(x(\lambda=0)) = f(x(\lambda=t))$$ because

$$\left(1 + t\frac{\partial}{\partial\lambda} + \frac{1}{2}t^2\left(\frac{\partial}{\partial\lambda}\right)^2 + \cdots\right) f(x(\lambda=0))$$

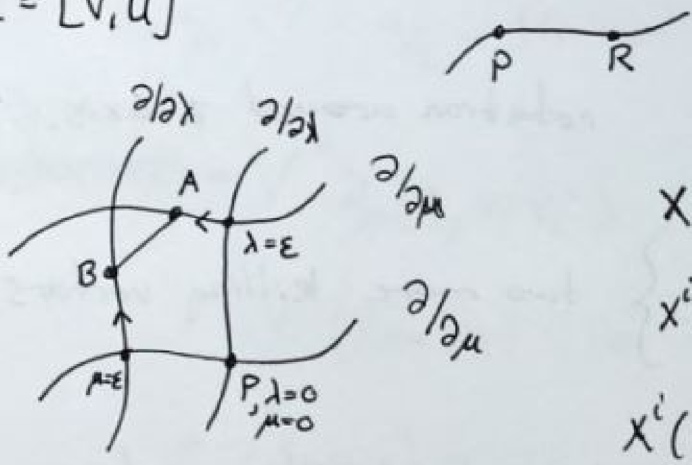
$$= f(x(0)) + t\left.\frac{\partial f}{\partial\lambda}\right|_{\lambda=0} + \frac{1}{2}t^2\frac{\partial^2 f}{\partial\lambda^2} + \cdots$$

$$= f(x(0)) + t\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial\lambda} + \frac{1}{2}t^2\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\left(\frac{\partial x^i}{\partial\lambda}\frac{\partial x^j}{\partial\lambda} f(x(0))\right) + \cdots$$

$$= f(x(t)).$$

Suppose U, V are two vector fields on a manifold M.

$\mathcal{L}_V U = [V, U]$



$$x^i(A) = x^i(B)$$

$$x^i(A) = e^{\varepsilon\partial/\partial\mu} e^{\varepsilon\partial/\partial\lambda} x^i(P)$$

$$x^i(B) = e^{\varepsilon\partial/\partial\lambda} e^{\varepsilon\partial/\partial\mu} x^i(P)$$

$$x^i(A) = \left(1 + \varepsilon\partial/\partial\mu + \tfrac{1}{2}\varepsilon^2(\partial/\partial\mu)^2\right)\left(1 + \varepsilon\partial/\partial\lambda + \tfrac{1}{2}\varepsilon^2(\partial/\partial\lambda)^2\right) x^i(P)$$

$$x^i(A) = \left(1 + \varepsilon\frac{\partial}{\partial\mu} + \varepsilon\partial/\partial\lambda + \frac{1}{2}\varepsilon^2\left(\frac{\partial}{\partial\lambda}\right)^2 + \frac{1}{2}\varepsilon^2\left(\frac{\partial}{\partial\mu}\right)^2 + \varepsilon^2\frac{\partial}{\partial\mu}\frac{\partial}{\partial\lambda}\right) x^i(P)$$

similarly for $x^i(B)$, so

$$x^i(A) - x^i(B) = \varepsilon^2\left(\frac{\partial}{\partial\mu}\frac{\partial}{\partial\lambda} - \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}\right) = \varepsilon^2[V, U] = \mathcal{L}_V U.$$

$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \varepsilon\begin{pmatrix}1&0\\0&0\end{pmatrix}$

$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \varepsilon\begin{pmatrix}0&1\\0&0\end{pmatrix}$

09/24/13

Lie Derivatives $\mathcal{L}_V g = 0$

Action for a point particle: $S = -m \int \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$

Choose coordinate system so that g is independent of x^i

$\left(V = \frac{\partial}{\partial x^i}\right)$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} \qquad \frac{d}{d\lambda}\frac{\partial L}{\partial x^i} = 0 \qquad \frac{d}{dt}P_i$$

$L = \frac{1}{2}m\vec{v}^2 - V(x)$, $V(x)$ is independent of x^i, then $F_i = \frac{\partial V}{\partial x^i} = 0$

Lie Group: is a manifold that is also a group.

$L_g : h \longmapsto gh$ must be smooth diffeomorphisms.

$R_g : h \longmapsto hg$

$: h \longmapsto h^{-1}$

$L_{g*} : T_h \longrightarrow T_{gh} \qquad R_{g*} : T_h \longrightarrow T_{hg}$

Left Invariant vector fields. $X_g =$ a tangent vector at g.

$L_{g*}X_h = X_{gh}$ left invariant

$R_{g*}X_h = X_{hg}$ right invariant

Consider $X_e \in T_e$, where e is the identity.

Manifold is covered by a set left-invariant vector fields generated from X_e by pushforward by group elements as L_{g*}.

Caveat: M must be ~~simply~~ connected and compact!

1-parameter subgroups:

$$g : \mathbb{R} \to G \quad \text{such that} \quad g(s+t) = g(s)\,g(t) = g(t)\,g(s)$$

$$\lim_{s \to 0} \frac{d}{ds} g(s+t) = g(t)\,g'(0) \Longrightarrow g'(t) = g(t)\,g'(0) \Rightarrow g(t) = e^{t g'(t)}$$

Exponential map: $e^{t\, d/dt}$

$$g'(0) = X_e \in T_e$$

$$\underset{\substack{\text{structure}\\\text{constants}\\\text{(basis dependent)}}}{\underbrace{[X_i, X_j] = c_{ijk} X_k}} \qquad \text{Lie Algebra}$$

$\mathcal{G} = \{ g : \mathbb{R} \to G \}$ is the Lie Algebra.

09/30/14

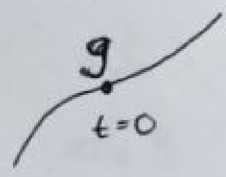
$\phi_t : \mathbb{R} \to G$ ← a lie group

$\phi_t = e^{tV}$ $\quad V \in T_e = \mathfrak{g}$ ← Lie algebra

$[T_i, T_j] = c_{ijk} T_k$ $\quad \dim \mathfrak{g} = \dim G$ as a manifold.

$L_g h = gh$
$R_g h = hg$ $\quad g, h \in G$ $\quad$ posh-forwards: $T_g \longrightarrow T_{g'}$
defined by L_{g*} ("induced map")

$\phi_t(g) = g e^{tV}$ $\quad$ 1-parameter subgroup which runs through g.



$$L_{g*} V = \lim_{t \to 0} \frac{d}{dt} g e^{tV} = gV =: X_v \big|_g$$

X_v = left invariant vector field generated by g.

<u>Example</u>: $A(1)$ group of affine line.

$\left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \,\middle|\, \begin{matrix} x, y \in \mathbb{R} \\ x, y > 0 \end{matrix} \right\}$

~~$t \mapsto$~~
$t \mapsto x(t)$
$t \mapsto y(t)$

$\begin{bmatrix} x(t) & y(t) \\ 0 & 1 \end{bmatrix}$ has tangent curve

$\begin{bmatrix} \frac{dx}{dt} & \frac{dy}{dt} \\ 0 & 0 \end{bmatrix}$

Left translate $\frac{\partial}{\partial x}$ to (x, y).

$h(t) = \begin{bmatrix} 1+t & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \frac{\partial}{\partial x}$ tangent at e ← origin = identity of group

$h'(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \frac{\partial}{\partial x}$

$$L_{g*} \frac{\partial}{\partial x} = \lim_{t \to 0} \frac{d}{dt} \left[g h(t) \right)$$

→

Left translate $\frac{\partial}{\partial x}$

$$L_{g*}\frac{\partial}{\partial x} = \lim_{t\to 0}\frac{d}{dt}\left(gh(t)\right) = \lim_{t\to 0}\frac{d}{dt}\left(\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1+t & 0 \\ 0 & 1\end{pmatrix}\right)$$

$$= \lim_{t\to 0}\frac{d}{dt}\begin{pmatrix} x+tx & y \\ 0 & 1\end{pmatrix} = \begin{bmatrix} x & 0 \\ 0 & 0\end{bmatrix} \longrightarrow x\frac{\partial}{\partial x}.$$

Left translate $\frac{\partial}{\partial y}$

$$h(t) = \begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix} \quad L_{g*}\frac{\partial}{\partial y} = \frac{d}{dt}\left(\begin{bmatrix} x & y \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}\right) = \frac{d}{dt}\begin{bmatrix} x & tx+y \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0\end{bmatrix}$$

$$\longrightarrow x\frac{\partial}{\partial y}.$$

Adjoint Representation

$$ad_h : g \longmapsto hgh^{-1}$$
$$G \longrightarrow G$$

$$Ad_h := ad_{h*}\big|_e$$

$$Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$\sigma_V(t) = e^{tV} \quad V \in \mathfrak{g}$$

$$ad_g\,\sigma_V(t) = g e^{tV} g^{-1} = e^{t(gVg^{-1})}$$

$$Ad_g V = \lim_{t\to 0}\frac{d}{dt}e^{t(gVg^{-1})} = gVg^{-1} \in \mathfrak{g}$$

$$Ad_* : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$Ad_*(X)(Y) = \lim_{t\to 0}\frac{d}{dt}e^{tX}Ye^{-tX} = XY - YX = [X,Y].$$

$Ad_*(X) = [X, \cdot]$ $\qquad (T^a)_{ij} \longrightarrow M_{ik}(T^a_{kj})$

For a matrix representation, generators T^a that are matrices. Structure constants $[T^i, T^j] = c_{ijk} T^k$.

Structure constants form a representation:

$$Ad_{*T_c}(V) = (T^C V^A)_{ab} - (V^A T^C)_{ab} = \text{[redacted]}$$

$\downarrow$

$[T^C, V^A]_{ab}$

V^B is component of basis vector T^B

$V \cdot T \in \mathfrak{g}$

10/01/14

$ad_a : G \to G$

$ad_a g = aga^{-1}$

$Ad_a := ad_{a*} : \mathfrak{g} \to \mathfrak{g}$

$Ad_g V = gVg^{-1} \in \mathfrak{g}$

Ad_g is a Lie Algebra HM

$Ad_g[X,Y] = [Ad_g X, Ad_g Y]$

$g[X,Y]g^{-1} = [gXg^{-1}, gYg^{-1}]$

$Ad^* : \mathfrak{g} \to \mathfrak{g}$ $\quad \mathcal{L}_X = Ad^*_X$

$Ad^*_X Y = [X,Y]$

If $X \in \mathfrak{g}$, then Ad_X should be unambiguous.

We should have that $[Ad_X, Ad_Y] = Ad_{[X,Y]}$.

gives us the Jacobi Identity.

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

c is a direction in tangent space, corresponds to $e^{iL_c} \in G$

$$\mathrm{Ad}_{e^{iL_c}} V = \text{"}\mathrm{Ad}_c V\text{"} = \cancel{\lim_{\theta_c \to 0} \frac{d}{d\theta_c}\left(e^{i\theta^e L_e} V^A e^{-i\theta^e L_e}\right)} L_A$$

$g = e^{i\theta_e L_e}$

$g^{-1} = e^{-i\theta_e L_e}$

$\{L_a\}$ basis for $\mathfrak{g}$

$$= V'^A L_A = \left(\underbrace{M(\theta_c)}_{n\times n \text{ matrix},\ n = \dim \mathfrak{g}} V\right)^B L_A$$

also

$$\cancel{= [iL_c V^A - iV^A L_c) L_A}$$

This is really just the statement that $\mathrm{Ad}_X Y = [X, Y]$

$$\mathrm{Ad}_c V = \lim_{\theta_c \to 0} \frac{d}{d\theta_c}\left(e^{i\theta^a L_a} L_A e^{-i\theta^a L_a}\right) V^A$$

$$= [iL_c L_A - iL_A L_c] V^A$$

$$= i[L_c, L_A] V^A \qquad [L_i, L_j] = iC_{kij} L_k$$

new components corresponding to c

$$= (i)(i) C^F{}_{AC} L_F V^A = V'^A(\theta_c) L_A$$

$$\Longrightarrow \quad V'^F = \underbrace{(i)^2 C^F{}_{AC} V^A}_{\text{matrix times a vector,}}$$

So the structure constants form the matrix for the adjoint representation!

$V' = MV.$

$M = M^F{}_A(\theta_c)$

$C^F{}_{AC}$ are explicit matrices representing the Lie Algebra!

Jacobi identity

$$[[L_a, L_b], L_c] + [[L_b, L_c], L_a] + [[L_c, L_a], L_b] = 0$$

$$[ic^f{}_{ab} L_f, L_c] + [ic^f{}_{bc} L_f, L_a] + [ic^f{}_{ca} L_f, L_b] = 0$$

$$ic^f{}_{ab}[L_f, L_c] + ic^f{}_{bc}[L_f, L_a] + ic^f{}_{ca}[L_f, L_b] = 0$$

$$(i)^2 c^f{}_{ab} c^g{}_{fc} L_g + (i)^2 c^f{}_{bc} c^g{}_{fa} L_g + (i)^2 c^f{}_{ca} c^g{}_{fb} L_g = 0$$

$$-\left(c^f{}_{ab} c^g{}_{fc} + c^f{}_{bc} c^g{}_{fa} + c^f{}_{ca} c^g{}_{fb}\right) L_g = 0$$

$$\Longrightarrow \quad c^f{}_{ab} c^g{}_{fc} + c^f{}_{bc} c^g{}_{fa} + c^f{}_{ca} c^g{}_{fb} = 0 \quad .$$

This one is opside-down.

10/03/14

Recall: adjoint rep of Lie algebra is defined by the structure constants $c^a{}_{bc}$ as matrices.

$\mathfrak{g}^*$ is dual space of $\mathfrak{g}$.

$\eta \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$, $\eta(\xi) \in \mathbb{R}$. Gives inner product on (η, ξ)

Ad_g^* is the dual map of Ad_g

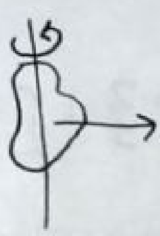
$$(Ad^*_g \eta)(\xi) = \eta(Ad^*_g \xi) = tr(\eta g \xi g^{-1})$$

$$= tr(g^{-1} \eta g \xi) = (g^{-1} \eta g)(\xi)$$

Rigid Body Rotation

If $\vec{r}_1, \vec{r}_2$ are points on the rigid body, $\vec{r}_2 - \vec{r}_1$ is fixed for all point.

This body only has 6 degrees of freedom: 3 translations
3 rotations



collective coordinates

If we want to think about a top, hold one point fixed. Define two distinct reference frames: the body-fixed frame and the lab-fixed frame.

Coordinates

lab frame: $\tilde{e}_a$

body frame: e_a

rotation depends on time.

At a given time, $e_a(t) = R_{ab}\,\tilde{e}_b$

in terms of lab frame, depends on t.

~~$\vec{r}(t) = r_a(t) = e$~~ $\quad \vec{r}(t) = \tilde{r}_a(t)\,\tilde{e}_a = r_a\,e_a(t)$

$$\Longrightarrow \frac{d\vec{r}}{dt} = \frac{d\tilde{r}_a}{dt}\,\tilde{e}_a = r_a\,\frac{de_a}{dt} = r_a\,\dot{R}_{ab}\,\tilde{e}_b$$

$$\Longrightarrow \frac{de_a}{dt} = \underbrace{\dot{R}_{ab}\,R^{-1}_{bc}}_{\omega_{ac} \,=\, \text{angular velocity}}\,e_c$$

$\omega_{ac} = -\omega_{ca}$

b/c $RR^T = \underline{I}$ in $SO(3)$, so

$$\frac{d}{dt}(RR^T) = 0$$

$$\Downarrow$$

$$\dot{R}R^T + R\dot{R}^T = 0$$

$$\Downarrow$$

$$\dot{R}R^{-1} + R\dot{R}^{-1} = 0$$

antisymmetric, so $\omega_{ac} = -\omega_{ca}$.

define $\omega_a = \frac{1}{2}\varepsilon_{abc}\,\omega_{bc}$ ("hodge dual").

$$\Longrightarrow \omega_{bc} = \omega_a\,\varepsilon_{abc}$$

Now ~~$\omega_{ac} e_c =$~~

$$\frac{de_a}{dt} = \omega_{ac}\,e_c = (\omega_d\,\varepsilon_{dac})e_c = (\omega_d\,\varepsilon_{dca})e_c$$

$$= -(\omega \times e)_a$$

Kinetic Energy of a Rigid Body.

$$KE = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2}\sum m_i (\omega_a (r_i)_b \varepsilon_{abc})(\omega_A (r_i)_B \varepsilon_{A\ldots}$$

$$= \frac{1}{2}\sum_i m_i (\omega_a \omega_A (r_i)_b (r_{iB}))(\delta_{aA}\delta_{bB} - \delta_{aB}\delta_{bA})$$

$$= \frac{1}{2}\sum_i m_i (\omega^2 r_i^2 - (\omega \cdot r_i)^2) = \frac{1}{2}\sum_i m_i ((r_i^2 \delta_{ab} - r_a r_b)\, \omega_a \omega_b)$$

$$= \frac{1}{2} I_{ab}\, \omega_a \omega_b$$

where $I_{ab} = \int d^3x\, \rho(x) (\vec{x}^{\,2} \delta_{ab} - x_a x_b)$ "Moment of inertia tensor"

Symmetric $\Longrightarrow$ diagonalizable

$$\begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$$

I_i is principle moment of inertia.

$$I_{ij} c_i c_j = \int (r^2 c^2 - (r \cdot c)^2)\, d^3r \geq 0$$

$\Longrightarrow$ pos. def. $\Longrightarrow$ $I_i \geq 0$.

10/06/14

Rigid Body:

$\tilde{e}_a$ inertial frame $\qquad \vec{r} = r_a e_a(t) = r_a(t)\tilde{e}_a$

e_a body-fixed frame

$$\omega_{ac} = R_{ab} R^{-1}_{bc} \qquad \omega_a = \frac{1}{2}\varepsilon_{abc}\omega_{bc}$$

$$\frac{de_a}{dt} = -\varepsilon_{aib}\omega_i e_b \qquad \frac{d}{dt}\vec{r} = \frac{d}{dt} r_a e_a(t) = r_a(-\varepsilon_{aib}\omega_i e_b)$$

$$= -(r \times \omega)_b e_b$$

$$\boxed{(\vec{V}\times\vec{U})_a = \varepsilon_{abc} V_b U_c} \qquad \frac{d\vec{r}}{dt} = (\vec{\omega}\times\vec{r})_b e_b$$

Kinetic Energy

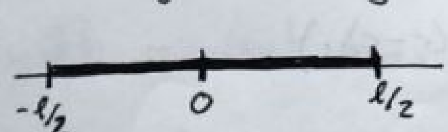
$\frac{1}{2} I_{ij}\omega_i\omega_j$

$I_{ij} = \int d^3x\, \rho(x)\, (\vec{r}^{\,2}\delta_{ij} - r_i r_j)$ $\qquad \rho$ is density

I is symmetric, can be diagonalized into $I = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$, $I_i \geq 0$. The basis defined by these is called the principal axes.

EXAMPLES:

rod of length ℓ along x axis

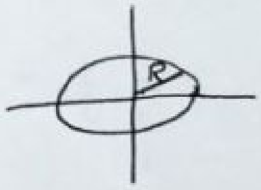


$$\rho = \frac{M}{\ell} \qquad I_{ij} = \int dx \frac{M}{\ell}\left[x^2\delta_{ij} - x_i x_j\right]$$

$I_x = 0 \quad I_y = I_z$ by symmetry $\qquad I_y = \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} dx\,(x^2) = \rho\frac{\ell^3}{12}.$

So therefore $I = \begin{bmatrix} 0 & & \\ & \frac{Ml^2}{12} & \\ & & \frac{Ml^2}{12} \end{bmatrix}$

<u>Disc:</u>



$$\rho = \frac{M}{\pi R^2}$$

$$I_{ij} = \int_0^{2\pi} d\theta \int_0^R r\,dr\, (\rho)(r^2\delta_{ij} - r_i r_j)$$

$$I_3 = 2\pi\rho \int_0^R r\,dr\,(x^2+y^2) = 2\pi\rho \int_0^R r^3 dr = 2\pi\rho\,\frac{R^4}{4} = \frac{MR^2}{2}.$$

$$I_1 = I_2 = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta\,(\vec{x}^2 - x^2) = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta\big((x^2+y^2) - x^2\big)$$

$$I_1 = I_2 = \frac{MR^4}{4}$$

<u>Parallel Axis Theorem:</u>

If the fixed point of rotation is not the center of mass, but displaced from the C.O.M. by $\vec{c}$, then

$$I_c = I_{com} + M(\vec{c}^2\delta_{ab} - c_a c_b)$$

<u>Proof:</u>

$$(I_c)_{ab} = \int \rho(x)\, d^3x\big((\vec{r}-\vec{c})^2\delta_{ab} - (r-c)_a(r-c)_b\big)$$

$$= \int \rho(x)\, d^3x\Big((\vec{r}^2\delta_{ab} - r_a r_b) + \underbrace{(\delta_{ab}(-2\vec{r}\cdot\vec{c}) + c_a r_b + c_b r_a)}_{=0 \text{ by defn of center of mass...}} + (\delta_{ab}\vec{c}^2 - c_a c_b)\Big)$$

<u>EXAMPLE:</u>

Rotating rod about not its center, say the endpoint.

$$I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2.$$

Derive the equations of motion

$$L = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$$= \sum_i m_i \left((r_i)_a \left(\omega_A (r_i)_B \varepsilon_{ABC} \right) \varepsilon_{aCD} \right)$$

$$= -\sum_i m_i \left(r_i \cdot \omega \, r_D - r_i^2 \omega_D \right) = \sum_i I_{D_i} \omega_i$$

$$L_D = (I_D)_i \, \omega_i .$$

$$\frac{d\vec{L}}{dt} = 0 \quad \text{(no torque)}$$

$$\frac{d}{dt} L_a e_a(t) = \frac{dL_a}{dt} e_a + L_a \frac{de_a}{dt} = \frac{dL_a}{dt} e_a + L_a (\omega \times e_a)$$

$$\frac{d\vec{L}}{dt} = \frac{dL_a}{dt} e_a + (\omega \times L) = 0$$

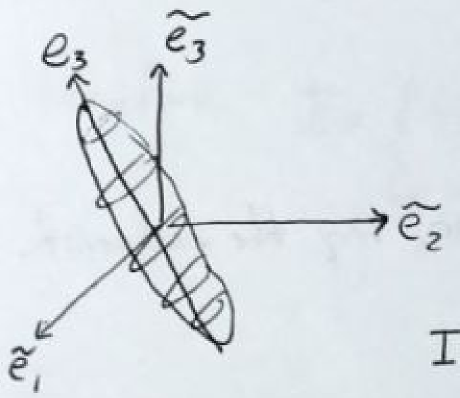
Three nonlinear PDE to solve.

Work with principal axes so I is diagonal.

$$I_1 \dot{\omega}_1 + (I_2 \omega_2)\omega_3 - (I_3 \omega_3)\omega_2 \Longrightarrow \boxed{I \dot{\omega}_1 = -(I_3 - I_2)\omega_2 \omega_3}$$

$$\boxed{\begin{aligned} I \dot{\omega}_2 &= (I_3 - I_1)\omega_1 \omega_3 \\ I \dot{\omega}_3 &= (I_1 - I_2)\omega_1 \omega_2 . \end{aligned}}$$

$\longleftarrow$ Euler Equations



$$I_1 = I_2 \neq I_3$$

ω_3 is constant by euler equations.

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \qquad \text{let } I = I_1 = I_2$$

$$I \dot{\omega}_2 = (I_3 - I)\omega_1\omega_3$$

$$\text{Let } \Omega = \frac{(I - I_3)\omega_3}{I} \qquad \dot{\omega}_1 = \Omega\omega_2$$

$$\dot{\omega}_2 = -\Omega\omega_1$$

differentiate

$$\ddot{\omega}_1 = -\Omega^2\omega_1$$

$$\ddot{\omega}_2 = -\Omega^2\omega_1$$

solutions

$$\omega_1 = \sin\Omega t$$

$$\omega_2 = \cos\Omega t$$

10/08/14

Define left/right invariant metric as preserving length of left/right vector fields.

$$\langle A, A\rangle = \langle A_g, A_g\rangle$$

vector field at identity $\quad$ vector field at g

$$\langle\ ,\ \rangle : T_g \times T_g \to \mathbb{R}$$

$$(\ ,\ ) : T_g^* \times T_g \to \mathbb{R}$$

$$A : \underbrace{\mathfrak{g} \to \mathfrak{g}^*}_{A : T_e \to T_e^*}$$

$$(A\xi, \eta) \in \mathbb{R} \quad \text{for} \quad \xi, \eta \in \mathfrak{g}$$

Metric at point $g \in G$ $\quad A_g : T_g \to T_g^*$

$\xi \in T_g \qquad A_g \xi = L_{g^{-1}}^* A L_{g*} \xi \qquad$ where L_g^* is map defined by

maps back to origin, transforms to one-form, puts back at g.

$$(L_g^* \eta, \xi) = (\eta_h, L_g \xi_{g \to h}) = (L_g^* \eta_h, \xi_{g^{-1}h})$$

$$\boxed{L_g^* \eta_h = \eta_{g^{-1}h}}$$

$$\begin{tikzcd}
T_e = \mathfrak{g} \arrow[d, "A"'] \arrow[rr, bend left, "Ad_g"] & T_g \arrow[l, "L_{g^{-1}*}"'] \arrow[r, "R_{g^{-1}*}"] \arrow[d, "A_g"] & \mathfrak{g} = T_e \arrow[d, "A"] \\
T_e^* = \mathfrak{g}^* \arrow[rr, bend right, "Ad_g^*"'] & T_g^* \arrow[l, "L_g^*"'] \arrow[r, "R_g^*"] & \mathfrak{g}^* = T_e^*
\end{tikzcd}$$

A defines a metric on every point of G.

Consider rigid rotation:

Any configuration is described by some $g \in SO(3)$.

$g(t)$ is a path in $SO(3)$

$\dot{g} \in T_g$ is generalized angular velocity when carried back to the identity. Two ways:

$g^{-1}\dot{g}$, $\dot{g}g^{-1}$ Which one is the right one to use?

One will be in body frame, the other will be in the inertial frame.

Return to Rigid Body

$$\left.\begin{aligned} e_a(t) &= R_{ab}(t)\tilde{e}_b \\ \frac{de_a(t)}{dt} &= \dot{R}_{ab}\tilde{e}_b = \underbrace{\dot{R}_{ab} R^{-1}_{bc}}_{\substack{\cancel{\omega_{ac} = \frac{1}{2}\varepsilon_{acb}\omega_b} \\ \omega_{ac} = \frac{1}{2}\varepsilon_{acb}\widehat{\omega_b}}} e_c \end{aligned}\right\} \text{NOT CONVENTION WE WILL USE}$$

The geometric convention is that left multiplication gives time translation. $g(t+\delta t) = g(\delta t)\,g(t)$

In this convention, $\widehat{\omega}_{body} = \frac{1}{2}\varepsilon_{acb}(\omega_{body})_{ac}$.

$(\omega_{fixed})_{ab} = \dot{R}R^{-1}$

$(\omega_{body})_{ab} = R^{-1}\dot{R}$

$T := KE = \frac{1}{2} I \omega^2_{body}$

I is constant in body frame $\longrightarrow$ I is left invariant (moves w/ body)

I is <u>not</u> right-invariant (changes w/t in lab)

The moment of inertia tensor I defines a metric on the manifold.

$$T = \frac{1}{2}\langle \omega_{body}, \omega_{body}\rangle = \frac{1}{2} I \omega_{body}^2 = \frac{1}{2}(A\omega_{body}, \omega_{body})$$

$$A_g : T_g \longrightarrow T_g^*$$

$M = A_g \dot{g} \in T_g^*$ is generalized angular momentum.

$$\left.\begin{array}{l} M_{body} = L_g^* M \in \mathfrak{g}^* \\ M_{fixed} = R_g^* M \in \mathfrak{g}^* \end{array}\right\}$$ translate back to origin so that this lives in the Lie Algebra.

Essentially, we're doing $L = I\omega$

10/13/14

Rigid Rotator

Angular Velocity: $\left.\begin{array}{l} \omega_{body} \\ \omega_{fixed} \end{array}\right\} \in \mathfrak{g}$

Angular momentum: $\left.\begin{array}{l} M_{body} \\ M_{fixed} \end{array}\right\} \in \mathfrak{g}^*$

Kinetic energy $= \frac{1}{2}\langle \omega_{body}, \omega_{body}\rangle$, where $\langle \cdot, \cdot \rangle$ is the metric

$(\cdot, \cdot)$ is contraction of one-form w/ vector

Use variational principle:

time independent case

$\longrightarrow$ locally shortest path between points

$$\text{length} = \int \left(\frac{dx^i}{ds} \frac{dx^j}{ds} g_{ij} \right)^{1/2} ds$$

$\Downarrow$ Euler-Lagrange

$$\frac{d}{ds} \frac{dL}{d\dot{x}^i} = \frac{dL}{dx^i} \longrightarrow \frac{d}{ds} \frac{dx^j/ds \; g_{ij}}{\sqrt{\frac{dx^i}{ds}\frac{dx^j}{ds} g_{ij}}} = \frac{\frac{1}{2} \frac{dx^a}{ds} \frac{dx^b}{ds} \frac{\partial g_{ab}}{\partial x^i}}{\sqrt{\frac{dx^i}{ds}\frac{dx^j}{ds} g_{ij}}}$$

Consider a coordinate system with a path along only one coordinate.

Parameterize path by y, the arc-length.

(Analogous to rest frame of moving observer).

In this case, $g_{ij}\, dx^i dx^j = dy^2$ and $\frac{dx^i}{ds} \frac{dx^j}{ds} g_{ij} = \left(\frac{dy}{ds}\right)^2$.

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{x}^i} = \underbrace{\frac{d}{ds} \frac{\frac{dx^j}{dy} \cancel{\frac{dy}{ds}} g_{ij}}{\cancel{\frac{dy}{ds}}}}_{\frac{d}{dy}\left(\frac{dx^j}{dy} g_{ij}\right)} = \underbrace{\frac{1}{2} \frac{\frac{dx^a}{dy} \frac{dx^b}{dy} \frac{\partial g_{ab}}{\partial x^i} \cancel{\left(\frac{dy}{ds}\right)^2}}{\cancel{(dy/ds)^2}}}_{} = \frac{\partial L}{\partial x^i}$$

$$\frac{d}{dy}\left(\frac{dx^j}{dy} g_{ij}\right) = \frac{1}{2} \frac{dx^a}{dy} \frac{dx^b}{dy} \frac{\partial g_{ab}}{\partial x^i}$$

$$\frac{d^2x^j}{dy^2} g_{ij} + \left(\frac{dx^c}{dy}\frac{\partial g_{ij}}{\partial x^c}\right)\frac{dx^j}{dy} = \frac{1}{2} \overset{\frac{\partial x^a}{\partial y}}{u^a} u^b \frac{\partial g_{ab}}{\partial x^i}$$

$$g_{ij}\,\dot{u}^j + u^c \frac{\partial g_{ij}}{dx^c} u^j = \frac{1}{2} u^a u^b \frac{\partial g_{ab}}{\partial x^i}$$

$$g_{ij}\,\dot{u}^j = \frac{1}{2} u^a u^b \frac{\partial g_{ab}}{\partial x^i} - u^c u^j \frac{\partial g_{ij}}{\partial x^c}$$

$$\boxed{g_{ij}\,\dot{u}^j = \frac{1}{2} u^a u^b \left(\frac{\partial g_{ab}}{\partial x^i} - \frac{\partial g_{ib}}{\partial x^a} - \frac{\partial g_{ia}}{\partial x^b}\right)}$$

geodesic equation

Multiply by inverse metric

$$\dot{u}^j = \frac{1}{2} g^{ij} u^a u^b \left(\frac{\partial g_{ab}}{\partial x^i} - \frac{\partial g_{ib}}{\partial x^a} - \frac{\partial g_{ia}}{\partial x^b}\right) = -\Gamma^i_{\ ab} u^a u^b$$

$$= \qquad \Gamma^i_{\ ab} = \left(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}\right) g^{ic}$$

$\Gamma^i_{\ ab}$ are the connection coefficients.

Not a tensor, but geodesic equation is a tensor.

Covariant Derivative

$$x^i(y), \quad V^i = \frac{dx^i}{dy}$$

$$V\cdot \underbrace{\nabla V^j}_{\text{covariant derivative}} = V^i\left(\partial_i V^j + \Gamma^j_{\ ia} V^a\right)$$

if $x^i(y)$ is the geodesic, then $\underbrace{V\cdot\nabla V^j}_{\frac{D}{Dy}} = 0$

What does it all mean?

Define change with respect to what?

"parallel transport" defines change in a vector or tensor. e.g.

Covariant derivative is coordinate-dependent

parallel transport depends on choice of connection

given a basis e_μ for a tangent space, define connection coefficients more generally as

$$\nabla_{e_\nu} e_\mu = e_\lambda \Gamma^\lambda_{\nu\mu} .$$

More Rotating Body

$K.E. = \frac{1}{2}\langle \omega_b, \omega_b \rangle$

Moment inertia tensor gives left-invariant metric I_{ij} on G.

spatial geodesic on spatial manifold

$$L = \int dy \sqrt{\frac{dx^i}{dy}\frac{dx^j}{dy} g_{ij}}$$

$$\frac{du^i}{dy} = \frac{d^2x^i}{dy^2} = -\left(\Gamma^i_{ab}\left(\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}\right) g^i_c\right) u^a u^b$$

connection coefficients
(Levi-civita)

Covariant Derivative $\nabla_u u^i = u \cdot \nabla u^i$

tangent is covariantly constant

$$\nabla_{e_\mu} e_\nu = \Gamma^\rho_{\mu\nu} e_\rho$$

Connection tells you how local coordinates changed as vectors move around the manifold.

How does Γ transform?

It's a tensor in the ρ and ν indices. $\left(\Gamma^\rho_{\mu\nu}\right)$

$e_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$

$f_\alpha = \partial'_\alpha = \frac{\partial}{\partial y^\alpha}$

$\longrightarrow$

$$\nabla_{e_\nu} e_\mu = \Gamma^\lambda_{\ \mu\nu}\, e_\lambda$$

$$\nabla_{f_\nu} f_\mu = \tilde{\Gamma}^\lambda_{\ \nu\mu}\, f_\lambda = \nabla_{f_\nu}\left(\overbrace{\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial}{\partial x^\beta}}^{f_\mu}\right) = \frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu}\frac{\partial}{\partial x^\beta} + \frac{\partial x^\beta}{\partial y^\mu}\nabla_{f_\nu}\frac{\partial}{\partial x^\beta}$$

$$\nabla_{f_\nu} f_\mu = \boxed{\frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu}\frac{\partial}{\partial x^\beta}} + \boxed{\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\alpha}{\partial y^\nu}\Gamma^\lambda_{\ \alpha\beta}\, e_\lambda}$$

$$\nabla_{f_\nu} f_\mu = \frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu}\frac{\partial}{\partial x^\beta} + \frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\alpha}{\partial y^\nu}\Gamma^\lambda_{\ \alpha\beta}\frac{\partial}{\partial x^\lambda}$$

$$\cancel{\nabla_{f_\nu}\frac{\partial}{\partial x^\beta} = f_\nu \cdot \nabla\frac{\partial}{\partial x^\beta} \neq f_\nu}$$

Also $= \tilde{\Gamma}^\lambda_{\ \mu\nu}\, f_\lambda$

$= \Gamma^\lambda_{\ \mu\nu}\dfrac{\partial x^\beta}{\partial y^\lambda}\dfrac{\partial}{\partial x^\beta}$

match components of vector e_β

$$\Longrightarrow \quad e_\beta\left(\frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\nu} + \frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\mu}\Gamma^\beta_{\ \alpha\rho}\right) = \tilde{\Gamma}^\lambda_{\ \nu\mu}\frac{\partial x^\beta}{\partial y^\lambda}\, e_\beta$$

$$\Longrightarrow \quad \boxed{\frac{\partial y^\lambda}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu} + \frac{\partial y^\nu}{\partial x^\beta}\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\mu}\Gamma^\beta_{\ \alpha\rho} = \tilde{\Gamma}^\lambda_{\ \nu\mu}.}$$

Note: $\nabla_W = W^\alpha \nabla_\alpha$

$\Gamma^\alpha_{\ \beta\gamma}$ transforms as a tensor in any single index, or any pair except (β, γ).

Parallel Transport

- No torsion (G.R. case)
  - Γ^i_{jk} fixed in terms of g.
  - geodesics are minimum path lengths. (Euler-Lagrange eqns)
  - Γ^i_{ab} is the Levi-Civita connection

$$\Gamma^i_{ab} = g^{ic}\left(\frac{\partial g_{cb}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c}\right)$$

  - Covariant Derivative

$$W = \frac{dV^j}{dy} = -\Gamma^j_{ia}u^a \qquad \left.\begin{array}{c} W\cdot\nabla V = 0 \\ W^i\nabla_i V^j = 0 \\ \nabla_W V = 0 \end{array}\right\} \text{ if one of these is true, then } V \text{ is parallel along } W.$$

- General Affine Connection

a map $T \times T \to T$ that is distributive and a derivation.

$$\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$$

$$\nabla_X (fY) = \underbrace{X(f)}_{X \text{ acts as a differential operator on } f} Y + f\nabla_X Y$$

On a manifold with a metric, the covariant derivative acts on g by

$$\nabla_\mu g_{\lambda\rho} = \partial_\mu g_{\lambda\rho} - \Gamma^\beta_{\mu\lambda} g_{\beta\rho} - \Gamma^\beta_{\mu\rho} g_{\beta\lambda} = 0$$

Action of ∇_V on a 1-form.

$$\cancel{\nabla_V \omega(W) = \omega(\nabla_V W).}$$

Derive this by $\nabla_X(\omega, Y) = (\nabla_X \omega, Y) + (\omega, \nabla_X Y)$

$$= X^\mu \partial_\mu (\omega_\nu Y^\nu) +$$

$$= X^\mu (\partial_\mu \omega_\nu) Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu$$

$$\nabla_X (\omega, Y) = (\nabla_X \omega, Y) + (\omega, \nabla_X Y)$$

$$= X^\mu \partial_\mu (\omega_\nu Y^\nu) + (\omega, \nabla_X Y)$$

$$= X^\mu (\partial_\mu \omega_\nu) Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu$$

$$(\nabla_X \omega, Y) - (\omega, \nabla_X Y) = X^\mu (\partial_\mu \omega_\nu) Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu - (\omega, \nabla_X Y)$$

$$= X^\mu (\partial_\mu \omega_\nu) Y^\nu + \cancel{X^\mu \omega_\nu \partial_\mu Y^\nu} - \left(\omega_\mu \left(\cancel{X^\rho \partial_\rho Y^\mu} + X^\rho \Gamma^\mu_{\ \rho\sigma} Y^\sigma\right)\right)$$

$$= X^\mu (\partial_\mu \omega_\nu) Y^\nu - \omega_\mu X^\rho \Gamma^\mu_{\ \rho\sigma} Y^\sigma$$

So $\nabla_X \omega_\nu = X^\mu (\partial_\mu \omega_\nu - \Gamma^\lambda_{\ \mu\nu} \omega_\lambda)$.

$$\nabla_\mu \omega_\nu := \nabla_{e_\mu} \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\ \mu\nu} \omega_\lambda .$$

Now

$$\nabla_\mu g_{\lambda\rho} = \partial_\mu g_{\lambda\rho} - \Gamma^\beta_{\ \lambda\rho} g_{\beta\rho} - \Gamma^\beta_{\ \mu\rho} g_{\beta\lambda} = 0 .$$

Although Γ doesn't transform as one, it defines two geometric objects.

<u>Torsion</u>: $T^\lambda_{\ \mu\nu} = \Gamma^\lambda_{\ \mu\nu} - \Gamma^\lambda_{\ \nu\mu}$ is a $(1,2)$ tensor.

Alternatively, T is a map from $TM \times TM \to TM$.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T(X,Y) = T^\lambda_{\ \mu\nu} X^\mu Y^\nu . \qquad \text{antisymmetric, bilinear}$$

Torsion Tensor $\quad T^{\lambda}_{\ \mu\nu} = \Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}$

$$T(e_\mu, e_\nu) = \left(\Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}\right) e_\lambda$$

$$T(X,Y) = X^\mu Y^\nu T(e_\mu, e_\nu).$$

Curvature Tensor $\quad TM \times TM \times TM \to TM.$

$$R(X,Y,Z) = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z$$

also written like this $\longrightarrow$ $R(X,Y)Z = (X^\mu Y^\nu Z^\rho)(R(e_\mu, e_\nu)e_\rho)$

10/22/14

On a manifold, structures $g_{\mu\nu} \leftarrow$ distance

$\Gamma^{\lambda}_{\mu\nu} \leftarrow$ parallel transport.

Γ should be metric compatible.

that is, $\nabla g = 0.$

$$\partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu} = 0$$

if $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$, Levi-Civita Connection.

In general, $T^{k}_{\mu\nu} = 2\Gamma^{k}_{[\mu\nu]} = [\Gamma^{k}_{\mu\nu} - \Gamma^{k}_{\nu\mu}]$

~~Tor~~ ~~Torsion: TM)~~

Torsion: $TM \times TM \longrightarrow TM.$ is a (1,2) - Tensor

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Since $T=0$ for Levi-Civita connection,

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

Since $T(X,Y)$ is geometric,

$$T(X,Y) = X^{\mu} Y^{\nu} T(e_{\mu}, e_{\nu})$$

$$T(e_{\mu}, e_{\nu}) = \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right) e_{\lambda}.$$

Riemann Tensor

$$R: TM \times TM \times TM \longrightarrow TM$$

$$R(X,Y)Z = R(X,Y,Z) = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z$$

$$R(X,Y)Z = X^{\mu} Y^{\nu} Z^{\rho} R(e_{\mu}, e_{\nu}) e_{\rho}$$

$$R(e_{\mu}, e_{\nu}) e_{\rho} = [\nabla_{e_{\mu}}, \nabla_{e_{\nu}}] e_{\rho} - \nabla_{[e_{\mu}, e_{\nu}]} e_{\rho}$$

coordinates commute, so vanishes.

$$= \nabla_{e_{\mu}} \left[\Gamma^{\lambda}_{\nu\rho} e_{\lambda}\right] - \nabla_{e_{\nu}} \left[\Gamma^{\lambda}_{\mu\rho} e_{\lambda}\right]$$

$$= \partial_{\mu} \Gamma^{\lambda}_{\nu\rho} e_{\lambda} + \Gamma^{\lambda}_{\nu\rho} \Gamma^{\beta}_{\mu\lambda} e_{\beta} - \partial_{\nu} \Gamma^{\lambda}_{\mu\rho} e_{\lambda} - \Gamma^{\lambda}_{\mu\rho} \Gamma^{\beta}_{\nu\lambda} e_{\beta}.$$

$R(e_{\mu}, e_{\nu}) e_{\rho}$

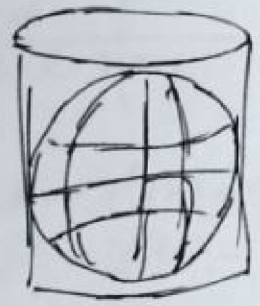
$$= \left(\partial_{\mu} \Gamma^{\lambda}_{\nu\rho} - \partial_{\nu} \Gamma^{\lambda}_{\mu\rho}\right) + \left(\Gamma^{\sigma}_{\nu\rho} \Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\sigma}_{\mu\rho} \Gamma^{\lambda}_{\nu\sigma}\right)$$

$R^{\lambda}_{\rho\mu\nu}$

antisymmetric in μ and ν.

Example: Mercator Projection

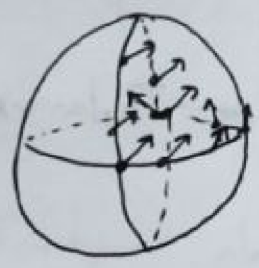
Projecting a sphere on to a cylinder



Define a connection so that Parallel Transport should maintain constant angles with meridians, ~~with no torsion~~

(In general, parallel transport A to B is path dependent.)

how ships navigate by compass. The paths that are straight on a sphere w/ Mercator connection (geodesics) are called "loxodromes"



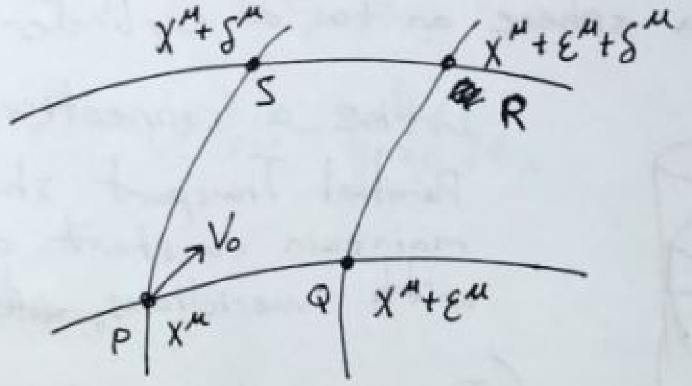
Different result of parallel transport around different paths

Holonomy group is group of transformations by parallel transport on V (a vector)

e.g. holonomy group of S^2 w/ Levi-Civita connection is $SO(2)$

holonomy is fixed by curvature.

Holonomy fixed by curvature



(1) Parallel Transport V_0 from P to Q two ways. Through S, and through R.

$$\varepsilon\cdot\nabla V^\mu = 0 \implies \varepsilon^\mu[\partial_\mu V^\rho + \Gamma^\rho_{\mu\nu}V^\nu] = 0$$

$$V^\mu(Q) = V^\mu(P+\varepsilon^\mu) = V^\mu(P) + \varepsilon^\nu\partial_\nu V^\mu(P) \leftarrow \text{Taylor expand}$$

$$V^\mu(P+\varepsilon^\mu) = V^\mu(P) - \varepsilon^\rho\Gamma^\mu_{\rho\nu}(P)V^\nu(P)$$

Similarly, $V^\mu(R) = V^\mu(Q) - \delta^\rho\Gamma^\mu_{\rho\nu}(Q)\,V^\nu(Q)$

$$= V^\mu(P) - \varepsilon^\rho\Gamma^\mu_{\rho\nu}(P)V^\nu(P) - \delta^\rho\Gamma^\mu_{\rho\nu}(P)V^\nu(Q)$$

$$- \delta^\rho\varepsilon^\lambda\partial_\lambda\Gamma^\mu_{\rho\nu}(P)V^\nu(Q)$$

$V^\nu(P) + O(\varepsilon)$

$$V^\mu_{PQR}(R) = V^\mu(P) - \varepsilon^\rho\Gamma^\mu_{\rho\nu}(P)V^\nu(P)$$

$$- \delta^\rho\Gamma^\mu_{\rho\nu}(P)V^\nu(P) - \delta^\rho\varepsilon^\lambda\Gamma^\mu_{\rho\nu}\partial_\lambda V^\nu$$

$$- \delta^\rho\varepsilon^\lambda\partial_\lambda\Gamma^\mu_{\rho\nu}V^\nu$$

If we go around the other way, switch ε, δ in the above.

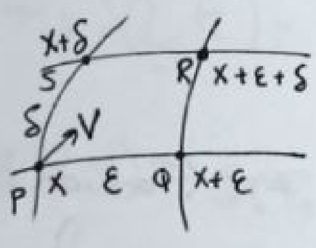
$V^\mu_{PSR}(R)$ is $V^\mu_{PQR}(R)$ with ε, δ interchanged.

$$V^\mu_{PSR} - V^\mu_{PQR} = \cancel{\delta^\rho\varepsilon^\lambda\Gamma^\mu_{\rho\nu}\partial_\lambda V^\nu} + \delta^\rho\varepsilon^\lambda\partial_\lambda\Gamma^\mu_{\rho\nu}V^\nu$$
$$- \cancel{\delta^\lambda\varepsilon^\rho\Gamma^\mu_{\rho\nu}\partial_\lambda V^\nu} - \delta^\lambda\varepsilon^\rho\partial_\lambda\Gamma^\mu_{\rho\nu}V^\nu.$$

(Lost some terms somewhere...)

10/24/14

Parallel Transport along two different paths



First transport along PQR

$$V^\rho(Q) = V^\rho(P) - \varepsilon^\mu \Gamma^\rho_{\mu\nu}(P) V^\nu(P)$$
$$V^\rho(R) = V^\rho(Q) - \delta^\mu \Gamma^\rho_{\mu\nu}(\varepsilon) V^\nu(\varepsilon)$$

with some taylor expansions.

Substitute

$$V^\rho(R) = V^\rho(P) - \varepsilon^\mu \Gamma^\rho_{\mu\nu}(P) V^\nu(P) - \delta^\mu \Gamma^\rho_{\mu\nu}(P) V^\nu(P) - \delta^\mu \varepsilon^\lambda \partial_\lambda \Gamma^\rho_{\mu\nu}(P) V^\nu(P) + \left(\delta^\lambda \Gamma^\rho_{\lambda\nu}\right)\left(\varepsilon^\mu \Gamma^\nu_{\mu\sigma} V^\sigma(P)\right)$$

Transport along PSR.

$$V^\rho_{PSR} - V^\rho_{PQR} = -\varepsilon^\mu \delta^\lambda \partial_\lambda \Gamma^\rho_{\mu\nu} V^\nu + \varepsilon^\lambda \delta^\mu \Gamma^\rho_{\lambda\nu} \Gamma^\nu_{\mu\sigma} V^\sigma + \delta^\mu \varepsilon^\lambda \partial_\lambda \Gamma^\rho_{\mu\nu} V^\nu - \delta^\lambda \varepsilon^\mu \Gamma^\rho_{\lambda\nu} \Gamma^\nu_{\mu\sigma} V^\sigma$$

Pull out V^ν out of every term and also an $\varepsilon^\mu \delta^\lambda$

$$= \varepsilon^\mu \delta^\lambda \left[-\partial_\lambda \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\hat{\sigma}} \Gamma^{\hat{\sigma}}_{\lambda\nu} + \partial_\mu \Gamma^\rho_{\lambda\nu} - \Gamma^\rho_{\lambda\hat{\sigma}} \Gamma^{\hat{\sigma}}_{\mu\nu} \right] V^\nu$$

$$= \boxed{V^\nu \varepsilon^\mu \delta^\lambda R^\rho_{\ \nu\mu\lambda}.}$$

What are the symmetries of the Rieman Tensor?

Bianchi Identities

- $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0 \leftarrow$ (Torsion Free)

in components: $R^k_{\lambda\mu\nu} + R^k_{\mu\nu\lambda} + R^k_{\nu\lambda\mu} = 0$

- $\nabla_X R(Y,Z)V + \nabla_Z R(X,Y)Z + (\nabla_Y R)(Z,X)V = 0$

$$(\nabla_\sigma R)^\rho{}_{\lambda\mu\nu} + (\nabla_\mu R)^\rho{}_{\lambda\nu\sigma} + (\nabla_\nu R)^\rho{}_{\lambda\sigma\mu} = 0$$

<u>Proof of first identity:</u>

Define $S(f(X,Y,Z)) = f(X,Y,Z) + f(Y,Z,X) + f(Z,X,Y)$

1^{st} identity is $S(R(X,Y)Z) = 0$.

use torsion free condition: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

<u>consider:</u> $\nabla_Z T(X,Y) = \nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_Z [X,Y]$

claim $\nabla_Z [X,Y] = \nabla_{[X,Y]} Z + [Z,[X,Y]]$

let $W = [X,Y]$. The above identity is $\nabla_Z W = \nabla_W Z + [Z,W]$.
follows from $T(X,Y) = 0$.

$$\nabla_Z T(X,Y) = \nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_{[X,Z]} Z + [Z,[X,Y]] = 0$$

We also know that

$$S(\nabla_Z T(X,Y)) = 0.$$

$$S\left(\nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_{[X,Y]} Z + [Z,[X,Y]]\right) = 0$$

Now, $S\left(\sum_i f_i\right) = \sum S(f_i)$, so distribute S over objects.

$$S(\nabla_Z \nabla_X Y) - S(\nabla_Z \nabla_Y X) - S(\nabla_{[X,Y]} Z) + \underbrace{S([Z,[X,Y]])}_{0 \text{ by jaccbi identity}} = 0$$

$$= S(\nabla_X \nabla_Y Z).$$

$$\Rightarrow S(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = 0.$$

Symmetry Properties of R where there is no torsion

$$R^{\lambda}_{\ \rho\mu\nu} = -R^{\lambda}_{\ \rho\nu\mu}$$

define $R_{k\rho\mu\nu} = g_{k\lambda} R^{\lambda}_{\ \rho\mu\nu}$. Then

$$R_{k\rho\mu\nu} = R_{\mu\nu k\rho}$$

$$R_{k\rho\mu\nu} = -R_{\rho k\mu\nu}$$

$\Rightarrow$ Rank of R is $\frac{1}{12} d^2(d^2-1)$

When $d=4$, rank of R is 20.

Sectional Curvature

Let $p \in M$ and $U, V \in T_pM$.

$$k(U,V) = \frac{\langle R(U,V)V, U\rangle}{\langle U,U\rangle\langle V,V\rangle - \langle U,V\rangle^2}$$

symmetric in U and V.

area of parallelogram defined by U, V.

Geodesic for the Rigid Body Problem

Euler-Arnold equations.

Assume we have a torsion-free metric.
Want to solve $\nabla_X X = 0$ to find the connection.

Torsion-free $\Longrightarrow$ $\nabla_X Y - \nabla_Y X = [X,Y]$ ① $= \mathcal{L}_X Y$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad ②$$

① and ② imply

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle + \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle.$$

proof on next page

~~Proof:~~ $\cancel{\nabla_X Y = [X,Y] + \nabla_Y X}$, ~~so~~

$$\langle \nabla_X Y, Z\rangle = \langle [X,Y],Z\rangle + \langle \nabla_Y X, Z\rangle$$

$$\cancel{= \langle [X,Y],Z\rangle + \langle [Y,X],Z\rangle + \langle \nabla_X Y, Z\rangle}$$

?

$$\cancel{= \langle [X,Y],Z\rangle + \langle [Y,X],Z\rangle + \langle}$$

$$= \langle [X,Y],Z\rangle + Y\langle X,Z\rangle - \langle X, \nabla_Y Z\rangle$$

$$\cancel{= \langle [X,Y],Z\rangle + Y\langle X,Z\rangle}$$

$$\cancel{\langle [X,Y],Z\rangle + Y}$$

$$= Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle \nabla_Z X, Y\rangle$$

Proof: $\nabla_X Y = [X,Y] + \nabla_Y X$

$$\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + \langle \nabla_Y X, Z\rangle \quad ⊛$$

using ②

$$\langle \nabla_Y X, Z\rangle = Y\langle X, Z\rangle - \langle X, \nabla_Y Z\rangle$$

substitute into ⊛

$$\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle + Y\langle X,Z\rangle - \langle X, \nabla_Y Z\rangle \quad \text{use ①}$$

$$= \langle [X,Y], Z\rangle + Y\langle X,Z\rangle - \left(\langle X, [Y,Z]\rangle + \langle X, \nabla_Z Y\rangle\right)$$

②

$$= \langle [X,Y], Z\rangle + Y\langle X,Z\rangle - \langle X, [Y,Z]\rangle - \cancel{\langle X,[Z,Y]\rangle + \langle X, \nabla_Y Z\rangle}$$

$$Z\langle X,Y\rangle + \langle \nabla_Z X, Y\rangle$$

$$= \langle [X,Y], Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \left(-\langle [Z,X], Y\rangle + \langle \nabla_X Z, Y\rangle\right)$$

$$= \langle [X,Y], Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle [Z,X], Y\rangle + +X\langle Z,Y\rangle + \langle Z, \nabla_X Y\rangle$$

$$\Longrightarrow 2\langle \nabla_X Y, Z\rangle = \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle$$
$$+ X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle. \qquad \blacksquare$$

For left invariant vector fields and left invariant metric,

$$\langle \nabla_X Y, Z\rangle = \left(\langle [X,Y], Z\rangle - \langle X, [Y,Z]\rangle + \langle [Z,X], Y\rangle\right)\left(\tfrac{1}{2}\right)$$

because the terms $X\langle Y,Z\rangle$ are the translations of left invariant ~~vector~~ vector fields, and so vanish.

Now if $\tilde{X}$ is a vector on the rigid body manifold,

$$\tilde{X} = \tilde{X}^i \partial_i = \partial_t + X^\alpha \partial_\alpha \qquad i=0 \text{ is time.}$$

$\Gamma^i_{0k} = 0$ b/c there is no time dependence on our manifold.

Claim:

$$\nabla_X Y = \frac{1}{2}\left(Ad_{*X} Y - Ad^*_X Y - Ad^*_Y X\right)$$

Recall

$Ad^*_X Y = [X,Y]$

$\langle Ad^*_X Y, Z\rangle = \langle Y, Ad_X Z\rangle$

Proof:

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(\langle Ad_{*X} Y, Z\rangle - \langle Ad^*_X Y, Z\rangle - \langle Ad^*_Y X, Z\rangle\right)$$

$$\langle X, [Y,Z]\rangle \qquad \langle Y, [X,Z]\rangle$$

Holds for all Z. ∎

Geodesic equation:

$$\nabla_X X = 0$$

Use the claim, and $Ad^*_X X = 0$, so the geodesic equation becomes

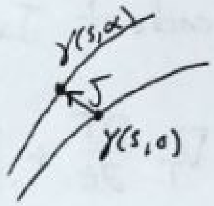
$$\boxed{\cancel{Ad_{*X} X = 0}} \quad \text{Euler Arnold Equation.}$$

$$\boxed{Ad^*_X X = 0}$$

Metric gives canonical isomorphism between tangent and cotangent spaces, so we may identify the two (insert metrics where needed).

10/31/14

J is the "geodesic deviation"



want $J \text{ "=" } \gamma(s,\alpha) - \gamma(s,0)$

infinitesimally, $= \alpha \dfrac{\partial \gamma(s)}{\partial \alpha} = \alpha J$

$J = \partial_\alpha \gamma$. Let T be the tangent to $\gamma(s)$

· $\nabla_\alpha \nabla_s T = 0$ ← because T is tangent to γ along s, so $\nabla_s T = 0$.

But also, $\nabla_\alpha \nabla_s T = \nabla_s \nabla_\alpha T - R(T,J)T$

$$= \nabla_s \nabla_\alpha (\partial_s \gamma) - R(T,J)T$$

$$= \nabla_s \nabla_s \partial_\alpha \gamma + R(J,T)T$$ ← follows from $[\partial_\alpha, \partial_s] = 0$ and $\Gamma_{ij}^k = \Gamma_{ji}^k$ (HW)

Thus, $\boxed{\nabla_s \nabla_s J + R(J,T)T = 0}$ ← Jacobi Equation

How does $|J|$ change?

$\partial/\partial s\, |J| = ?$

Consider $\frac{1}{2} \frac{\partial}{\partial s} \frac{\partial}{\partial s} |J|^2 = \frac{1}{2} \frac{\partial}{\partial s} \frac{\partial}{\partial s} \langle J, J \rangle = \frac{1}{2} \frac{\partial}{\partial s} \left(\langle \frac{\partial J}{\partial s}, J \rangle + \langle J, \frac{\partial J}{\partial s} \rangle \right)$

$$= \frac{1}{2} \frac{\partial}{\partial s} 2 \langle J, \frac{\partial J}{\partial s} \rangle = \frac{\partial}{\partial s} \langle J, \frac{\partial J}{\partial s} \rangle = \langle \frac{\partial J}{\partial s}, \frac{\partial J}{\partial s} \rangle + \langle J, \frac{\partial^2 J}{\partial s^2} \rangle$$

$$= \langle \frac{\partial^2 J}{\partial s^2}, J \rangle + |\nabla_T J|^2.$$

Combine with the Jacobi Equation

$$= \langle -R(J,T)T, J \rangle + |\nabla_T J|^2 = |\nabla_T J|^2 - \underbrace{k(T,J)}_{\text{sectional curvature (by defn)}}$$

To add time dependence, replace T by $\tilde{T}$

$$T \longmapsto \tilde{T} = T_0 \partial_0 + T^i \partial_i = \partial_t + T^i \partial_i.$$

$$\nabla_T \nabla_T J \longmapsto (\partial_t + \nabla_T)(\partial_t + \nabla_T) J$$

Time-dependent Jacobi Equation

$$\frac{\partial^2 J}{\partial t^2} + \nabla_T \frac{\partial J}{\partial t} + \frac{\partial}{\partial t}(\nabla_T J) + \nabla_T \nabla_T J + R(J,T)T = 0.$$

<u>Sperical Symmetry</u>

$I = c\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ set $c=1$

The inner product on the manifold is $\langle X, Y \rangle = \hat{X} \cdot \hat{Y} = tr(XY)$

<u>Claim</u>: I is bi-invariant.

By definition, it's left invariant.

$$\langle L_{*g} X, L_{*g} Y \rangle = \langle X, Y \rangle$$

$$= tr((L_{*g}X)(L_{*g}Y)) \stackrel{?}{=} tr((R_{*g}X)(R_{*g}Y))$$

To show this, we will demonstrate Ad invariance of the metric.

$$\langle X, Y \rangle \to \langle gXg^{-1}, gYg^{-1} \rangle = tr(gXg^{-1}gYg^{-1}) = tr(gXYg^{-1})$$
$$= tr(XY).$$

trace is cyclic.

11/03/14

Spherically symmetric top with metric proportional to the identity. $J = c\mathbb{1}$. ← bi-invariant metric.

(A) 1-parameter subgroups are geodesic

(B) Sectional curvature is positive definite.

<u>Theorem</u>: For any compact Lie group there is a unique bi-invariant metric $\langle A, B \rangle = \frac{1}{2} tr(AB)$ called the Cartan-Killing form.

Proof of (A):

$$\nabla_X Y = \frac{1}{2}\left(Ad_{X*} Y - Ad^*_Y X - Ad^*_X Y\right)$$

Claim last two terms vanish

$$\langle Ad^*_Y X, Z\rangle = \langle Y, [X,Z]\rangle$$
$$\langle Ad^*_X Y, Z\rangle = \langle X, [Y,Z]\rangle$$

→ Use dual vectors

$$\langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle$$
$$= \hat{Y}^a(\hat{X}^i \hat{Z}^j)\varepsilon^{ija} + \hat{X}^a \hat{Y}^i \hat{Z}^j \varepsilon^{ija}$$
$$= \hat{Y}^a(\hat{X}^i \hat{Z}^j)\varepsilon^{ija} + \hat{X}^a \hat{Y}^i \hat{Z}^j(-\varepsilon^{aji})$$
$$= 0$$

Therefore, $\nabla_X Y = \frac{1}{2}[X,Y]$

$\gamma = e^{\lambda X}$ ~~[illegible]~~ 1-parameter subgroup parameterized by X in the Lie Algebra.

$\nabla_X X = 0 \longrightarrow X$ is a geodesic (solves Euler Arnold equation)

Time Dependent equation $\partial_t X = 0$ "stationary flow"

Proof of (B)

Consider sectional curvature for unit vectors

$$K(X,Y) = \langle R(X,Y)Y, X\rangle$$

$$R(X,Y)Y = \nabla_X \overset{0}{\cancel{\nabla_Y Y}} - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y$$

$$= -\frac{1}{2}\nabla_Y(\hat{X}\times\hat{Y}) - \frac{1}{2}(\hat{X}\times\hat{Y})\times\hat{Y}$$

$$= -\frac{1}{4}\hat{Y}\times(\hat{X}\times\hat{Y}) - \frac{1}{2}(\hat{X}\times\hat{Y})\times\hat{Y}$$

$$= \frac{1}{4}\hat{Y}\times(\hat{X}\times\hat{Y})$$

$$K(X,Y) = \tfrac{1}{4}\langle \hat{Y}\times(\hat{X}\times\hat{Y}), \hat{X}\rangle$$

$$= \tfrac{1}{4}\left(X^a\left(Y^i\left(X^cY^d\varepsilon^{cdj}\right)\varepsilon^{ija}\right)\right)$$

$$= -\tfrac{1}{4}\left(X^a\left(Y^iY^dX^c\right)\left(\delta^{ci}\delta^{ad} - \delta^{ac}\delta^{di}\right)\right)$$

$$= -\tfrac{1}{4}\left((X\cdot Y)^2 - (|X|^2|Y|^2)\right) = \tfrac{1}{4}\left(|X|^2|Y|^2 - (X\cdot Y)^2\right) > 0$$

Cauchy Schwarz

$$= \tfrac{1}{4}|\hat{X}\times\hat{Y}|^2 = \tfrac{1}{4}|\nabla_X Y|^2$$

Jacobi Equation

$$\frac{1}{2}\frac{\partial}{\partial s}\frac{\partial}{\partial s}|J|^2 = |\nabla_T J|^2 - K(T,J) = 0$$

Since we have t independence, the second derivative of $|J|^2$ is zero. Soln to this equation shows $|J|^2$ is polynomial in s.

Called "neutral stability".

<u>Assymmetric Top</u>

$$\langle X, Y\rangle = J\hat{X}\cdot\hat{Y}$$

previously, $Ad_X^* Y = J^{-1}(J\hat{Y}\times\hat{X})$

So $\nabla_X Y = \frac{1}{2}\left(J[X,Y] - J^{-1}(J\hat{Y}\times\hat{X}) - J^{-1}(J\hat{X}\times\hat{Y}\right)$

$$= \tfrac{1}{2}J^{-1}\left(J[X,Y] - (J\hat{Y})\times\hat{X} - (J\hat{X})\times\hat{Y}\right)$$

Write J as $J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}$.

Equation becomes

$$\nabla_x Y = \frac{1}{2} J^{-1}(K\hat{X} \times \hat{Y})$$

$$K = \begin{pmatrix} J_2+J_3-J_1 & & \\ & J_1+J_2-J_3 & \\ & & J_1+J_3-J_2 \end{pmatrix}$$

$J_i = J_{ii}$ ← no sum

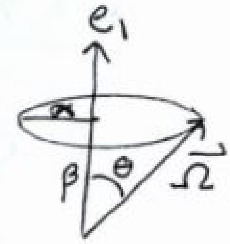
all elements > 0.

Recall $J_{ij} = \int \rho \, (-x_i x_j + |x|^2 \delta_{ij})$

$$J_2 + J_3 = \int \rho \, ((|x|^2 - x_2^2) + (|x|^2 - x_3^2))$$

$$J_{22} + J_{33} = \int \rho \, (x_1^2 + x_3^2 + x_1^2 + x_2^2) \geq 0.$$

<u>Symmetric Top.</u>



$$J_2 = J_3 = J_\perp \neq J_1$$

<u>Euler Eqns:</u>

$$J_1 \dot{\Omega}_1 - (J_2 - J_3)\Omega_2\Omega_3 = 0$$
$$J_2 \dot{\Omega}_2 - (J_3 - J_1)\Omega_1\Omega_3 = 0$$
$$J_3 \dot{\Omega}_3 - (J_1 - J_2)\Omega_1\Omega_2 = 0$$

Set $\Omega_2 + i\Omega_3 = \alpha e^{i\omega t}$ where $\omega = \dfrac{\Omega^1(J_1 - J_\perp)}{J_\perp}$

$\Omega_1 = \beta$ $\quad \alpha = |\Omega| \sin\theta$
$\quad \beta = |\Omega| \cos\theta$

Metric is $\begin{bmatrix} J_1 & & \\ & J_\perp & \\ & & J_\perp \end{bmatrix}$, has a Killing vector of the form

$\xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & R_{e_1} \end{bmatrix}$ where R_{e_1} is ~~the~~ a generator of rotation around the e_1-axis.

Conserved quantity corresponding to this killing vector is

$$\langle X, \xi \rangle = \text{const}$$

geodesic

Using $\widehat{\nabla_X Y} = -\frac{1}{2} J^{-1}(K\hat{X} \times \hat{Y})$, with $K = \begin{bmatrix} 2J_\perp - J_1 & & \\ & J_1 & \\ & & J_1 \end{bmatrix}$

dual of this guy so we can write it as a vector

$$\widehat{\nabla_Y X} = \frac{1}{2} J^{-1}(K\hat{Y} \times \hat{X})$$

$$= \frac{1}{2} J^{-1}\left[0,\ J_1 \hat{Y}^3 \hat{X}^1,\ -J_1 Y^2 X^1\right]$$

$$\langle \nabla_Y X, Z \rangle = \frac{1}{2}(J_1 Y^3 X^1 Z^2 - J_1 Y^2 X^1 Z^3) = \frac{J_1}{2}(Y^3 Z^2 - Y^2 Z^3) X_1 .$$

$$\Rightarrow \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0.$$

Choose $Z = Y$ (geodesic).

$\nabla_Y Y = 0.$

$$\langle \nabla_Y X, Y \rangle + \langle \nabla_Y X, Y \rangle = 0 \Longrightarrow \langle \nabla_Y X, Y \rangle = 0 \Rightarrow \nabla_Y \langle X, Y \rangle = 0$$

So $\langle X, Y \rangle$ is constant along Y.

Therefore, $\langle X, Y \rangle = J_1 X^1 Y^1 = \text{const.} \Longrightarrow J_1 Y^1 = \text{const.}$

if $Y = \Omega$, $J_1 \Omega^1 = \text{const.}$

Stability Analysis: exchange $J \to B$ in Jacobi Equation.

Solve the second order equation for B, find Neutral Stability.

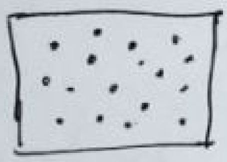
11/05/14

Fluid Motion (Hydrodynamics)

What is a fluid?

a system of particles that have Mean Free Path $\ll L$, where L is any observable of interest.

Continuum approximation is good.

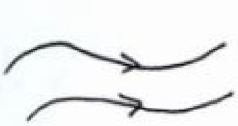


Each particle is identical

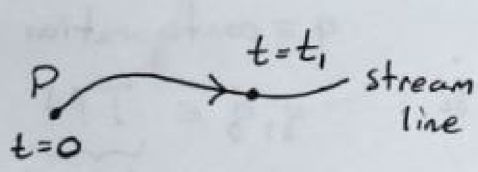
No two can occupy same position.

STEADY-STATE FLOW.

particles move along streamlines



b/c particles are identical, looks the same at any time.

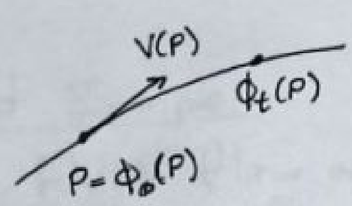


Map ϕ_t that takes a particle at P and takes it to $\phi_t(P)$

"Lagrangian Representation"

In terms of local coordinates, x^j, j^{th} component of particle is

$$x^j \circ \phi_t(P) = x^j_t(P) = x^j(t, x_{t=0})$$



$$V(P) = \frac{d}{dt}\phi_t(P)\Big|_{t=0} = V^i \partial_{x^i}$$

$$V^i = \frac{dx^i_t(P)}{dt}\Big|_{t=0}$$

Stream lines $V^i(x)$ are tangent to integral curves.

$$\frac{dx^i}{dt} = V^i(x(t))$$

ϕ_t is a diffeomorphism, (we assume this)

The group we care about is Diff^n (infinite dimensional group)

The algebra is the simplest example of a Kac-Moody Algebra.

elements of Diff^n are diffeomorphisms on the manifold

n is the dimension of ~~spacetime~~ space.

Time dependent flow: add time component

$$V^i\partial_i \rightarrow \partial_0 + V^i\partial_i$$

Streamlines change shape over time.

FIBER BUNDLES on manifold M

Lagrangian is function of $q_i, \dot{q}_i$

$q \equiv$ configuration variables.

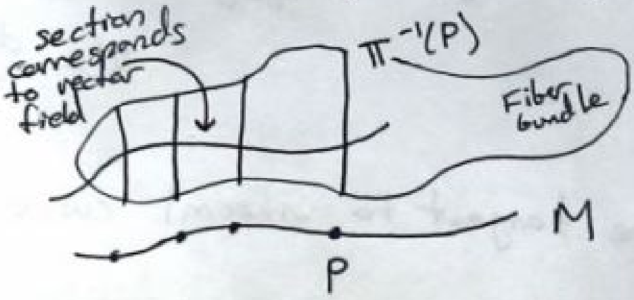
$q, \dot{q} \in \underbrace{TM}_{\text{tangent bundle}}$

$$TM = \bigsqcup_{p \in M} T_pM = \text{collection of } \underline{\text{all}} \text{ tangent vectors at all points.}$$

Associated with any bundle space is a projection map

$$\pi_i(Q) = q_i \qquad \pi_i : TM \longrightarrow T_{P_i}M$$
$$Q \longmapsto \text{component at } p_i .$$

$\pi_i^{-1}(q)$ = "fiber at q" for $q \in M$



Example:

If $M = \mathbb{R}^n$, has a tangent vectors in $\mathbb{R}^n$.

The ~~[illegible]~~ tangent bundle is $\mathbb{R}^n \otimes \mathbb{R}^n$

describes position on manifold

describes tangent vectors.

"Tangent Bundle" and "Vector Bundle" are interchangeable.

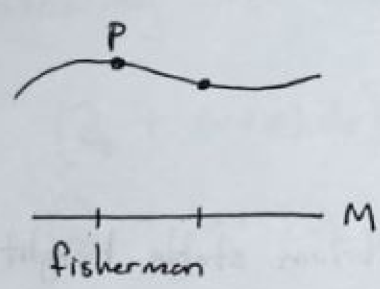
Parallel Transport is a map from one fiber to a nearby one.

Lie Derivatives in context of fluids

given $X \in TM$ can associate it with a flow

$$\xi_t : M \longrightarrow M \qquad \xi_0 = id \qquad X(x) = \frac{d}{dt}\xi_t(x)$$

Consider a scalar function on M.



"fisherman derivative"

$$\mathcal{L}_X f(x) = \frac{d}{dt}\left(\xi_t^* f\right)(x)\Big|_{t=0}$$

$$= \frac{d}{dt}(f \circ \xi_t)(x)\Big|_{t=0}$$

$$= X^i \frac{\partial}{\partial x^i} f(x)$$

Also called "lagrange derivative".

For unsteady flow, $\left(\partial_t + X^i \partial_{x^i}\right) f(x)$.

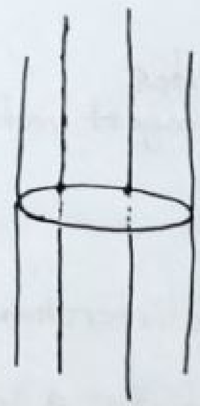
Easiest Example:

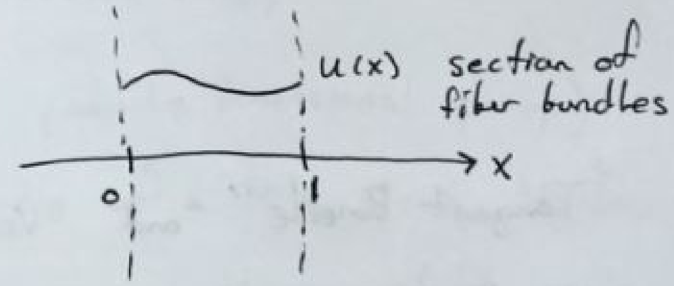
Fluid flow on S^1: Group is $\mathrm{Diff}\, S^1$ "loop group"

Coordinatize by x such that $x = x+1$.

Tangent space at each point is a line.

TM is cylinder

TM = 



Infinitesimal Diffeomorphism, ~~$x \mapsto x + \xi(x)$~~ $x \longmapsto x + \varepsilon(x)$

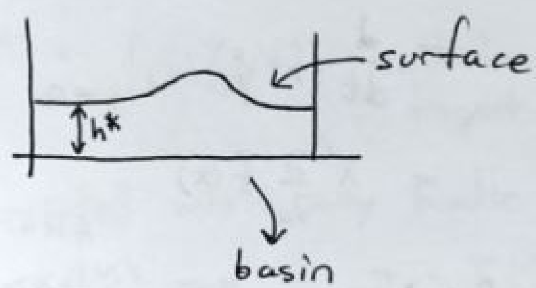
Algebra generated by $\varepsilon(x)\frac{\partial}{\partial x}$

Consider two generators $u(x)$, $v(x)$

$$\left[u(x)\frac{\partial}{\partial x}, v(x)\frac{\partial}{\partial x}\right] = \left(u'(x)v(x) - v'(x)u(x)\right)\frac{\partial}{\partial x}$$ "Witt Algebra"

Infinite dimensional because these are functions.

Shallow Water Waves in 1D



h^* = "equilibrium stable height"

$u(x,t)$ velocity of particle at x and time t.

Ignoring non-linear effects, (all amplitude differences) a first approximation is the wave equation.

$$\partial_t^2 u - c_*^2 \partial_x^2 u = 0 \qquad c_* = \sqrt{g h_*}$$, where g = gravitational constant.

11/07/14

Fluid Mechanics

Group Manifold: Diff S^1

- calculate the geodesics
- calculate sectional curvature

Shallow Water Waves

h^* is undisturbed wave height

h is height of wave above undisturbed height

Small waves! $\frac{h}{h^*} \ll 1$ leads to wave eqn $\partial_t^2 u - c_*^2 \partial_x^2 u = 0$.

$u(x,t)$ is velocity field.

in this approximation $u \sim h$

Allowing for nonlinearities, the equations of motion are

Ⓐ $\left(\partial_t + (u+c)\partial_x\right)(u+2c) = 0$ where $c = \sqrt{gh}$

Ⓑ $\left(\partial_t + (u-c)\partial_x\right)(u-2c) = 0$

Solve these by the method of characteristics:

Consider a curve $(t(s), x(s))$ such that

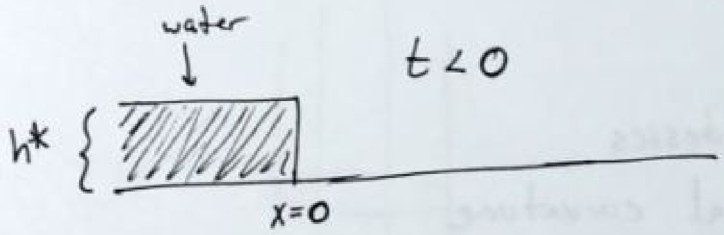
$\pm$ for ~~...~~ the two cases Ⓐ and Ⓑ

$\frac{dx}{ds} = u \pm c$ $\frac{dt}{ds} = 1$. This guarantees that Ⓐ, Ⓑ have constant solutions along characteristic curves:

$$Ⓐ = \left(\frac{dt}{ds}\frac{d}{dt} + \frac{dx}{ds}\frac{d}{dx}\right)(u+2c) = \frac{d}{ds}(u+2c) = 0$$

$$Ⓑ = \left(\frac{dt}{ds}\frac{d}{dt} + \frac{dx}{ds}\frac{d}{dx}\right)(u-2c) = \frac{d}{ds}(u+2c) = 0$$

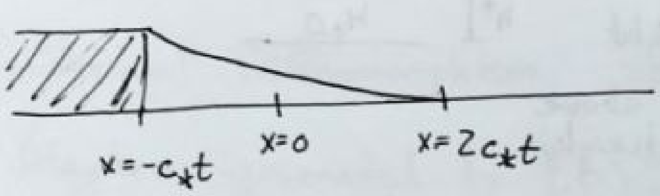
The Dam Problem: water blocked by a dam, removed at $t=0$



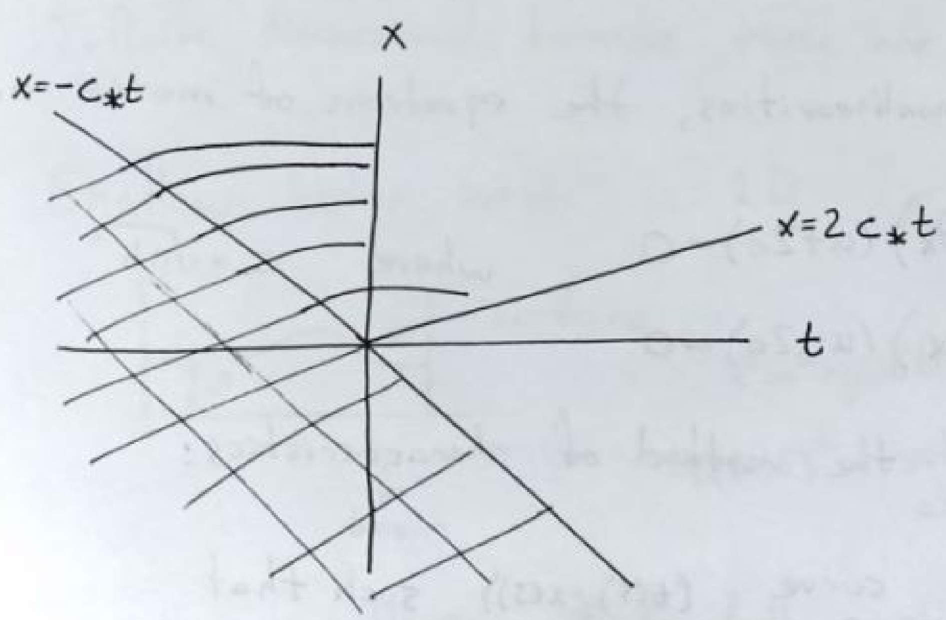
$x>0 \quad u=0$

$c = c_* = \sqrt{gh}$

$\downarrow \; t>0$



Characteristic Curves



In the region $x < -c_* t$,

then $u + 2c = 2c_* \implies c = \dfrac{2c_* - u}{2} = c_* - \dfrac{u}{2}$.

Plug in to equation Ⓐ,

$$\left(\partial_t + (u \mp (c_* - u/2))\partial_x\right)(u \mp (2c^* - u)) = 0$$

In equation B, instead have $u-2c=2c_*$

$$c=\frac{u}{2}-c_*$$

$$\left(\partial_t+\left(u-\left(\frac{u}{2}-c_*\right)\right)\partial_x\right)\left(u-2\left(\frac{u}{2}-c_*\right)\right)=0$$

Both of these give us the equation $\left(\partial_t+\left(\frac{u}{2}+c_*\right)\partial_x\right)(c_*)=0$.

But flipping them and substituting $c=\frac{u}{2}-c_*$ into A or $c=-\frac{u}{2}+c_*$ into B gives

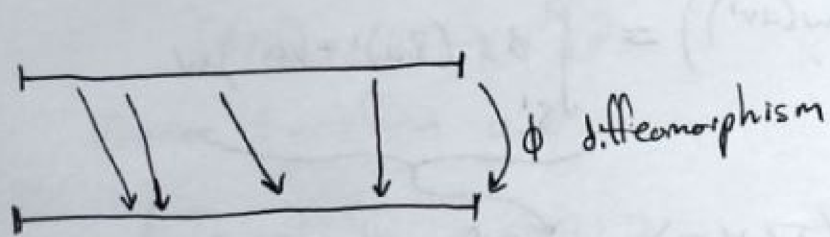
$$\left(\partial_t+\left(\frac{3u}{2}-c_*\right)\partial_x\right)(u)=0$$

Let $v=\frac{3u}{2}-c_*$, so we have now $\partial_t v+v\partial_x v=0$

unstable waves only from this equation.

The reason that people didn't find the stable wave solutions is because they didn't add a central extension to the Lie algebra $\mathrm{Diff}\, S^1$.

Unroll S^1



Algebra is $[X,Y]=(uv'-vu')\partial_x$

Introduce a right-invariant metric

Right - Invariant Metric

$$U_g g^{-1} = U_e g g^{-1} = U_e$$

$$\langle U,V\rangle_g = \int_{S_1} dx \left(U_g \circ g^{-1}, V_g \circ g^{-1}\right)_g = \langle U,V\rangle_e$$

U_g, V_g right invariant vector fields on S'.

Let $X, Y \in T_e(\mathrm{Diff}\, S')$.

As before, we have $\nabla_X Y = \frac{1}{2}\left(Ad_X Y - Ad_X^* Y - Ad_Y^* X\right)$

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\langle [X,Y], Z\rangle - \frac{1}{2}\langle X, [Y,Z]\rangle - \frac{1}{2}\langle Y, [X,Z]\rangle$$

let $X = U\partial_x$, $Y = V\partial_x$, $Z = W\partial_x$

$$= \int_{S'} dx\, \frac{1}{2}\left((uv' - vu')w - u(vw' - wv') - v(uw' - wu')\right)$$

$$= \frac{1}{2}\int_{S'} dx\, \left(-2uvw' + 2uwv' + vwu' - vwu'\right) = \int_{S'} dx\, \left(uwv' - uvw'\right)$$

$$= \int_{S'} dx\, \left(w \frac{\partial}{\partial x}(uv) + w(uv')\right) = \underbrace{\int_{S'} dx\, (2uv' + vu')w}$$

integrate by parts, term vanishes b/c periodic boundary

So $\langle \nabla_X Y, Z\rangle =$ for all z, so

$$\nabla_X Y = 2uv' + vu'$$

Geodesic Equation: $\nabla_X X = 0$

$$\Longrightarrow \partial_t u + 3u\partial_x u = 0 \quad \text{(adding time dependence)}$$

11/10/14

Kortwig de-Vries Equations

$$d_\tau u + \frac{3}{2} u \partial_\xi u + \frac{1}{6} \partial_\xi^3 u = 0$$

$$\tau \propto c_* t$$
$$\xi \propto (x - c_* t)$$

u is the surface height of the water which is also proportional to its velocity.

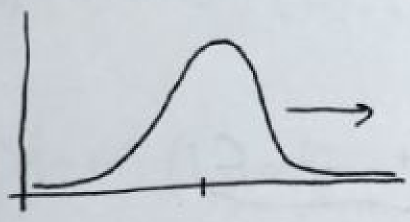
substitute $u = \frac{2}{3} v$

$$\partial_t v + v \partial_\xi v + \frac{1}{6} \partial_\xi^3 v = 0.$$

This equation has soliton solutions of the form

$$A \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}\left(\xi - \frac{A\tau}{3}\right)\right)$$

for some parameter A that depends on initial conditions.



Retains shape, moves with constant speed over time.

Derive from geodesic motion on Lie group?

Add a central extension

$$[u,v] = (uv' - vu')\partial_x \qquad x = x+1$$

Fourier transform u, v

$$L_n = ie^{in\theta}\frac{\partial}{\partial\theta} = -z^{n+1}\frac{\partial}{\partial z} \qquad \cancel{z = e^{in\theta}} \quad z = e^{i\theta} \qquad \frac{d}{d\theta} = \frac{dz}{d\theta}\frac{\partial}{\partial z}$$

$$= ie^{i\theta}\frac{\partial}{\partial z} = iz\frac{\partial}{\partial z}$$

$$[L_n, L_m] = \left[z^{n+1}\frac{\partial}{\partial z}, z^{m+1}\frac{\partial}{\partial z}\right] = z^n z^{m+1}(m+1)\frac{\partial}{\partial z} - z^{m+1} z^n (n+1)\frac{\partial}{\partial z}$$

$$= z^{n+m+1}(m-n)\frac{\partial}{\partial z} = (m-n) L_{n+m}\frac{\partial}{\partial z}.$$

The Witt Algebra

$$[L_n, L_m] = (n-m) L_{n+m}$$

Central Extension

$$[L_n, L_m] = (n-m) L_{n+m} + a(n,m) C$$ "central charge"

$$\hat{S}^1_{Diff} = S^1_{Diff} \oplus_c C$$

Want to set $a(0,n) = 0$. We can do this by changing the basis.

$$[L_0, L_n] = -n L_n + a(0,n) C \qquad \text{set } L_n' = L_n + \frac{-a(0,n)}{n} C$$

$$\boxed{[L_0, L_n'] = -n L_n'}$$

$$[L_n', L_m'] = (n-m) L_{n+m} + \underbrace{a(n,m) C + \_\_\_\, C}_{a'(n,m) C}$$

Drop the primes in new basis.

Jacobi Identity constrains $a(n,m)$.

$$[L_0, [L_m, L_n]] = [[L_n, L_0], L_m] + [L_n, [L_0, L_m]]$$

$$-[L_0, (n-m) L_{n+m} + a(n,m) C] = [n L_n, L_m] + [L_n, m L_m]$$

$$-(n-m)(-(n+m)) L_{n+m} = (n+m) [L_n, L_m]$$

$$(n-m)(n+m) L_{n+m} = (n+m) \left((n-m) L_{n+m} + a(n,m) C \right)$$

$$\Longrightarrow \quad a(n,m)(n+m) C = 0$$

$$\Longrightarrow \quad a(n,m) \propto \delta_{n,-m}$$

Write $a(m,n) = \delta_{m,-n}\, a(m)$.

$$[L_m, L_n] = (m-n) L_{n+m} + \delta_{m,-n}\, a(m) C$$

if m,n switch, then $a(-m) = -a(m)$.

generate a recursion relation for $a(m)$

Consider the Jacobi equation with L_ℓ, L_n, L_m, $L_{\ell+n+m} = L_0$.

$$[L_\ell,[L_m,L_n]] + [L_m,[L_n,L_\ell]] + [L_n,[L_\ell,L_m]] = 0$$

$$(m-n)[L_\ell, L_{m+n}] + (n-\ell)[L_m, L_{n+\ell}] + (\ell-m)[L_n, L_{m+\ell}] = 0$$

$$(m-n)(\ell-(m+n))\left(L_{\ell+m+n} + \delta_{\ell,-(m+n)}\, a(\ell) C\right) +$$
$$+ (n-\ell)(m-(n+\ell))\left(L_{\ell+m+n} + \delta_{m,-(n+\ell)}\, a(m) C\right)$$
$$+ (\ell-m)(n-(m+\ell))\left(L_{\ell+n+m} + \delta_{n,-(m+\ell)}\, a(n) C\right) = 0$$

$$-2(m-n)\left((m+n)(L_0) + \left(\delta_{m+n,+(m+n)}\, a(-m-n) C\right)\right)$$
$$+ + (m+2n)\left((2m)(L_0) + \left(\delta_{m,m}\, a(m) C\right)\right)$$
$$+ -(n+2m)\left((2n)(L_0) + \delta_{n,n}\, a(n) C\right) = 0$$

$$\left((-2m+2n)(m+n) + (2m^2+4mn) - 2n^2 - 4nm\right) L_0$$

$$2n^2 - 2m^2 + 2m^2 + 2mn - 2n^2 - 2mn = 0$$

So we're left with

$$\cancel{-2(m^2-n^2)\, a(-m-n) + (2m^2+4nm)\, a(m) - (2n^2+4nm)\, a(n) = 0}$$

$$-2(m-n)\, a(-m-n) + (m+2n)\, a(m) - (n+2m)\, a(n) = 0$$

$$2(m-n)\, a(m+n) + (m+2n)\, a(m) - (n+2m)\, a(n) = 0$$

Correct Answer:

$$(n-m)\,a(n+m) + (m+2n)\,a(m) - (n+2m)\,a(n) = 0$$

11/13/14

$$[L_n, L_m] = (n-m) L_{n+m}$$

$[L_0, L_n] = -n L_n$ can always choose L to make this so.

$$(n+m)\,a(m,n) = 0$$

$$a(n,m) = \delta_{m,-n}\, a(m)$$

Jacobi Eqn $\Rightarrow$ $(m-n)\,a(m+n) - (2n+m)\,a(m) + (n+2m)\,a(n) = 0$

for $n=1$, $(m-1)\,a(m+1) - (m+2)\,a(m) + (2m+1)\,a(1) = 0$

Combined with $a(0)=0$ and $a(m) = -a(-m)$, need only to find $a(1)$ and $a(2)$.

By inspection $a(m)=m$, $a(m)=m^3$ are solutions, the most general solution is $\alpha m + \beta m^3$.

Use the freedom to shift L's by constants, set $\alpha=\beta=1$.

$$\boxed{[L_n, L_m] = (m-n) + \delta_{m,-n}\left(\frac{m^3-m}{12}\right)}$$

Virasoro Algebra: The algebra of the conformal group in 2 dimensions.

$$L_n = -z^{n+1}\frac{\partial}{\partial z}$$

Central Extensions / Projective Representations.

$$\mathcal{U}: G \to GL(V) \qquad \mathcal{U}(g_1)\mathcal{U}(g_2) = \mathcal{U}(g_1 g_2)$$

normal representation

projective representation $\quad \mathcal{U}(g_1)\mathcal{U}(g_2) = e^{i\phi(g_1,g_2)}\,\mathcal{U}(g_1 g_2)$

impose associativity $\quad \left(\mathcal{U}(g_1)\mathcal{U}(g_2)\right)\mathcal{U}(g_3) = \mathcal{U}(g_1)\left(\mathcal{U}(g_2)\mathcal{U}(g_3)\right)$

$$\Rightarrow \; e^{i\phi(g_1,g_2)}\,\mathcal{U}(g_1 g_2)\,\mathcal{U}(g_3) = \mathcal{U}(g_1)\, e^{i\phi(g_2,g_3)}\,\mathcal{U}(g_2 g_3)$$

$$e^{i\phi(g_1,g_2)}\, e^{i\phi(g_1 g_2, g_3)}\,\mathcal{U}(g_1 g_2 g_3) = e^{i\phi(g_2,g_3)}\, e^{i\phi(g_1, g_2 g_3)}\,\mathcal{U}(g_1 g_2 g_3)$$

$$\phi(g_1,g_2) + \phi(g_1 g_2, g_3) = \phi(g_1, g_2 g_3) + \phi(g_2, g_3)$$

Can often remove the phase, e.g. if $\phi(g_1,g_2) = \alpha(g_1 g_2) - \alpha(g_1) - \alpha(g_2)$

Terminology: $\phi(g_1,g_2)$ is called a "two-cocycle"

$\phi(g_1,g_2)$ is related to the existence of central extensions.

Relate the central charge to the two cocycle

$$\phi(g,1) = \phi(1,g) = 0$$

Consider $\phi(g(\theta), g(\bar\theta))$. Taylor expansion starts at $O(\theta\bar\theta)$

$$\approx \theta^a \bar\theta^b f_{ab} \quad \text{where } f_{ab} \in \mathbb{R}$$

$$U(g(\theta)) \approx 1 + i\theta^a T^a + \frac{1}{2}\theta^a\theta^b T^{ab} \qquad T^{ab} = T^{ba}$$

$$U(g(\theta))\,U(g(\bar\theta)) = 1 + i(\theta^a + \bar\theta^a)T_a + \frac{1}{2}(\theta^a\bar\theta^b + \bar\theta^a\bar\theta^b)T_{ab} - \theta^a\bar\theta^b T_a T_b$$

$$U(g(\theta)g(\bar\theta)) = U(\tilde f(\theta,\bar\theta)^a T_a)$$

$$\cancel{\tilde f(\theta,\bar\theta)^a = \bar\theta^a} \qquad \cancel{\tilde f(\theta,\bar\theta)} \qquad \tilde f^a(0,\bar\theta) = \bar\theta^a$$

$$\tilde f^a(\theta,0) = \theta^a$$

$$\tilde f(\theta,\bar\theta)^a \approx \theta^a + \bar\theta^a + f^a{}_{bc}\,\bar\theta^c\theta^b$$

Now expand the expression

$$e^{i\phi(g(\theta),g(\bar\theta))}\,U(g(\theta)g(\bar\theta)) = U(g(\bar\theta))\,U(g(\theta))$$

LHS: $e^{i\phi}U(g(\theta)g(\bar\theta)) \approx (1 + (i)\theta^a\bar\theta^b f_{ab})\left(1 + i(\theta^a + \bar\theta^a + f^a{}_{bc}\theta^b\bar\theta^c)T_a + \frac{1}{2}(\theta^a + \bar\theta^a)(\theta^b + \bar\theta^b)T_{ab}\right)$

RHS:

$$1 + i(\theta^a + \bar\theta^a)T_a + \frac{1}{2}(\theta^a\theta^b + \bar\theta^a\bar\theta^b)T_{ab} - \theta^a\bar\theta^b T_a T_b$$

$$i\theta^a\bar\theta^b f_{ab} + f^a{}_{bc}\theta^b\bar\theta^c T_a + \theta^a\bar\theta^b T_{ab} = -\theta^a\bar\theta^b T_a T_b$$

$$\Rightarrow \boxed{i f_{ab} + i f^c{}_{ab} T_c + T_{ab} = -T_a T_b} \leftarrow \cancel{\text{centr}}$$

central charge term

No central charge case

$$T_{ab} = -T_a T_b - i f^c{}_{ab} T_{ab} \Longrightarrow \boxed{T_a T_b - T_b T_a = i\underbrace{(f^c_{ba} - f^c_{ab})}_{\text{structure constants. } \tilde c_{ab}} T_c}$$

Allow for central charge

$$+T_a T_b + i f^c_{ab} T_c + i f_{ab} = T_b T_a + i f^c_{ba} + i f_{ba}$$

$$i(f_{ab} - f_{ba}) = [T_b, T_a] + i\underbrace{(f^c_{ba} - f^c_{ab})}_{C^c_{ab}} T_c$$

$$\boxed{[T_a, T_b] = i C^c_{ab} T_c \mp i(f_{ab} - f_{ba})}$$

central charge term

Nontrivial 2-cocycle $\Rightarrow$ Algebra admits a central extension.

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Associativity constraint on two-cocycles

$$\phi(g_1, g_2) + \phi(g_1 g_2, g_3) = \phi(g_1, g_2 g_3) + \phi(g_2, g_3)$$

<u>Theorem:</u> Any semisimple Lie algebra can have its central charge removed. (All two-cocycles are trivial)

by rescaling the generators

Central extension of $D \equiv \mathrm{Diff}\, S^1$. $\hat{D} = D \oplus \mathbb{R}$

if $\hat{g}, \hat{f} \in \hat{D}$, $\quad \hat{g} \circ \hat{f} = (g \circ f, a + b + B(g, f))$

$\hat{g} = (g, b)$ — group cocycle. (Bott)

$\hat{f} = (f, a)$

generators: $\hat{u} = (u(x)\partial_x, \alpha) \in T_e D \oplus \mathbb{R}$.

Lie Algebra $[\hat{u}, \hat{v}] = ((uv' - vu')\partial_x, c(u, v))$ — algebra cocycle. (Gelfand - Fuchs)

What is the group cocycle? prime is <u>not</u> derivative.

$$B(g'',g') + B(g''g',g) = B(g'',g'g) + B(g',g)$$

Bott: $B(g',g) = \frac{1}{2}\int_{S^1} \log(\partial_x(g'\circ g))\, d(\log(\partial_x g))$

Consider $B(g'',g'g) = \frac{1}{2}\int_{S^1} \log(\partial_x(g''\circ(g''g))\, d(\log(\partial_x g'g))$

Notation

$g_x = \frac{d}{dx} g(x)$

$g'_x = \frac{d}{dx} g'(x)$

derivative w.r.t. argument

$$= \frac{1}{2}\int_{S^1} \log(g''_x \cdot (g'g)_x)\, d(\log(\partial_x g'g))$$

$$= \frac{1}{2}\int_{S^1} (\log g''_x + \log(g'g)_x)(d(\log g'_x) + d(\log g_x))$$

So RHS

$$= \frac{1}{2}\int_{S^1} \Big[\overbrace{(\log g'_x + \log g_x)\, d\log g_x}^{=0} + \log g''_x (d\log g'_x + d\log g_x) + \underbrace{\log(g'g)_x\, (d\log(g'g)_x)}_{=0} \Big]$$

Some of these terms vanish, namely the ones that are total derivatives:

$\frac{1}{2}\int_{S^1} \log(g'g)_x\, d\log(g'g)_x = \int_{S^1} d(\log^2 g_x)$ periodic boundary conditions make $\int_{S^1}$ vanish.

Also, $\log g_x\, d\log g_x$ vanishes.

Can check that we are left with the same thing as the ~~RHS~~ LHS.

What is the algebra cocycle?

$\hat{\xi}_t$, $\hat{\eta}_s$ are flows on the group.

$\downarrow \quad \downarrow$

$g_t(x) \quad g'_s(x)$

Say that $\hat{u}$ is the generator of ξ_t

Say that $\hat{v}$ is the generator of η_s

$$\hat{u} = (u(x)\partial_x, \alpha) = \frac{d}{dt}\hat{\xi}_t\Big|_{t=0}$$

$$\hat{v} = (v(x)\partial_x, \beta) = \frac{d}{ds}\hat{\eta}_s\Big|_{s=0}$$

$$[\hat{u},\hat{v}]f = \hat{u}\hat{v}(f) - \hat{v}\hat{u}(f) = \left(\partial_t\partial_s\,\hat{\eta}_s\circ\hat{\xi}_t - \partial_s\partial_t\,\hat{\xi}_t\circ\hat{\eta}_s\right)$$

Why is this the case?

$$\eta_s\xi_t \approx \mathbb{1} + t\hat{u} + s\hat{v}(\hat{u}) \approx \mathbb{1} + t\hat{u} + s\hat{v} + st\,\hat{u}\hat{v}$$

Consider $\hat{\eta}_s\circ\hat{\xi}_t \xmapsto{\text{ext}} a_t + b_s + B(\eta_s,\xi_t)$ the "extension" (central charge component).

$$\partial_t\partial_s\,\mathrm{Ext}(\hat{\eta}_s\circ\hat{\xi}_t) = \partial_t\partial_s\,B(\eta_s,\xi_t)$$

$$\mathrm{Ext}([\hat{u},\hat{v}]) = \partial_t\partial_s\left(\mathrm{Ext}[\hat{\eta}_s\circ\hat{\xi}_t]\Big|_{s=t=0} - \partial_s\partial_t\mathrm{Ext}\left([\hat{\xi}_t\circ\hat{\eta}_s]\Big|_{s=t=0}\right)\right)$$

$$\Longrightarrow \boxed{\mathrm{Ext}([\hat{u},\hat{v}]) = \left(\partial_t\partial_s\,B(\eta_s,\xi_t) - \partial_s\partial_t\,B(\xi_t,\eta_s)\right)\Big|_{s=t=0}}$$

Using the definition of $B(f,g)$, we can find the algebra cocycle. (Ext is algebra cocycle)

$$[\hat{u},\hat{v}] = (uv' - vu')\partial_x + \mathrm{Ext}([\hat{u},\hat{v}])$$

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Group (Bott) Cocycle

$$B(g',g) = \frac{1}{2}\int_{S^1} \log(g'_x g_x)\, d\log(g_x)$$

Algebra (Gelfand-Fuchs) 2-cocycle

$$\hat{u} = (u\partial_x, \alpha) \quad \hat{v} = (v\partial_x, \beta) \qquad \alpha, \beta \in \mathbb{R}$$

$$\hat{u} = \frac{d\hat{\xi}_t}{dt}\Big|_{t=0} \qquad \hat{v} = \frac{d\hat{\eta}_s}{ds}\Big|_{s=0} \qquad \xi_t = e^{ut},\ u = u(x)$$

$$\mathrm{Ext}([\hat{u},\hat{v}]) = \Big[\partial_t \partial_s B(\eta_s, \xi_t) - \partial_s \partial_t B(\xi_t, \eta_s)\Big]_{s=t=0}$$

$$\partial_s B(\eta_s, \xi_t) = \frac{1}{2}\partial_s \int_{S^1} \log(\partial_x(\eta_s \circ \xi_t))\, d\log \partial_x \xi_t$$

$$= \frac{1}{2}\int_{S^1} \frac{\partial_s(\partial_x(\eta_s \circ \xi_t)(x))\, d\log\partial_x \xi_t(x)}{\partial_x((\eta_s \circ \xi_t)(x))}$$

Recall ξ_t is a map $x \longmapsto e^{u(x)t}$, taylor expand in t to get

$\cancel{\xi_t}$ .... $\xi_t \approx id + tu + \ldots$

$\xi_t(x) \approx x + tu(x) + \ldots$

So $\log \partial_x \xi_t(x) \approx \log \partial_x(x + tu(x) + \ldots)$

$= \log(1 + tu' + \ldots)$

$d \log \partial_x \xi_t \approx \frac{tu''}{1+tu'}\, dx \approx t u_{xx}\, dx + O(t^2)$

Since we will send s,t to 0, only keep terms linear in t, s

Now consider $\qquad\qquad\qquad\qquad\qquad\qquad x \equiv$ identity map

$$\eta_s \circ \xi_t \approx (x+sv+\dots)\circ(x+tu+\dots)$$
$$= x + tu(x) + sv(x) + o(ts)$$

$$\partial_x(\eta_s\circ\xi_t) = \partial_x(x+tu+sv) = 1 + tu_x + sv_x$$
$$= 1 + tu_x + sv_x$$

Recall $v = \frac{\partial}{\partial s}$, so putting it all together

$$\partial_t\partial_s B(\eta_s,\xi_t) = \frac{1}{2}\partial_t\int_{S^1}\frac{\partial_s(\partial_x(\eta_s\circ\xi_t))\,d\log\partial_x\xi_t}{\partial_x(\eta_s\circ\xi_t)}$$

$$? = \frac{1}{2}\int_{S^1}\frac{\partial_t\partial_s(1+tu_x+sv_x)tu_{xx}\,dx}{1+tu_x+sv_x}$$

turns out this is correct

denominator is higher order in t, s (taylor expand), so we can pretend that it's just 1 anyway.

for some reason the ∂_s commutes with the ∂_x, and if we don't commute them then we don't get the right answer

$$\cancel{= \frac{1}{2}\int_{S^1}\frac{\partial_x v(x+O(s)+O(t)) t u_{xx}\,dx}{1+tu_x+sv_x}}$$

$$= \frac{1}{2}\int_{S^1}\frac{\partial_t\partial_s((1+tu_x+sv_x)tu_{xx}\,dx)}{1+tu_x+sv_x} \quad 1+tu_x+sv_x \approx 1$$

$$= \frac{1}{2}\int_{S^1}\partial_t(v_x\,tu_{xx}\,dx) = \boxed{\frac{1}{2}\int_{S^1}v_x u_{xx}\,dx}$$

Similarly $\quad \partial_s\partial_t B(\xi_t,\eta_s) = \frac{1}{2}\int_{S^1}u_x v_{xx}\,dx$

integrate by parts $= \frac{1}{2}\int_{S^1}v_x u_{xx}\,dx$

Therefore, $\boxed{\mathrm{Ext}([\hat{u},\hat{v}]) = \int_{S^1} v_x u_{xx}\, dx}$ Gelfand–Fuchs Cocycle.

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Derivation of KdV equation

$\hat{u} = (u, \alpha)$

Right-invariant metric $\hat{v} = (v, \beta)$

$$\langle \hat{u}, \hat{v} \rangle_g = \int_{S^1} dx (u_g \circ g^{-1}, v_g \circ g^{-1}) + \alpha\beta$$

$$\langle \hat{u}, \hat{v} \rangle_e = \int_{S^1} u(x) v(x)\, dx + \alpha\beta$$

$$\nabla_X Y = \frac{1}{2}\left[\mathrm{Ad}_X Y - \mathrm{Ad}^*_X Y - \mathrm{Ad}^*_Y X\right]$$

$$\mathrm{Ad}_X Y = [X, Y]$$

$$\mathrm{Ad}_{\hat{u}} \hat{v} = [\hat{u}, \hat{v}] = (uv' - vu', \mathrm{Ext}([\hat{u},\hat{v}]))$$

$$\langle \mathrm{Ad}^*_{\hat{u}} \hat{v}, \hat{w} \rangle = \langle \hat{v}, \mathrm{Ad}_{\hat{u}} \hat{w} \rangle = \langle \hat{v}, [\hat{u}, \hat{w}] \rangle$$

$$= \langle \hat{v}, (\hat{u}w' - wu', \mathrm{Ext}[\hat{u},\hat{w}]) \rangle$$

$$= \int_{S^1} dx\, (vuw' - vwu') + \beta\, \mathrm{Ext}([\hat{u},\hat{v}])$$

$$= \int_{S^1} dx \left(-\frac{d}{dx}(vw) w - (uv)w\right) + \beta\, \mathrm{Ext}([\hat{u},\hat{v}])$$

$$= \int_{S^1} dx\, (-(v'u + u'v) - vu' - u'''\beta)\, \omega$$

Therefore,

$$Ad^{*}_{\hat{u}}\hat{v} = \left[(-2u'v - uv' - \beta u''')\partial_x , 0\right]$$

Similarly, $Ad^{*}_{\hat{v}}\hat{u} = \left((-2vu' - vu' - \alpha v''')\partial_x , 0\right)$

Use this to compute

$$\nabla_{\hat{u}}\hat{v} = \frac{1}{2}\left((4uv' + 2vu' + \beta u''' + \alpha v''')\partial_x , \; Ext[\hat{u},\hat{v}]\right)$$

The geodesic equation

$$\nabla_{\hat{u}}\hat{u} = 0 \implies (2uu_x + uu_x + \alpha\, u_{xxx} , 0) = 0$$

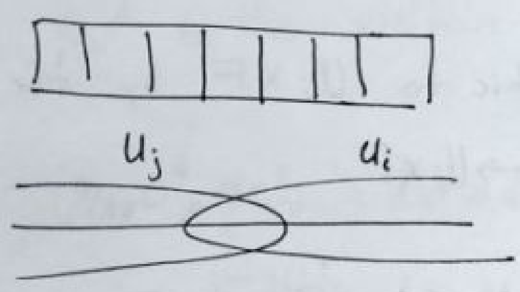
$$\implies 3uu_x + \alpha u_{xxx} = 0$$

Add time dependence $\quad u_t + 3uu_x + \alpha u_{xxx} = 0$

For $\alpha = 1/6$, we get KdV.

<u>More Fiber Bundles:</u>

Tangent Bundle



Fiber: $\mathbb{R}^n = T_p M$

locally, tangent bundle is $\mathbb{R}^n \times \mathbb{R}^n$

Non-triviality $\Longleftrightarrow$ non-trivial topology of base space

$\{U_i\}\{V_j\}$ two open covers of M

$p \in U_i \cap V_j$

Consider $v \in T_pM$

X^μ : U^i coordinates

y^ν : V^i coordinates

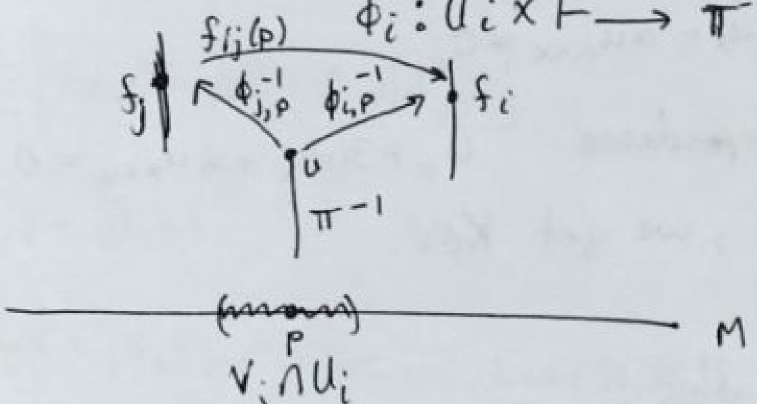
Structure group.

$\frac{\partial y}{\partial x}$ non-singular, $\in GL(n, \mathbb{R})$

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\nu \frac{\partial}{\partial y^\nu} \qquad \tilde{v}^\nu = \frac{\partial y^\nu}{\partial x^\mu} v^\mu$$

$\pi \phi_i = p$ define ϕ_i = "global trivialization"

$$\phi_i : U_i \times F \longrightarrow \pi^{-1}(U_i)$$



$f_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}$

transition functions

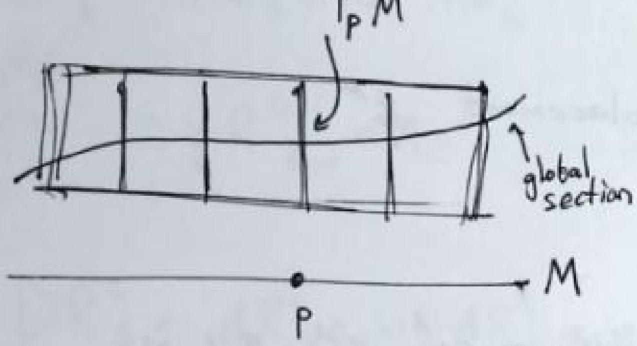
$\pi^{-1}(U_i)$ is diffeomorphic to $U_i \times F$ by the diff $\phi^{-1}: \pi^{-1}(u_i) \to U_i \times F$

e.g. trivial bundle over S^1 is $M = S^1$, $F = [-1, 1]$

Say $G = \mathbb{Z}/2\mathbb{Z}$. Open up strip 

identify edges in opposite orientation, with transition function $f_{ij} = -1$.

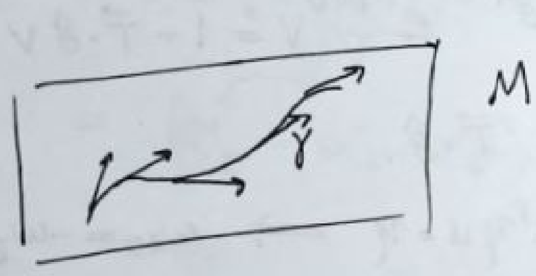
Tangent Bundle:



Fiber Bundle $E(F, M)$

M covered by open sets U_i

on $U_i \cap U_j$, $t_{ij} : U_i \to U_j$
transition function



a curve γ on M has a tangent at each point, which defines a global section in TM.

How do you transition between tangent spaces? The connection coefficients do this for us.

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$$

let $e_\mu{}^A$ be the vielbein, $\eta^{AB} = \cancel{g^{\mu\nu} e_\mu^A \otimes e_\nu^B} = g^{\mu\nu} e_\mu{}^A e_\nu{}^B$

$$ds^2 = (\eta_{AB} e^A{}_\mu \cdot e^B{}_\nu)\, dx^\mu \otimes dx^\nu = \eta_{AB}\, e^A \otimes e^B \qquad e^A = e^A_\mu\, dx^\mu$$

just multiply, don't tensor

Let $E = e^{-1}$, $e^B{}_\nu E^\nu_A = \delta^B_A$

A, B are local orthonormal coordinates
μ, ν are globac coordinates.

Connection One Form:

$$\omega^A{}_B = \omega^A{}_{B\mu}\, dx^\mu \qquad \text{"frame connection"}$$

Consider infinitesimal displacement ξ^μ

How does e^B transform?

$$e^{A'} = \omega^A{}_B(\xi) e^B = \omega^A{}_{B\mu}\xi^\mu e_\nu{}^B dx^\nu$$

Recall $e^{i\vec{T}\cdot\hat{\theta}} V = 1 + \vec{T}\cdot\hat{\theta}\, V + o(\hat{\theta}^2)$

$\omega^A{}_{B\mu}\xi^\mu$ looks like $\vec{T}\cdot\hat{\theta}$.

Maintain orthogonality, $u^T \eta u = \eta \implies \omega_{AB} = -\omega_{BA}$

$$\eta_{AC}\omega^C{}_A + \eta_{BC}\omega^C{}_A = 0$$

ω_{AB} = generators of Lorentz transformation

$\omega_{AB} \in$ the lie algebra of the Lorentz group.

(Lie Algebra valued 1-forms)

Define a geometric Derivative (exterior covariant derivative)

New exterior differentiation

$$D := d + [\omega, \cdot] : \Omega_p \longrightarrow \Omega_{p+1}$$

$$[\omega, \Sigma] = \left[\omega, \Sigma^{B_1,\dots,B_n}_{A_1,\dots,A_m} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_n}\right] =$$

sigma is a tensor on n-vectors taking values in m-vectors.

$$\omega^{B_1}{}_{B_1'}\Sigma^{B_1',B_2,\dots,B_n}_{A_1,\dots,A_m} + \omega^{B_2}{}_{B_2'}\Sigma^{B_1,B_2',B_3,\dots,B_n}_{A_1,\dots,A_m} + \dots + \omega^{B_n}{}_{B_n'}\Sigma^{B_1,\dots,B_n'}_{A_1,\dots,A_m}$$

$$-(-1)^p\Sigma^{B_1,\dots,B_n}_{A_1',\dots,A_m}\omega^{A_1'}{}_{A_1} - (-1)^p\Sigma^{B_1,\dots,B_n}_{A_1,A_2',\dots,A_m}\omega^{A_2'}{}_{A_1} - \dots - (-1)^p\Sigma^{B_1,\dots,B_n}_{A_1,\dots,A_m'}\omega^{A_m'}{}_{A_m}.$$

Curvature two-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$
$$= \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$

Torsion form

$$T^a = de^a + \omega^a{}_b \wedge e^b$$
$$= De^a.$$

$$DR^a{}_b = dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b$$

$$dR^a{}_b = \cancelto{0}{d^2\omega^a{}_b} + \underbrace{d(\omega^a{}_c)} \wedge \omega^k{}_b - \omega^a{}_c \wedge \underbrace{d(\omega^c{}_b)}$$

defn of curvature form.

$$= \left(R^a{}_k - \omega^a{}_{b'} \wedge \omega^{b'}{}_k\right) \wedge \omega^c{}_b - \omega^a{}_c \wedge \left(R^c{}_b - \omega^c{}_d \wedge \omega^d{}_b\right)$$

$$= R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b - \cancel{\omega^a{}_{b'} \wedge \omega^{b'}{}_c \wedge \omega^c{}_b} + \cancel{\omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b}$$

Then $$DR^a{}_b = \underset{(1)}{\cancel{R^a{}_c \wedge \omega^c{}_b}} - \underset{(2)}{\cancel{\omega^a{}_c \wedge R^c{}_b}} + \underset{(3)}{\cancel{\omega^a{}_c \wedge R^c{}_b}} - \underset{(4)}{\cancel{R^a{}_c \wedge \omega^c{}_b}}$$

(1) and (4) cancel

(2) and (3) cancel

"Bianchi Identity"

Thus, $DR^a{}_b = 0$ identically!!

<u>just</u> an algebraic identity!

Connection on a principle G-bundle.

define "Associated Bundle"

sometimes called "internal symmetry group"

given a representation of G, ~~represen~~ associated bundle is the tangent bundle for the ~~assoc~~ concrete vector spaces on the manifold. (A vector bundle)

<u>Formally:</u> Given a principal bundle P with structure group G and transition functions $C_{UV} \in G$ Between open sets U, V, define a new vector bundle associated to P via a representation ρ of G, with the ~~same~~ transition functions $\rho(C_{UV})$

$\vec{v} \mapsto \rho(C_{UV})\vec{v}$ transition function in associated bundle.

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gauge connection 1-form $A^{AB}{}_{\mu} dx^{\mu}$

where $A^{AB} \in T_e G$ = lie group.

In this case the metric is δ^{ij}, which is okay because every compact Lie group admits a bi-invariant metric.

Exterior Covariant Derivative

$$D := d + [\omega, \cdot] \qquad D : \Omega_r \longrightarrow \Omega_{r+1}$$

if something is a tensor, then the exterior covariant derivative is as well.

"Crash Course in E&M"

Curvature 2-form

$$F = DA \qquad F_{\mu\nu} = \left(\tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{2}[A_\mu, A_\nu]\right) dx^\mu dx^\nu$$

$DF = 0$ is two of Maxwell's equations

Yang-Mills Theory is an associated bundle with a non-abelian gauge group. Of course the bundle associated to a theory depends on choice of representation.

The gauge connection A transforms under the adjoint representation.

For this rep, $\dim \rho = \dim \underbrace{\mathcal{L}G}_{\text{Lie Algebra } T_e G}$, with structure constants $[T^a, T^b] = f^{ab}{}_c T^c$

Now put some structure on the manifold. Take a section of the principle bundle, put a "scalar field" on M that transforms under ρ, $M \longrightarrow$ (Section of Principle Bundle)

ρ^a representation

$\phi^a(\vec{x},t) \in$ Section. $(\vec{x},t)$ are local coordinates for M.

Gauge Transformation: <u>local</u> change of basis at each point

$$\phi^a(x,t) = \left[e^{i\alpha(x)^A T_A}\right]^{ab} \phi_b, \quad (T^A)_{ab} \text{ generator of representation } \rho$$

$A \longmapsto g^{-1}(x)(A+d)g(x)$ in analogy to ω (<u>Not</u> a tensor)

$F \longmapsto g^{-1}(x) F g(x)$ transforms as a matrix

$\hookrightarrow$ element of adjoint representation

$g(x) \in G$, so $g(x) = e^{i\alpha(x)^A T_A}$ for some $\alpha(x)$.

For an abelian theory, $g = e^{i\alpha(x) q}$, where q is the sole generator of the Lie Algebra.

F is invariant

$$A \longmapsto A + i(d\alpha(x))$$

<u>Magnetic</u> <u>Monopole:</u> Abelian $\Longrightarrow D = d$. Gauge group is $U(1)$

$$F = dA$$

$$dF = 0 \Longrightarrow \vec{\nabla} \cdot \vec{B} = 0$$

"closed" form: ω obeys $d\omega = 0$

"exact" form: ω can be written as $d\eta$ for some η.

Not all closed forms are **exact**. Closed = $\ker d$
Exact = $\operatorname{im} d$

Two forms which differ by an exact form are called homologous.

Exact/ Closed forms are a group under addition

Closed r-forms are $Z^r(M)$ $\qquad H^r(M) := Z^r(M) / B^r(M)$

Exact r-forms are $B^r(M)$

$\uparrow$ r^{th} De-Rahm Cohomology group.

$$H^r(M) = \ker d_{r+1} / \operatorname{im} d_r$$

<u>Poincare Lemma:</u> in any neighborhood U of M (open set) that is simply connected, then $H^r(M) = 0$.

However, because we are using the group $G = U(1)$, and manifold $M = \mathbb{R}^3 \setminus \{0\}$.

The relevant bundle is $G = S^1$ and $M = S^2$.

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Associated Bundle with Structure Group G

$A \equiv$ connection 1-form

$F \equiv$ curvature 2-form

$$F = DA = dA + A \wedge A$$
$$= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right) dx^\mu dx^\nu$$

A transforms under adjoint representation

$A = A^a_\mu T^a dx^\mu$ where T^a are generators of adjoint rep

Metric is Euclidean, $g = e^{i\alpha(x)}$

$$\left.\begin{aligned} A &\mapsto g^{-1}(A+d)g \\ F &\mapsto g^{-1}Fg. \end{aligned}\right\} \xRightarrow[\text{Abelian}]{} \begin{aligned} A &\longmapsto A + g^{-1}dg \\ &\quad = i\partial_\mu \alpha(x) dx^\mu \\ F &\longmapsto F \end{aligned}$$

De Rahm Cohomology

$$H^r(M) = \frac{\ker d_{r+1}}{\operatorname{im} d_r} = \frac{Z^r(M)}{B^r(M)} \qquad \dim H^r(M) = \text{Betti Numbers}$$

Claim: if $H^r(M)$ is trivial, then there are no magnetic monopoles.

We look for forms which are closed but not exact.

On $\mathbb{R}^2 \setminus \{0\}$, an example is $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$

Integration on Manifolds

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega. \qquad \text{Stokes Theorem}$$

An n-form ω is a volume element.

Integrating a function $f: M \longrightarrow \mathbb{R}$

- Define $\omega = f(x_1, \dots, x_n)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$
- Divide region into cells spanned by $\left(\Delta x^1 \frac{\partial}{\partial x^1}, \Delta x^2 \frac{\partial}{\partial x^2}, \dots, \Delta x^n \frac{\partial}{\partial x^n}\right)$
- $$\int \omega := \int f(x_1, \dots, x_n)\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n \left(\Delta x^1 \frac{\partial}{\partial x^1}, \Delta x^2 \frac{\partial}{\partial x^2}, \dots, \Delta x^n \frac{\partial}{\partial x^n}\right)$$
$$= \int f(x_1, \dots, x_n)\, dx^1 dx^2 \dots dx^n$$

Example: change of coordinates

$$\int f(\lambda, \mu)\, d\lambda \wedge d\mu = \int f(\lambda, \mu) \left(\frac{d\lambda}{dx} dx + \frac{d\lambda}{dy} dy\right) \wedge \left(\frac{d\mu}{dx} dx + \frac{d\mu}{dy} dy\right)$$

$$d\lambda = \frac{d\lambda}{dx} dx + \frac{d\lambda}{dy} dy \qquad = \int f(\lambda, \mu) \underbrace{\left(\frac{d\lambda}{dx}\frac{d\mu}{dy} - \frac{d\lambda}{dy}\frac{d\mu}{dx}\right)}_{\text{Jacobian}} dx \wedge dy$$

Back to the monopole stuff.

$$\vec{B} = g \frac{\hat{r}}{|r|^2} \qquad \text{Magnetic charge } g \text{ at origin.}$$

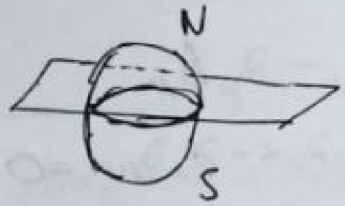
$$g = \int_{S^2} \nabla \cdot B \, d^3\vec{x} = \int_{S^2} \nabla \cdot (\nabla \times A) \underset{\substack{\text{div.}\\ \text{thm.}}}{=\!=} \cancel{\int_{\partial S^2} (\nabla \times A) \cdot \hat{n}\, dA = 0}$$

Then it has magnitude zero. $\int_{\partial S^2} A \cdot dl = 0$

What about if we have nontrivial topology?

Can we find a gauge field ~~those~~ corresponding to a magnetic charge?

Split S^2 into southern and northern hemispheres



So then

$$\int_{S^2} \nabla\times A = \int_N \nabla\times A_N + \int_S \nabla\times A_S$$

$$= \oint_{\partial N} A_N\cdot dl + \oint_{\partial S} A_S\cdot dl$$

$$= \oint_{\partial N} (A_N - A_S)\, dl$$

More precisely, define a northern chart and southern chart

$$U_N = \{0 \le \theta \le \tfrac{\pi}{2} + \varepsilon\}$$
$$U_S = \{\tfrac{\pi}{2} - \varepsilon \le \theta \le \pi\}$$
$$\longrightarrow U_N \cap U_S = S^1$$

$$A_N = i g(1-\cos\theta)\, d\phi$$
$$A_S = -i g(1+\cos\theta)\, d\phi$$
$$dA = \frac{g}{|r|^2}\, d\theta\wedge d\phi$$

Now consider a transition function $t_{us} \in G = U(1)$. So t_{us} is a map from $S^1 \to S^1$, $t_{us}(x) = e^{i\alpha(\theta)}$.

$(g = e^{i\alpha(\phi)})$

Under coordinate transformation, $A \longmapsto A + g^{-1}dg = A + id\alpha$

$$A_S = A_N + id\alpha. \implies A_S - A_N = id\alpha = i\frac{d\alpha}{d\phi}\, d\phi$$

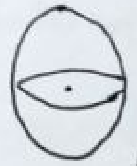
$$g = \oint_{S^1} (A_N - A_S)\, d\phi = \oint_{S^1} i\frac{d\alpha}{d\phi}\, d\phi = i(\alpha(2\pi) - \alpha(0))$$

To be single valued, impose $e^{i\alpha(2\pi)} = e^{i\alpha(0)}$.

Therefore $\alpha(\phi) = n\phi$, so $g = 2\pi n i$

Wu-Yang Construction 12/1/14

$$A_N = ig(1-\cos\theta)\,d\phi$$



$$A_N - A_S = \text{"pure } \cancel{\text{group}}\ \text{gauge"}$$

$$A_S = ig(1+\cos\theta)\,d\phi$$

$$F_{\mu\nu} = 0 = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$A_\mu = \partial_\mu \alpha \qquad = \partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha = 0$$

For Gauge Theories, A_μ need not vanish at ∞ (pure gauge)

in 2+1 dimensions, spatial infinity is S^1, $A_n : S^1 \longrightarrow S^1$

$\pi_1(S^1) = \mathbb{Z}$ A_N is pure gauge at ∞

$U(1) = \{e^{i\alpha x}\}$ $A_\mu = \partial_\mu \alpha$

<u>More</u> <u>Monopole</u>

at $\theta = \pi/2$, $A_N - A_S = 2gi\,d\phi = i\dfrac{d\psi}{d\phi}\,d\phi$ $\psi = 2g\phi$

$$= i\,d\psi(\theta)$$

$$A \longrightarrow A + i\,d\psi(\phi)$$

$$\int_{S^1} d\psi = \int_0^{2\pi} d\psi(\phi) = \int_0^{2\pi} d\psi(\phi) = \int_0^{2\pi} 2g\phi\, d\phi \implies \cancel{g\in\pi\mathbb{Z}} \qquad g = \frac{n}{2},\ n\in\mathbb{Z}$$

Existence of a <u>single</u> monopole is sufficient to prove quantization of electric charge.

Quantum Mechanics: $\psi(x) \xrightarrow{\text{gauge theory}} e^{iq(\alpha(x))}\psi(x)$

(invariance of Shrödinger Eqn)

$\psi(x) \to e^{iq(2g)\phi}\psi(x)$ in the case of a monopole,

$2gq \in \mathbb{Z}$

$q = \frac{n}{2g} = \frac{n}{m} \in \mathbb{Q}.$

Shapire-Wilczek

- How do bacteria move in liquid?
- Force-free motion of a deformable object ↗ astronaut in space ↘ bacteria in water
- Motion of deformable object in liquid with large viscosity (low Reynolds number)
  - when bacteria stops, it has no inertia in water
  - what is final orientation after motion.

there is ambiguity in the choice of orientation for any given shape.

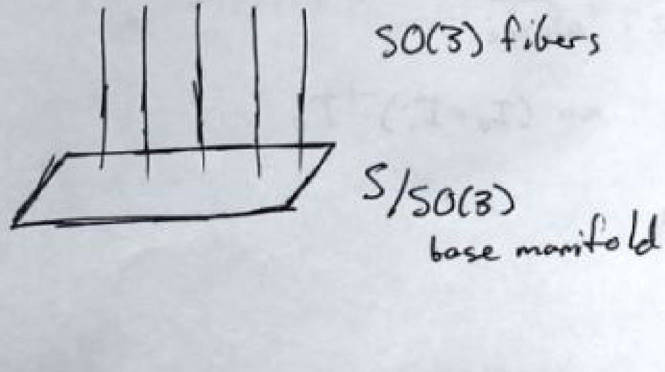
call shapes equivalent if related by rotations in $SO(3)$

all elements of the same class get one set of axes, S_0.

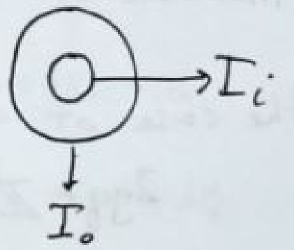
the physical orientation is $S = RS_0$ for $R \in SO(3)$.

The system is described by an associated $SO(3)$ bundle.

Base Manifold is $S/SO(3)$

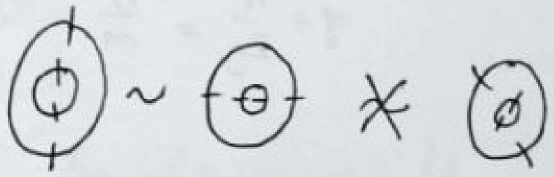


Example: concentric spheres



$L = I_i\dot{\theta} + I_o\dot{\theta} = 0$

Conservation of angular momentum: inner one rotates means outer one goes opposite direction.



"Shapes"

The standard configuration we choose is with the inner sphere oriented along the z-axis. (choice of gauge)

$S(t) = R\,S_o(t)$

Free to make local change in S_o

$S_o \longmapsto \Omega(t) S_o$, $\quad \Omega(t) \in SO(3)$

but S must be invariant, so

$R \longmapsto R\,\Omega(t)^{-1}$. "gauge transformation"

Let J_i be the generator of <u>relative</u> rotations of the two spheres.

Net rotation = rotation of inner sphere.

$$I_i\dot{\theta} = I_o((\dot{\theta}-\dot{\theta}') - \dot{\theta})$$

$$\frac{I_o(\dot{\theta}-\dot{\theta}')}{I_i+I_o} = \dot{\theta}$$

Vector equation, b/c I is a matrix

$$(I_i+I_o)^{-1} I_o(\dot{\theta}-\dot{\theta}') = \dot{\theta}$$

Infinitesimal Shape Change: $\omega_i J_i$

Net Rotation: $\alpha\omega_i J_i$ $\qquad \alpha = (I_o+I_i)^{-1} I_o$

↑ gauge field.

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SO(3) principal bundle over our configuration S

$S(t) = R\,S_0(t)$ equivalent if related by rotation

$S_0(t)$ is a path in the base manifold

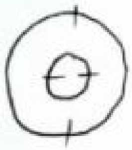
$R \in SO(3)$

local gauge transformation: $S_0(t) = \Omega(t) S_0(t)$

How to calculate gauge potential A:

A := Lie-Algebra valued 1-form on S_0

Generators of relative angular rotations J_i



Inner sphere has moment of inertia I, outer has I'

change in shape: $\omega_i(t) J_i$

(infinitesimal) Net rotation $A = \alpha \omega_i J_i$, where α is fixed by $\frac{\partial L}{\partial t} = 0$

$$\alpha = \frac{(\dot{\theta} - \dot{\theta}')I'}{(I + I')}$$

Calculate net rotation after transversing path in S_0.

$$d(\theta) = \alpha(\omega_i(t) J_i)dt\,\theta \implies \frac{d\theta}{dt} = \alpha \omega_i(t) J_i\,\theta$$

"Time dependent Schrödinger with time dependent Hamiltonian"

$$i\frac{d\psi}{dt} = H(t)\psi. \longrightarrow \text{solution } \psi(t') = \psi(t)\,\tau\left(e^{\int_t^{t'} dt'' H(t'')}\right)$$

Therefore, we can solve

"Wilson Line"
Parallel Transport
not just infinitesimally

$$\theta(t_2) = \theta(t_1)\, T e^{\int_{t_1}^{t_2} dt'\, \omega_i(t')\,dt'}$$

$$= \theta(t_1)\left(1 + \int_{t_1}^{t_2} dt'\, \omega_i(t')\, \mathcal{J}_i + \frac{1}{2}\int_{t_1}^{t_2} dt'\, \omega_i(t')\, \mathcal{J}_i \int_{t_1^*}^{t'} dt''\, \omega_i(t'')\, \mathcal{J}_i(t'') + \ldots\right.$$

$$+ \frac{1}{2}\int dt'' \int dt' \,(\omega_i(t')\, \mathcal{J}_i)\,(\omega_i(t'')\, \mathcal{J}_i) + \ldots$$

Wilson Line $\quad W = P e^{i\int A^\mu dx_\mu}$ $\quad$ parallel transpoint from point A to point B if distance between them not infinitesimal

Under gauge transformation, $\quad W(t_1, t_2) = \Omega^{-1}(t_1)\, W(t_2, t_1)\, \Omega(t_2)$

More generally, $\quad \dfrac{dR}{dt} = \dot{R} = R R^{-1} \dot{R} = R\underbrace{(R^{-1}\dot{R})}_{\in\, so(3)} = RA$ $\quad$ lie Algebra

$$R(t_2) = R(t_1)\, T e^{\int_{S_0(t_1)}^{S_0(t_2)} dS_0 \cdot A_{\dot{S}_0}(S_0)}$$

$\downarrow$

initial configuration $S_0(t_1)$
final configuration $S_0(t_2)$

$S_0 :=$ direction of tangent in configuration space.

"Just Yang-Mills theory but not with spacetime but another spacetime"

How does A transform?

$S_0 \to R S_0 \implies R \to R\, \Omega^{-1}(t)$

So $\frac{dR}{dt} = RA$, and we get $\frac{d}{dt}(R\Omega^{-1}) = R\Omega^{-1}A'$

$$\Longrightarrow \left(\dot{R}\Omega^{-1} + R\dot{\Omega}^{-1} = R\Omega^{-1}A'\right)\Omega$$

$$\dot{R} + R\dot{\Omega}^{-1}\Omega = R\Omega^{-1}A'\Omega$$

$$0 = \frac{d}{dt}(\Omega\Omega^{-1}) = \dot{\Omega}\Omega^{-1} + \Omega\dot{\Omega}^{-1} \Longrightarrow \dot{R} - R\dot{\Omega}\Omega^{-1} = R\Omega^{-1}A'\Omega$$

Hence $\dot{R} = R\Omega^{-1}A'\Omega + R\dot{\Omega}\Omega^{-1}$

$$RA = R\Omega^{-1}A'\Omega + R\dot{\Omega}\Omega^{-1}$$

$$\Longrightarrow A = \Omega^{-1}A'\Omega + \dot{\Omega}\Omega^{-1}$$

$$A' = \Omega A\Omega^{-1} - \Omega\dot{\Omega}\Omega^{-1}\Omega^{-1}$$

$$\boxed{= \Omega A\Omega^{-1} + \Omega\dot{\Omega}^{-1}}$$

better to use $\frac{d}{dt}(\Omega^{-1}\Omega) = 0$, substitute that instead.
Answer is easier to work with.

Transformation of Wilson Line

$$R(t_2) = R(t_1)\, Te^{\int_{t_1}^{t_2} A\,dt'}$$

$$R(t_2)\Omega(t_2) = R(t_1)\Omega(t_1)\underbrace{Te^{\int_{t_1}^{t_2} A\,dt'}}_{W}$$

$$W \longrightarrow \Omega^{-1}(t_1)\, W\, \Omega(t_2)$$

"Named after some aquatic creature, I can't remember which one"

Scallop Theorem: Scallops cannot swim!

can only flap

because their curvature tensor is 1D and antisymmetric, so vanishes.