Extremal Combinatorics

Random Topic

Ramsey Number: $R(k,t)$ = minimum N such that any 2-edge-coloring of K_N has either a black K_k or ~~a~~ white K_t.

in general, for graphs G, H

$R(G,H)$ = minimum N s.t. any 2-edge-coloring of K_N has black G or white H.

Theorem: $\forall\, G, H$, $R(G,H)$ is finite

What is known?

$$(1+o(1))\frac{\sqrt{2}\,s}{e}\,2^{s/2} \le R(s,s) \le \left(s^{-\frac{c\log s}{\log\log s}}\right)(4^s)$$

Spencer   cohen(?)

$$R(3,t) = \Theta\left(\frac{t^2}{\log t}\right) \quad \text{(Kim)}$$

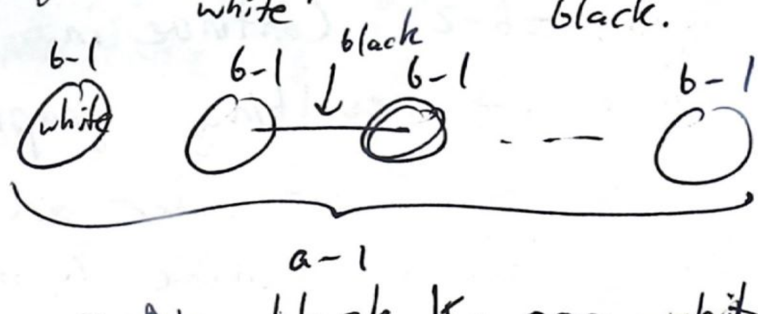
$R(3,3) = 6$
$R(3,4) = 9$
$R(4,4) = 18$
$R(4,5) = 25$
$43 \le R(5,5) \le 49$

Theorem: For any $a, b \ge 2$, let T_b be any tree on b vertices, then $R(K_a, T_b) = (a-1)(b-1)+1$

Proof:

lower bound: $R(K_a, T_b) > (a-1)(b-1)$

Color $K_{(a-1)(b-1)}$ by partitioning into $(a-1)$ cliques of size $(b-1)$. Edges within cliques ~~black~~ white, others ~~white~~ black.



$\Longrightarrow$ No black K_a nor white T_b

<u>upper bound:</u> Consider any 2-edge-coloring of K_N, $N = (a-1)(b-1)+1$.

Suppose No black K_a. Let G be the graph induced by white edges, so the maximum independent set of G is of size $\leq (a-1)$

<u>Fact:</u> $\chi(G) \cdot \left| \begin{matrix} \text{max independent} \\ \text{set in } G \end{matrix} \right| \geq |V(G)|$

$$\Longrightarrow \chi(G) \geq \left\lceil \frac{(a-1)(b-1)+1}{\underbrace{(\max IS)}_{\leq (a-1)}} \right\rceil \geq \left\lceil \frac{(a-1)(b-1)+1}{(a-1)} \right\rceil = b$$

$= (a-1)(b-1)+1$

<u>Two claims:</u> (1) If $\chi(G) \geq b$, then G has a subgraph H whose min-degree $\geq b-1$

(2) If $\delta(H) \geq b-1$, then H contains any tree on b vertices (in particular, T_b).

<u>Proof of (1):</u> Suppose not. Then any subgraph of G has a vertex with degree $\leq b-2$

We can find a vertex $v_n \in G$ s.t. $d_G(v_n) \leq b-2$

Let $G_{n-1} := G \setminus \{v_n\}$, find v_{n-1} with degree $d_{G_{n-1}}(v_{n-1}) \leq b-2$. Continue in this manner until the resulting graph is empty.

Defines a linear order on vertices such that $\forall i$, $d_{G_i}(v_i) \leq b-2$ where $G_i = G[\{v_1 \dots v_i\}]$

Then we can color G by $(b-1)$ colors using the greedy algorithm. ~~Assign~~

Assuming we have colored G_{i-1} by $b-1$ colors, consider v_i, which has $\leq b-2$ neighbors in G_{i-1}, so gives new color to use on v_i. So $\chi(G) \leq b-1$

$\divideontimes$ Then, since $\chi(G) \geq b$ from before, this cannot be.

$\blacksquare$ (1)

<u>Proof of (2):</u>

<u>Mini Claim:</u> For any tree on b vertices, we can arrange the vertices $v_1, v_2, \ldots, v_b$ s.t. for each i, v_i has precisely one neighbor within $v_1, v_2, \ldots, v_{i-1}$

<u>**Proof:**</u> Choose a leaf as v_b, remove, recurse.

Assuming $\delta(H) \geq b-1$, we will choose $v_1, v_2, \ldots v_b$ such that $H[\{v_1, \ldots, v_b\}] \supseteq T_b$.

Choose any vertex of H to be v_1. Assume we have picked $v_1, v_2, \ldots, v_{i-1} \in H$ such that $H[\{v_1, \ldots, v_{i-1}\}] \supseteq T[\{v_1, \ldots, v_{i-1}\}]$

Next we want to choose a vertex of H to be v_i.
Let v_j be the unique neighbor of v_i for $j > i$.
Since v_j has $\leq b-2$ neighbors in $T[\{v_1, v_2, \cdots v_{i-1}\}]$, but v_j has $\geq b-1$ neighbors in H, we can just pick an unused vertex (also neighbor of v_j) in H to be v_i.

$\Longrightarrow$ H has ~~the~~ tree T_b. $\blacksquare$

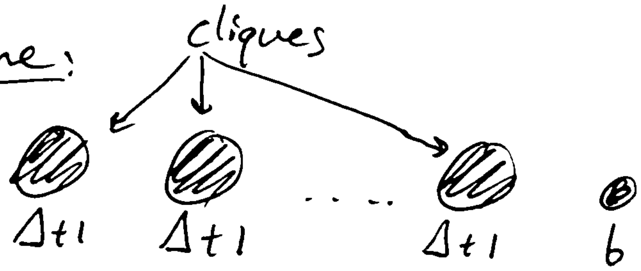
Schur's Theorem: For any integer c, there is $N = N(c)$ such that the following holds for $n \geq N$:

Any c-coloring of integers $1, 2, \ldots, n$, will provide 3 numbers $1 \leq a, b, c \leq n$ s.t. $a + b = c$, and a, b, c get same color.

Proof:

Question: N-vertex graph w/ max degree Δ, what is max # of triangles?

Conjecture:



Theorem: if # vtxs $\leq 2(\Delta+1)$, then true.

Proof: If # vtxs $\leq \Delta + 1$, true. (Take K_n)

If # vtxs $= (\Delta+1)z$, $z \in \mathbb{Z}^+$, true.

Since each vertex has $\leq \Delta$ neighbors, so

$$\triangleleft \leq \Delta \implies \# K_3 @ v \leq \binom{\Delta}{2}$$

So $$\sum_v \# K_3 @ v \leq n \binom{\Delta}{2}$$

$$\implies \# K_3 \leq \frac{1}{3} n \binom{\Delta}{2}$$

triple counting

Proof continued.

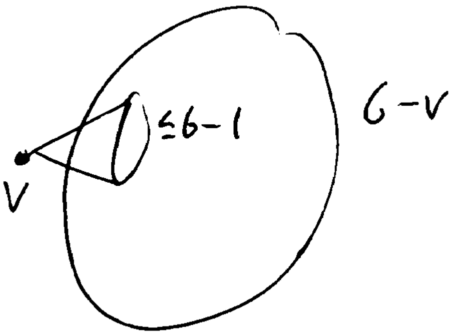
Remains: $n=(\Delta+1)+b$, where $0\le b\le\Delta$.

Proof: by induction on b.

Base case $b=0$ done above.

Have some $1\le b\le\Delta$. Use to establish min-degree condition. show $\#K_3\le\binom{\Delta+1}{3}+\binom{b}{3}$

$\boxed{\text{If } \exists v \text{ with degree } \le b-1:}$



$$(\#K_3 \text{ in } G-v)\le\binom{\Delta+1}{3}+\binom{b-1}{3}$$ by induction

$$\#K_3 \text{ in } G = \#\text{ edges in nbhd of } v\le\binom{b-1}{2}$$

$$\text{Total } \#K_3\le\binom{\Delta+1}{3}+\binom{b-1}{2}+\binom{b-1}{3}=\binom{\Delta+1}{3}+\binom{b}{3}$$

Last case: all degrees $\ge b$ (seek contradiction)

Use: $\#$ (edge / no edge) $= 2\left[\#\ \therefore \text{ inhomogeneous}\right]$ not all edges/non edges

$$=2\left[\binom{n}{3}-\#K_3-\#\overline{K_3}\right]$$

$$\le 2\left[\binom{n}{3}-\#K_3\right]$$

$$=\sum_v \deg(v)\,(n-1-\deg(v))$$

$$\ge n(b)(n-1-b)$$ since $\deg(v)\in[b,\Delta]$ $\forall v$

$$=n(b)(\Delta)$$

Proof continued:

Have

$$nb\Delta \le 2\left[\binom{n}{3} - \#K_3\right]$$

In 

: $\sum_v d_v(n-1-d_v) =$

$$= (\Delta+1)\Delta b + b\underbrace{(b-1)(\Delta+1)}_{\le b\Delta}$$

In G^*: $\sum_v d_v(n-1-d_v) \le 2\left[\binom{n}{3} - \#K_3^{in}(G)\right]$

$$= 2\left[\binom{n}{3} - \#K_3(G^*) - \cancel{\#\overline{K_3}(G^*)}^{\,0}\right] \le 2\left[\binom{n}{3} - \#K_3(G)\right]$$

$$\Longrightarrow \#K_3(G) \le \#K_3(G^*)$$

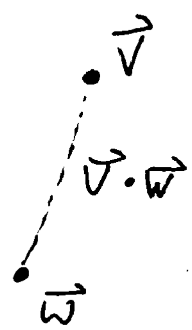
∎

<u>Theorem:</u> n vtx graph, edges colored in 2 colors
$\exists$ monochromatic clique of order $\geq (\frac{1}{2} + o(1)) \log_2 n$
"Quantitative Ramsey Theorem"

<u>Theorem:</u> $\exists$ coloring where all monochromatic cliques $\leq (2+o(1)) \log_2 n$.
<u>Proof:</u> Randomly ~~ee~~ color each edge with equal probability.
With probability >0, all sets of ~~vertices~~ vertices of order $\sim (2+o(1)) \log_2 n$ not monochromatic.

<u>Theorem:</u> There is a coloring ~~of~~ w/ all monochromatic cliques size $\leq (8+o(1)) \lg n$ $\left(\leq (6+o(1)) \lg n\right)$.

<u>Proof:</u> Say $n = 2^d$. Vertices are vectors over $\mathbb{F}_2^d$.
Color edges



gives color (either 0 or 1) by dot product

Can we make big 0-clique?
Set S of vectors w/ pairwise orthogonal $|S| \leq d$
not quite
Set S' of vectors pairwise $\vec{v} \cdot \vec{w} = 1$? $|S'| \leq d+1$
Define $\vec{v}' = \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix}$ $w' = \begin{pmatrix} \vec{w} \\ 1 \end{pmatrix}$ $w' \cdot v' = 0$

In $\mathbb{F}_2^d$, can take $S = \begin{pmatrix} a \\ a \\ b \\ b \\ \vdots \\ z \\ z \end{pmatrix}$ > equal $\quad |S| = 2^{d/2}$

So if $\begin{array}{l} v\cdot v=1 \\ v\cdot w=0 \end{array}$ is additional condition, then #vecs $\leq$ dim space

Try 2: $n = 2^d/2$: all v such that $v\cdot v=0$.

1 cliques $|S'| \leq d+1 = |S|$

Try 3: Take only vertices $v\cdot v=0$ and each only with probability p. (Where $p = 2^{-7/8 d}$)

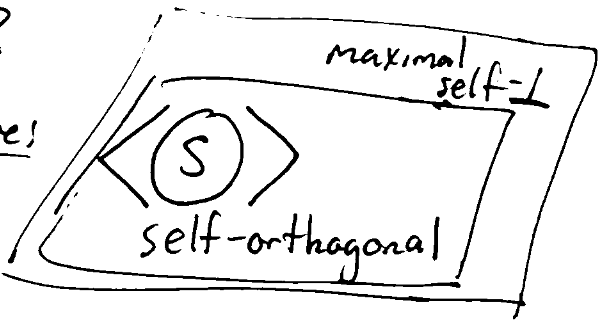
Goal: Each 0-clique has $\leq d$ surviving vertices.

Follows from:

(★) Each maximal self-orthogonal subspace has $\leq d$ vertices surviving

(★) $\Rightarrow$ Goal?

0-clique:



Prove: $\exists$ 2-coloring of K_N where all monochromatic cliques have size $\leq 8 \lg N$.

Recall Coloring: vtxs are vectors in $\mathbb{F}_2^d$ s.t. $v \cdot v = 0$
color of edge is $v \cdot w$.

Immediately 1-Clique is set where $\left.\begin{array}{l} v\cdot v = 0 \\ v \cdot w = 1 \end{array}\right\} \leq d+1$

0-cliques are self-orthogonal sets, which can be exponential size in d. Randomly pick vectors from self orthogonal sets w/ probability $p = 2^{-7/8 d}$

To do this, need

① # maximal self-$\perp$ subspaces $\leq \underline{2^{3/8 d^2}}$ ← Last Time

② $\mathbb{P}$[in set of $2^{d/2}$ vectors, $\geq d$ surviving] $\leq \underline{\binom{2^{d/2}}{d} p^d}$

Goal: product ①·② ≤ 1.

Approximation: $\binom{n}{m} \approx \left(\frac{en}{m}\right)^m$

Multiply ① and ②

$$\leq 2^{3/8 d^2} \left(\frac{e 2^{d/2}}{d}\right)^d p^d = \left(2^{3/8 d} 2^{d/2} p \cdot \frac{e}{d}\right)^d$$

Choose $p = 2^{-7/8 d}$ to make the above ≤ 1

So $N = 2^{d \cdot 1/2} \cdot 2^{-7/8 d} \approx 2^{1/8 d}$ so $d \sim 8 \lg N$

Optimize Previous Result:

Sparsity so that all maximal self-$\perp$ subspaces have $\leq Cd$ vectors surviving.

(2') $\mathbb{P}[\text{in set of } 2^{d/2} \text{ vectors}, \geq Cd \text{ surviving}] \leq \binom{2^{d/2}}{Cd} p^{Cd}$

Now (1) x (2')

$$\leq 2^{3/8 d^2} \left(\frac{e 2^{d/2}}{Cd}\right)^{Cd} p^{Cd}$$

$$= \left(2^{3/8 d} \cdot 2^{C/2 d} \, p^C \left(\frac{e}{Cd}\right)^C\right)^d$$

<u>Choose</u>: $p = 2^{-(\frac{1}{2} + 3/8C)d}$

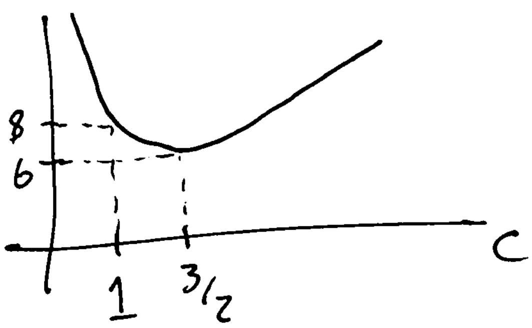
Now $N = 2^d \frac{1}{2} p \approx 2^{(1/2 - 3/8C)d} \implies d \sim \lg N \left(\frac{1}{1/2 - 3/8C}\right)$

<u>Got</u>: N vertices, 1-cliques size $\leq d$

0-cliques size $\leq Cd$.

All monochromatic cliques have order $\leq \lg N \left(\frac{C}{1/2 - 3/8C}\right)$

<u>Plot it!</u>



Improve bound to $d \sim 6 \log_2 N$.

<u>Next</u>: Clever Argument to get (1) $\leq 2^{d^2/4}$

~~From G-bound~~

From G-bound, get All cliques $\leq (\lg N)\left(\frac{C}{1/2 - 1/4C}\right)$

Plot this! Get $d \sim 4(\lg N)$

Can actually get $d \sim 8/3(\lg N)$

Use reference that has been lost to get ① $\leq 2^{d^2/8}$.

All cliques size $\leq (\lg N)\left(\frac{C}{\frac{1}{2} - 1/8C}\right)$ Take $C = 1$ to get smallest possible

Chernoff Bound:

$X \sim Bin(n,p)$

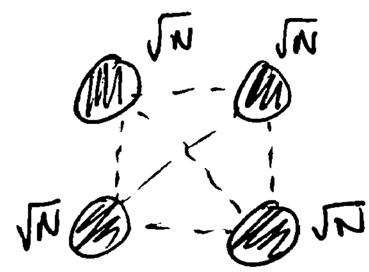
$$\mathbb{P}[\,|X - \mathbb{E}[X]| \geq t\,\mathbb{E}X\,] \leq e^{-C_t \mathbb{E}[X]}$$

Erdős: There exist 2-colorings of edges with all monochromatic cliques $\leq 2 \lg N$.

non-constructive

Constructive results?

1930's



all monochromatic cliques $\leq \sqrt{N}$

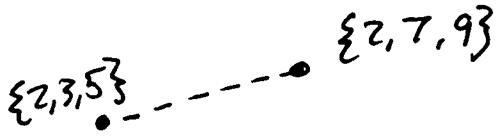
1970's
Nagy

all monochromatic cliques $\leq \sqrt[3]{N}$

Constructions:

1970's Nagy — N = one vertex per 3-set of $\{1, 2, \dots, M\}$.

So $N = \binom{M}{3}$



color edges by parity of size of intersection

0-cliques will be small

Think of vertices as vectors in $\mathbb{F}_2^M$, characteristic vectors

Key observation: $1_S \cdot 1_T = |S \cap T|$

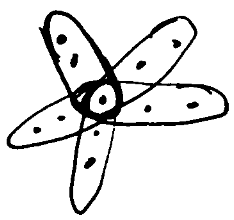
Bunch of vectors, each $1_A \cdot 1_B = 0$ over $\mathbb{F}_2$

$1_A \cdot 1_A = 1$ over $\mathbb{F}_2$

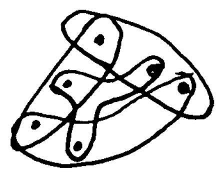
Orthonormal, Hence linearly independent, at most M.

size of 0-cliques $\leq M$, as we did before.

1-cliques:

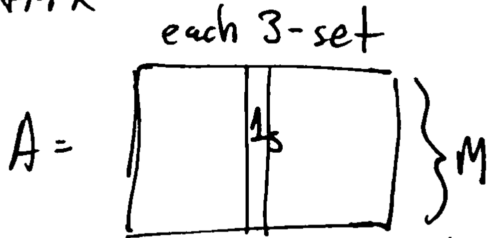


or

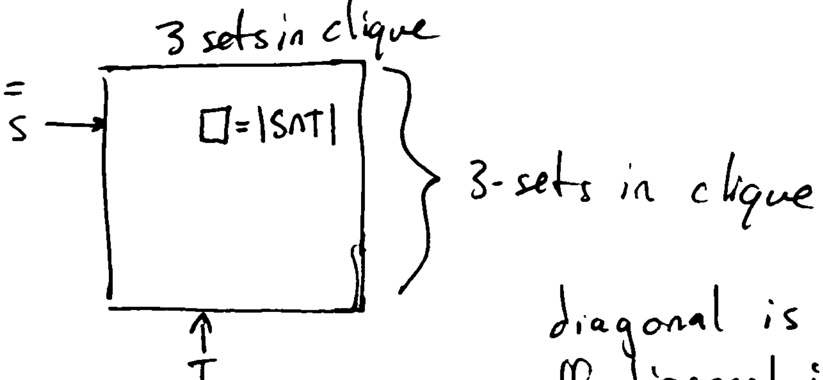


Take $1_S \in \mathbb{R}^M$ for characteristic vectors

Make matrix



Consider $A^T A$ =



diagonal is $|S \cap S| = 3$
off-diagonal is $|S \cap T| = 1$.

Show A^TA is nonsingular

Write $A^TA = J + 2I$, which turns out to be positive definite

$\Longrightarrow$ nonsingular

$\Longrightarrow$ Rank = # 3-sets in clique

3-sets in clique $=$ rank $A^TA \leq$ rank $A \leq M$

So 1-cliques have size $\leq M$.

~~Hence~~ ~~one~~

Last time: Found coloring of edges of K_n s.t. all monochromatic cliques size $\leq \sqrt[3]{n}$

Nagy 70's

3-sets, color by parity of intersection size

Frankl-Wilson 70's: all monochromatic cliques $\leq e^{\sqrt{\log n}\sqrt{\log\log n}}$

halfway between $\log(e^{\log\log n})$ and $\text{poly}(e^{c\log n})$.

Defn: A family of sets is called L-intersecting if any two of them have intersection size $\in L$, where ~~$L \subseteq \bigcup S_i$ where $\{S_i\}$ is family of sets.~~ $L \subseteq \mathbb{N}$.

Nagy: Cliques of color "odd" ~~d~~ were $\{1\}$ intersecting families.
Cliques of "even" color were $\{0,2\}$-intersecting.

Q: If $|L| = s$, how big can an L-intersecting family be?

Q: If $|L| = s$, how big can an L-intersecting family be?
(In terms of n and s, where ground set is $\{1, 2, \ldots n\}$.)

Construction: Use $L = \{0, 1, 2, \ldots, s-1\}$
Can get $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s}$ many by taking all sets of size $\leq s$.

Theorem (Ray-Chaudhuri-Wilson):
Any L-intersecting family of subsets of $\{1, 2, \ldots, n\}$ has size $\leq \sum_{i=0}^{s} \binom{n}{i}$

Theorem (Frankl-Wilson): Given prime p, set L of residues mod p, $|L| = s$. Then any L-intersecting family (mod p) of $\{1, 2, \ldots, n\}$ has size $\leq \sum_{i=0}^{s} \binom{n}{i}$

where each set in family has size $\notin L \pmod p$.

Proof: For each set S, associate a polynomial $p_S \in \mathbb{F}_p[x_1, \ldots, x_n]$. Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and ~~set $p_S = 1_S \cdot \vec{x}$.~~

$$p_S = (1_S \cdot \vec{x} - \ell_1)(1_S \cdot \vec{x} - \ell_2) \cdots (1_S \cdot \vec{x} - \ell_s)$$

where $L = \{\ell_1, \ldots, \ell_s\}$ and 1_S is the characteristic vector of S.

$$p_S = \prod_{\ell \in L} (1_S \cdot \vec{x} - \ell)$$

Examples: If $L = \{0, 2\}$
and $S = \{1, 2\} \subseteq \{1, 2, \ldots n\}$
then $p_S = ((x_1 + x_2) - 0)((x_1 + x_2) - 2)$.

Observe: If S and T are in L-intersecting family, then $p_S(1_T) = 0$, because if $|S \cap T| = \ell_1$, then

$$p_S(1_T) = (1_S \cdot 1_T - \ell_1)(\ldots)(\quad)$$
$$= 0\,(\,)\,(\,) = 0.$$

Also: $p_S(1_S) = (|S|-l_1)\cdots(|S|-l_s) \neq 0$ in $\mathbb{F}_p$.

① Want to show that $\{p_S : S \in \mathcal{F}\}$ (family) are linearly independent.

Suppose

$$\sum_{S\in\mathcal{F}} c_S p_S = 0.$$

Plug in 1_T to both sides:

$$\sum_{S\in\mathcal{F}} c_S p_S(1_T) = 0 \Longrightarrow c_T p_T(1_T) = 0$$

Since $p_T(1_T) \neq 0$, then $c_T = 0$ for any T.

So yes, lin-indep.

② Control dimension of vector space?

Observe $\dim \leq n^{s+1}$, b/c ~~degree~~ degree x_i in p_T is at most s, there are n variables.

Reduce further

Multilinearization:

$$p_T(\vec{x}) \longmapsto q_T(\vec{x}) \text{ by map } x_i^k \mapsto x_i, \text{ also HM.}$$

Doesn't affect what we care about b/c we only plug in characteristic vectors (0's and 1's)

q's Also linearly independent.

So Vector Space is multilinear polynomials of degree $\leq s$.

Basis: $\{1, x_1, x_2, \ldots, x_n, x_1x_2, x_1x_3, \ldots x_nx_{n-1}, \ldots x_1x_2\cdots x_n\}$

has cardinality $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s}$

Frankl-Wilson coloring of vertices of K_N:

Each vertex is subset of size $p^2+(-1)$ of $\{1,2,\dots,n\}$.

$S \bullet \text{- - - - - - - -} \bullet T$

$\uparrow$ intersection size $\in \{0,1,2,\dots,p^2-2\}$

red: size $\not\equiv -1 \pmod p$

blue: size $\equiv -1 \pmod p$

red clique: family of subsets of $\{1,2,\dots,n\}$ each of size $=p^2-1$ but with intersections $\not\equiv -1 \pmod p$

$\in \{0,1,2,\dots,p-2\} \pmod p$

So red clique has at most $\leq p\binom{p^2-1}{p-1} \leq \left(\frac{ep^2}{p}\right)^p \sim p^p$

blue clique: family, each set size $=p^2-1$ $\leftarrow$ $p\binom{n}{p-1} \leq \left(\frac{en}{p}\right)^p$

intersection sizes $\in \{p-1, 2p-1, \dots, p^2-1-p\}$

Using Frankl-Wilson w/ prime $q > p^2-1$ ~~$\Rightarrow$ blue cliques $\leq p$~~

blue cliques $\leq \left(\frac{en}{p}\right)^p$

$N=\binom{n}{p^2-1}$ choose $n=p^3$

$N=\binom{p^3}{p^2-1} \sim (p)^{p^2}$ clique size $\sim (p^2)^p$

$\Downarrow$

$\log N \sim p^2 \log p$

choose $p \sim \frac{\sqrt{\log N}}{\sqrt{\log\log N}} \longrightarrow \left(\frac{\log N}{\log\log N}\right)^p$

$e^{p\log p}$

$\downarrow$

$e^{\sqrt{\frac{\log N}{\log\log N}}\log\log N}$

$\downarrow$

$e^{\sqrt{\log N}\sqrt{\log\log N}}$

Ramsey Theory on Sequences

Thm (Erdős-Szekeres): If we have a sequence of n distinct numbers, then there is an increasing/decreasing subsequence of length $\geq \sqrt{n}$.

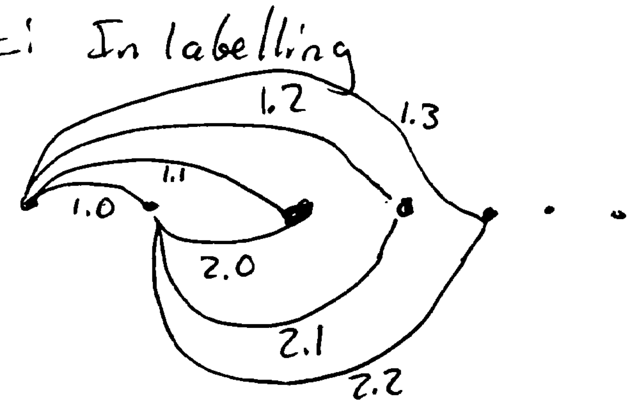
Q: What if each edge of K_n is labelled with $1, 2, \ldots, \binom{n}{2}$?
How long of an increasing path is guaranteed to exist?

Path doesn't re-use vertex

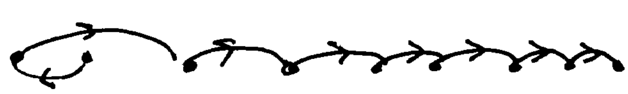
A: Unknown, open.

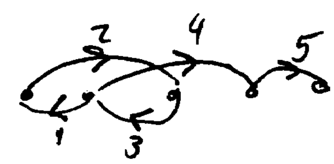
Theorem: If each edge of K_n is labelled with $1, 2, \ldots, \binom{n}{2}$, then there ~~are~~ is always an increasing walk of length $\geq n-1$
(# edges)

Observe: In labelling



All walks have length $\leq O(n)$

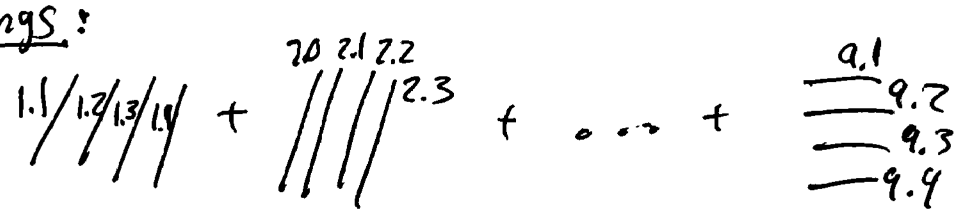




n even

Fact: Edges of K_n can be decomposed into $n-1$ perfect matchings.

Matchings:

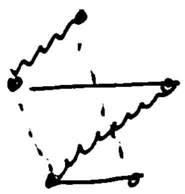


1st 2nd $(n-1)^{st}$

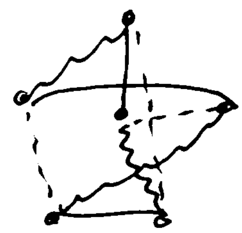
So, once you exit a layer, can't return
$\Rightarrow$ max. # of edges in increasing walk is $\le n-1$ (#layers)

First for N=odd:

Get n almost perfect matchings $\to$ edges $\binom{n-1}{2}$



For N even:
put extra vertex in middle



Proof: (There always exists an increasing walk of length $n-1$.)

Given graph w/ labelled edges, put person on each vertex. Person w/ megaphone shouts label and people move across ~~w~~ edge w/ that label.

Shout labels in increasing order.

People only take increasing walk.

Total steps taken $2\binom{n}{2} = n(n-1)$ There are n people walking, average # of steps taken is $(n-1)$.

Theorem: There is always an increasing path of length $\geq c\sqrt{n}$

Theorem: There exists a labelling with all increasing path length $\leq n-1$

Theorem: $\exists$ a labelling w/ all increasing path length $\leq \left(\frac{1}{2}+o(1)\right)n$

Remark: Need to force path to hit itself.

Theorem:

K_N, all edges black or white.

Always exists monotone monochromatic path of length $\gtrsim \sqrt{N}$

Proof: Suppose all monochromatic monotone paths have length at most n. Then $N \leq (n+1)^2$ (will show)

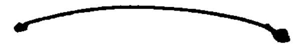
length of longest black path ending here → x

length of longest white path ending here → y

$\binom{x}{y}$ $\binom{1}{0}$ $\binom{?}{1}$

Observe: Never repeat an ordered pair.

Why? If two have the same, the edge between one adds one to one of coordinates



$\binom{5}{7}$ $\binom{5}{7}+1$

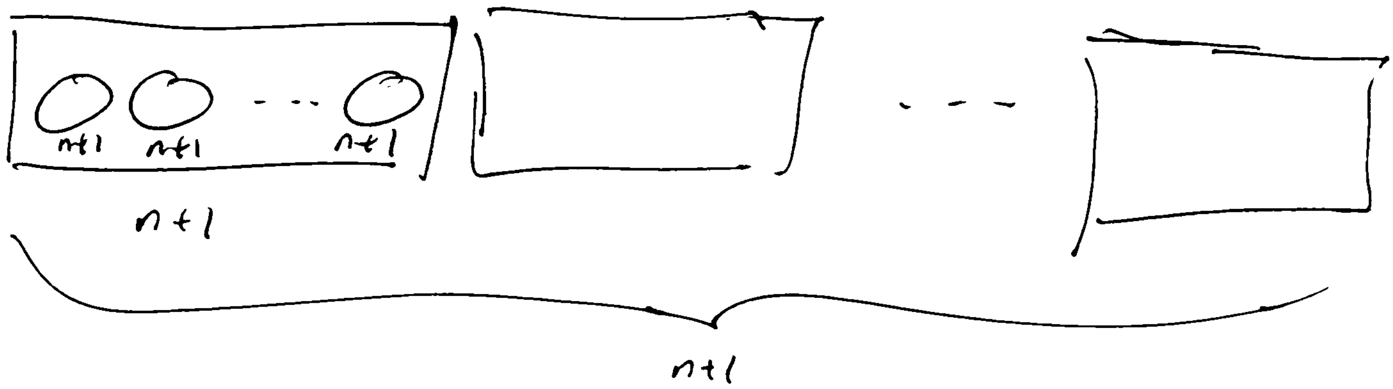
Also, $\exists$ coloring that gets $N=(n+1)^2$

What about 3 colors?

Theorem: vertex-ordered K_N, all edges colored $1,2,3$.
There is always a monochrome, monotone path of length $\geq \sqrt[3]{N}$

Proof is the same; show if all paths are length $\leq (n)$ then $N \leq (n+1)^3$.

Bound is also sharp, same reason.



Can also talk about color-avoiding paths.

Q: K_N, vertex ordered, all edges colored $1,2,3$, then $\exists$ path which avoids one color of length $\geq \sqrt{N}$

Apply previous result, merge two colors

Also: there is coloring where all ≥ 1-color avoiding paths have length $\leq N^{2/3}$

8a

Thm: K_n, vtx ordered, 3-colors for edges, $\exists$ path which avoids one color of length $\geq \sqrt{N} \log^* N$

Question: How long of a sequence of triples can I write, where each triple has elements $\in \{1 \dots n\}$ such that for any two triples, there are at least two coordinates where the later triple is strictly greater?

$\boxed{\text{Non-transitive order relation} \quad 1,2,3 < 2,3,1 < 3,1,2 < 1,2,3}$

Last time: 3-color edges of ordered vertex K_N
wanted to show $\exists$ long ≥ 1 color avoiding path.

~~1st triple~~ 1^{st} elt in triple: longest path avoiding color 1
2^{nd} elt: longest path avoiding color 2

$(\_ , \_ , \_)$

Know a sequence of triples of length $N \approx n^{3/2}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ & \vdots & \\ 1 & \sqrt{n} & \sqrt{n} \end{bmatrix} \quad \begin{bmatrix} \sqrt{n}+1 & \sqrt{n}+1 & 1 \\ & \vdots & \\ \sqrt{n}+1 & 2\sqrt{n} & \sqrt{n} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & \sqrt{n}+1 \\ 2 & 2 & \sqrt{n}+2 \\ & \vdots & \\ 2 & \sqrt{n} & 2\sqrt{n} \end{bmatrix} \quad \begin{bmatrix} \sqrt{n}+2 & \sqrt{n}+1 & \sqrt{n}+1 \\ & \vdots & \\ \sqrt{n}+2 & 2\sqrt{n} & 2\sqrt{n} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2\sqrt{n}+1 \\ 3 & 2 & 2\sqrt{n}+2 \\ & \vdots & \\ 3 & \sqrt{n} & 3\sqrt{n} \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} \sqrt{n} & 1 & (\sqrt{n}-1)(\sqrt{n})+1 \\ \vdots & & \vdots \\ \sqrt{n} & \sqrt{n} & n \end{bmatrix}$$

Get $\sqrt{n} \cdot n$ many triples.

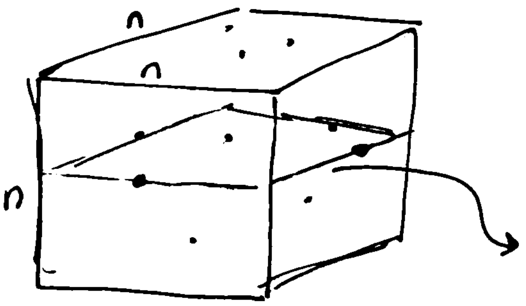
Is $N \approx n^{3/2}$ the best?

Other side:

Theorem: Length of a triples sequence $\leq \dfrac{n^2}{\log^* n}$

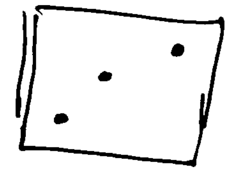
Proof Idea: Observe a connection to famous problem.

Put triples inside cube, plot them.



Swiss Cheese **Condition**

remove slice of cube, triples must increase to top right.

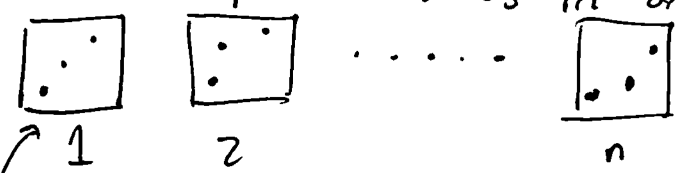


All have same 3rd coordinate, both of others must grow.

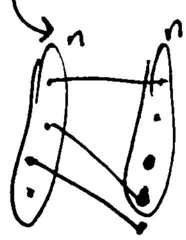
Every slice gets an increasing pattern ↗

How many points can you have?

Look at all n parallel slices in one direction.



Bipartite graph



Bijection between bipartite graph w/ edges between vtx i and j if (i,j) is coordinate of point in slice.

Want upper bound on total # of edges in all of these graphs?

Q: Same edge twice? $\Longrightarrow$ at most n^2 edges.

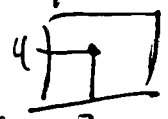
$(x, y, 5)$ No!

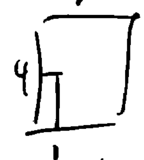
$(x, y, 7)$

Observation: these bipartite graphs are matchings, so each has $\leq n$ edges $* \, n$ graphs.

Obs: edges can't cross

Obs: layer 5   layer 7

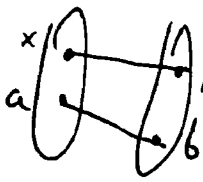




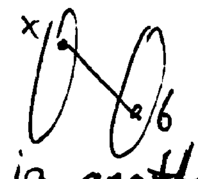
if this   can't have this

(3,4,5)   (1,4,7)

Obs: If in one matching,



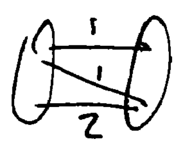
cannot have

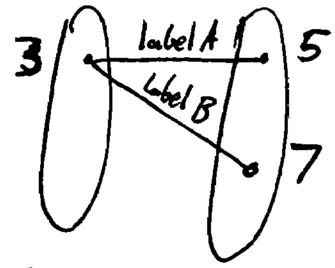


in another.

Why?

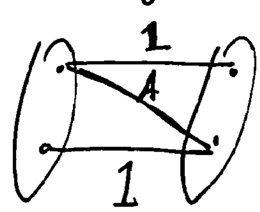
Stick all matchings on top of one another, label by which layer they come from





$(3,5,\underline{A})$
$(3,7,\underline{B})$
$\Rightarrow B > A.$

So if all edges on one graph G



$1 > A > 1$ ✱.

G is union of n induced matchings.

So can find n sets $V_1, V_2, \ldots, V_n$ so that each $G[V_i]$ is a matching, and these cover all edges

Q: How many edges in a graph which is union of n induced matchings?

(i,j)-matching $1 \leq i \leq \sqrt{n}$, $1 \leq j \leq \sqrt{n}$

Connect i^{th} vertex of each left chunk w/ j^{th} vertex of each right chunk.

gets $n^{3/2}$ matchings.

Known construction for induced matching problem that gets $n^2/e^{\sqrt{\log n}}$ many edges. $e^{\sqrt{\log n}} \ll e^{.001 \log n} = n^{.001}$

Theorem: Union of N-induced matchings has at most $\frac{n^2}{\log^* n}$.

Uses "Regularity Lemma".

Question: n-vertex graph, disjoint union of n induced matchings.

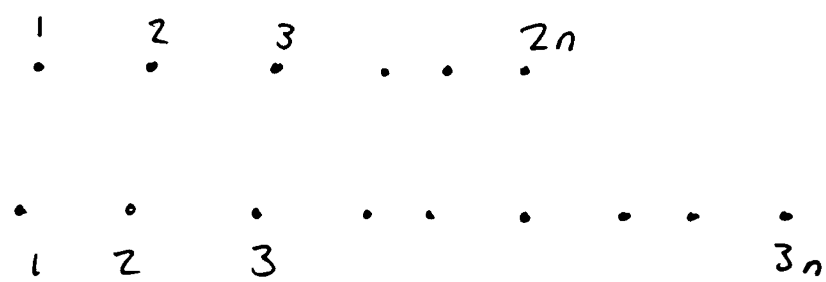
How many edges can you get?

Theorem: Number of edges $\le \frac{n^2}{\log^* n}$

but exist constructions with $\frac{n^2}{e^{\sqrt{\log n}}}$ edges

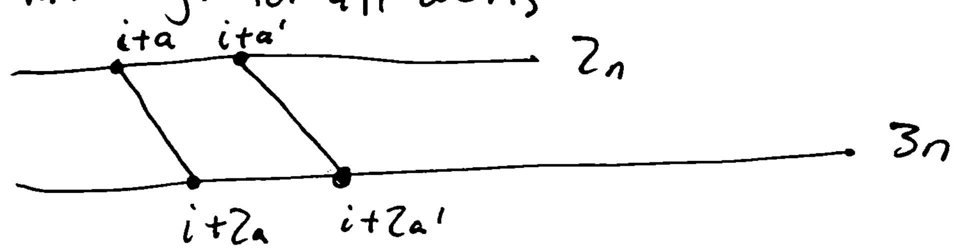
Construction (Rusza-Szemerédi):

Say $5n$ vertices in 2 rows



Build ~~~~ n matchings using a set $A \subseteq \{1, 2, \dots, n\}$

i^{th} matching: for all $a \in A$,

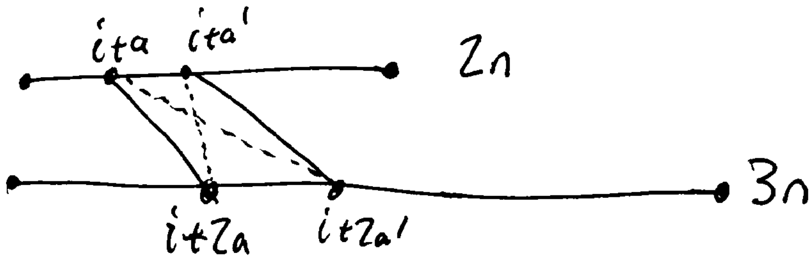


Observation: If G is a graph on n vertices, $\exists$ bipartite graph G' on n vertices which is a subgraph of G and

$e(G') \geq \frac{1}{2} e(G)$

Proof: each vertex independently goes to left or right w/ probability $= 1/2$, only keep edges that cross left $\to$ right. $\mathbb{E}$[# edges kept] = 1/2 * # edges.

Check: given two edges of i^{th} matching, won't find an edge of j^{th} matching that uses 2 of the vertices.



dotted lines in jth matching.

$i+a' = j+a''$
$i+2a = j+2a''$
OR
$i+a = j+a''$
$i+2a' = j+2a''$

Why can't we have either of these edges?
What condition on A to be certain?

Suppose we really do have

$$\begin{cases} i+a' = j+a'' & (1) \\ i+2a = j+2a'' & (2) \end{cases} \xrightarrow{(2)-(1)} 2a - a' = a'' \iff a = \frac{a'+a''}{2}$$

So if A has no three term arithmetic progression, then the construction makes the n induced matchings ~~and~~ legit.
Total # of edges in $\underline{n}$ induced matchings is $n|A|$.

How big of a subset of $\{1, 2, \dots, n\}$ can we get w/ no 3AP?

(1) $\log_2 n$ size $1, 2, 4, 8, \dots$

(2) Greedy approach

Given T many so far $x_1, x_2, \dots, x_T$

$\le \binom{T}{2}$ blocked, $+T < T^2$

Can add another within $\{1, 2, 3, \dots, T^2\}$

3AP free, size $\ge \sqrt{n}$.

Random: pick each # w/ prob p

For a fixed 3AP, probability that it stays in subset we picked $= p^3$

3APs in $\{1, \dots, n\}$ is $\le n^2$.

$\mathbb{E}[\#\text{ 3AP in selected subset}] \le n^2 p^3$.

pick $n^2 p^3 \le 1/2$. $p \sim c n^{-2/3} \Longrightarrow \mathbb{E}[\#\text{ 3AP in selected}] \le 1/2$

Probability that 3AP survives $\le 1/2$

picked $\sim \mathrm{Bin}(n, p)$

$\mathbb{E}[\#\text{ picked}] \approx np \sim n^{1/3}$

$\mathbb{P}[\mathrm{Bin}(n,p) \text{ deviates from mean by factor } \ge c] \le e^{-c\mathbb{E}[\cdot]}$

Random apprach 2: (Alterations Method)

ⓐ Pick each # w/ prob $= p$

ⓑ For each 3AP surviving, throw out one element.

$\mathbb{E}[\#\text{ of nums that survive ⓐ and ⓑ}] \ge \mathbb{E}[\#\text{ nums at ⓐ}] - \mathbb{E}[\#\text{ 3AP surviving past ⓐ}]$.

$\mathbb{E}[\text{selected in ⓐ}] = np$

$E[\#\text{3AP surviving past ⓐ}] \le n^2 p^3$ as above

$\mathbb{E}[\text{size of } A] \ge np - n^2 p^3$

$n - 3n^2p^2 = 0$

$\sqrt{\frac{n}{3n^2}} = p$

Choose p: to matimize $np - n^2 p^3$, between 0 and 1.

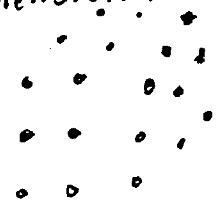
Ballpark estimate: $\frac{1}{2} np = n^2 p^3 \Longrightarrow p \approx n^{-1/2}$.

And then get $np - n^2 p^3 = \frac{1}{2} np \approx n^{1/2}$.

Behrend Construction: 3AP-free sets $\subseteq \{1, \ldots, n\}$ of size $\approx \frac{n}{e^{c\sqrt{\log n}}}$

First, build a 3AP-free subset of $\{1,2,\ldots,r\}^d$

A d-dimensional lattice cube



3AP in $\mathbb{R}^d$ is two points and their midpoint.

Lets find subset of $\{1,2,\ldots,r\}^d$ which has no 3 points on a line.

Pick a sphere, so that $\sum x_i^2 = T$. $S_T = \{x_1^2 + x_2^2 + \cdots + x_d^2 = T\}$

All r^d lattice points in the cube are are in

$$\bigcup_{i=0}^{r^2 d} S_i .$$

Use pigeonhole $\Longrightarrow \exists$ sphere with $\geq \frac{r^d}{r^2 d}$ lattice points from cube

This creates a 3AP-free d-dimensional subset L of $\{1,2,\ldots,r\}^d$.

Map to integers from d-dimensional points $\in L$.

$$(\underline{0 \ldots r}, \underline{0 \ldots r}, \ldots, \underline{0 \cdots r}) \longmapsto \text{integer}$$

Map to base $2r$. Why no 3AP in integer side?

Consider 3AP of integers?

Write in base $2r$, note that addition need not carry since each coordinate is <u>less than</u> r.

If 3AP in $\mathbb{Z} \Longrightarrow$ vector 3AP $\ast$ L is 3AP free.

Behrend Construction

$\{0,1,2,\dots,r-1\}^d$ took points on a sphere $\to$ pigeonhole $\geq \frac{r^d}{r^2 d}$ points, 3AP free.

Map $\vec{x} \in \{0,1,\dots,r-1\}^d \longmapsto \mathbb{Z}$ reading in base $2r$, gives 3AP-free subset of $\mathbb{Z}$.

Try to find 3AP-free subset of $\{1,2,\dots,N\}$. So $N = (2r)^d$.

Get 3AP-free subset of size $\geq \frac{r^d}{r^2 d}$.

Intuition: ~~maximize~~ maximize $\frac{\text{size}}{N} \geq \frac{\cancel{r^d}/r^2 d}{2^d \cancel{r^d}} = \frac{1}{2^d d r^2}$.

So minimize $2^d d r^2$ subject to $(2r)^d = N$. (☆)

For large d, $2^d d r^2 \approx 2^d r^2$ taking logs, $d \log r \approx \log N$

minimize $\log(2^d r^2) \approx \log r + d$.

Try $d \approx \log r \approx \sqrt{\log N}$. Specifically, take $d = \sqrt{\log N}$!

(☆) $\Longrightarrow 2r = e^{\sqrt{\log N}} \Longrightarrow r = \frac{1}{2} e^{\sqrt{\log N}}$

Now find $\frac{r^d}{r^2 d}$ as a function of $N \longrightarrow \dfrac{\left(\frac{1}{2} e^{\sqrt{\log N}}\right)^{\sqrt{\log N}}}{\sqrt{\log N}\ \frac{1}{4} e^{2\sqrt{\log N}}}$

$$= \frac{(1/2)^{\sqrt{\log N}}\ N}{\sqrt{\log N}\ \frac{1}{4} e^{\sqrt{\log N}}} = \frac{N}{e^{C\sqrt{\log N}}}.$$

More careful optimization gives size $\geq \dfrac{N}{(\log N)^{1/4} e^{2\sqrt{2}\sqrt{\log N}}}$

subgroup of units mod p.

Question: What's the size of the largest $\underbrace{\text{sum-free}}_{\text{no } x+y=z,\ (x,y,z) \in \text{subgroup}}$ subgroup of $\frac{\mathbb{Z}}{p\mathbb{Z}}^*$?

Paper: Alon + Bourgain

Observation: without subgroup requirement, can get sum-free subset of $\mathbb{F}_p$. Take $\left[\frac{3}{4}p, \frac{p}{2}\right]$ interval to get linear size in p, the best is $\left[\frac{1}{3}p, \frac{2}{3}p\right] \cap \mathbb{F}_p$.

Theorem (Alon & Bourgain): If S is a subgroup of $\mathbb{F}_p^*$, and $|S| > p^{3/4}$, then S is not sum-free.

Thm (A+B): There are infinitely many primes p s.t. $\exists$ sum-free subgroup of $\mathbb{F}_p$ of size $\geq p^{1/3}$.

Open Question: Close gap between $1/3$ and $3/4$.

Ideas to build large sum-free subgroup:

Note the multiplicative subgroup of $\mathbb{F}_p$ is cyclic, find generator α.

Want order of α to be $\approx m$ (bigger is better)

In a subgroup: $x + y = z \iff 1 + x^{-1}y = x^{-1}z$

let $a = x^{-1}y$, $b = x^{-1}z$, sum free subgroup $\iff$ no consecutive numbers subgroup.

If an elt. α of order m is ~~not a~~ root of

$$P_m(x) = \prod_{0 \leq i < j < m} (x^i - x^j - 1),$$ then $\langle \alpha \rangle$ is sum-free subgroup of order $\in \left(\frac{m}{2} \ldots m\right)$

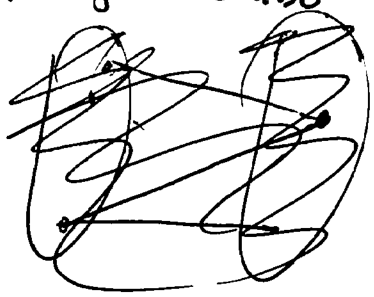
How to know there's a non-root of this polynomial.

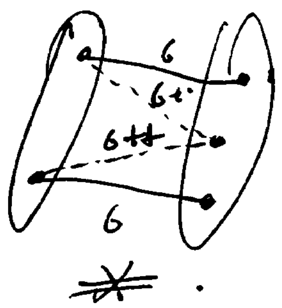
$\deg(P_m) < m^3$

If instead take $\frac{m}{2} - m$ as order of α

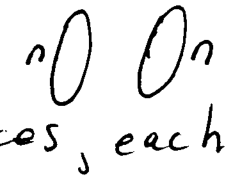
Triples Problem: longest sequence of triples under wacky order that isn't transitive.

Reduced to swiss cheese problem, and then to bipartite graphs. Matings are also Σ-free





cannot happen.



Theorem: n-matchings on $2n$ vertices, each one is

 -free (also Σ-free), also induced.

$\Longrightarrow$ total # edges $\leq n^{3/2}$

Problems!

Ramsey-Type: decomposing "complete" structures with colors and ~~[illegible]~~ finding nicely structured monochromatic things, as long as it's big enough.

Turán-Type: Given a sub-structure of a complete structure, show it has nice properties as long as it's dense enough.

If you could find this nice structure in any structure with at least $1/2$, then in any coloring of complete structure w/ 2 colors, one color class has property.

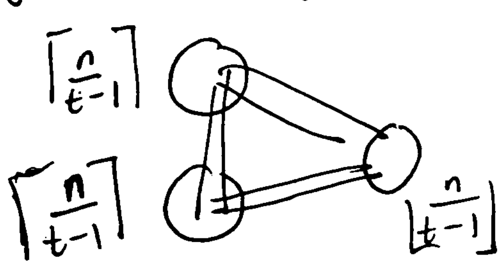
Defn: $ex(n,H)$ = max # of edges in any n-vertex graph w/ ~~no~~ H as a subgraph.

Eg: $ex(n, C_4)$

① Find construction w/ no C_4, lots of edges.

② Prove: if at least $f(n)$ edges, then exists C_4.

Eg: $ex(n, K_t)$ = # edges in this $\approx \left(\frac{n}{t-1}\right)^2 \binom{t-1}{2} = \frac{n^2}{(t-1)(t-1)} \frac{(t-1)(t-2)}{2}$



$$= \frac{n^2}{2}\left(\frac{t-2}{t-1}\right)$$

$$\approx \frac{n^2}{2}\left(1 - \frac{1}{t-1}\right)$$

Observe: As t increases, $ex(n,K_t)$ does too.

monotonic b/c $K_t \subseteq K_{t+1}$, so $ex(n,K_t) \leq ex(n,K_{t+1})$

Try to show $ex(n, C_4) \leq$ ____ . Upper Bound.

If > ___ edges, guarantee a C_4

Use $C_4 \subseteq K_4 \Longrightarrow ex(n, C_4) \leq ex(n, K_4) \approx \frac{n^2}{2}\left(1-\frac{1}{3}\right) = \frac{n^2}{3}$

In general $ex(n, H) \leq ex(n, K_{v(H)})$.

So $ex(n, C_4) \leq \frac{n^2}{3}$

Lower Bound:

Random graph: probability p for edge

$\mathbb{E}[\# \diamond] \leq n^4 p^4 \leq 1/2$ So $p \sim \frac{1}{n}$ So total edges $\binom{n}{2}p \sim n$.

n^4: fixing 4 ~~edges~~ vrtxs

p^4: all 4 edges there

Alterations Methods:

For each C_4 in random graph, throw out edge.

$\mathbb{E}[C_4] \leq n^4 p^4$ $\quad \mathbb{E}[\text{edges}] = \frac{n^2}{2}p$ $\quad \mathbb{E}[\text{edges surviving}] \geq \frac{n^2}{2}p - n^4p^4$

Choose $p \sim \frac{1}{n^{2/3}}$ $\quad \frac{n^2 p}{4} \sim n^4 p^4$

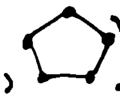
$\cancel{\mathbb{E}[\# C_4] \geq n^{4/3}}$ $\quad \mathbb{E}[\text{edges surviving}] \geq n^{4/3}$

Eg: $ex(n, \triangle) \geq$ get constructions w/no $\triangle$ yet lots of edges.

Use construction w/ no $\triangle$

$ex(n, H) \geq ex(n, K_{\omega(H)})$, where $\omega(H)$ is biggest clique in H.

Know: $ex(n, K_{\omega(H)}) \leq ex(n, H) \leq ex(n, K_{v(H)})$

Eg: $ex(n,$  $) \geq$

Construct graph with no 5-cycle yet lots of edges

One way is to use $K_{n/2, n/2}$. It has ~~odd~~ no odd cycles.

Also $ex(n, C_{2k+1}) \geq$ # edges in $K_{n/2, n/2}$

$ex\left(n, \begin{matrix}\text{not}\\ \text{bipartite}\\ \text{graph}\end{matrix}\right) \geq$ # edges in $K_{\frac{n}{2},\frac{n}{2}}$.

Also: $ex\left(n, \begin{matrix}\text{not } k\\ \text{colorable}\\ \text{graph}\end{matrix}\right) \geq$ # edges in complete k-partite graph

So $ex(n, H) \geq$ # edges in complete $(\chi(H)-1)$-partite graph

$= ex(n, K_{\chi(H)})$

Shown:

$$ex(n, K_{\chi(H)}) \leq ex(n, H) \leq ex(n, K_{v(H)})$$

Thm (Erdős - Stone - Simonovits):

$$ex(n, H) = ex(n, K_{\chi(H)}) + o(n^2)$$

Precisely, ~~for~~ given H, $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n > N$

$$ex(n, H) \leq ex(n, K_{\chi(H)}) + \varepsilon n^2$$

$ex(n, \text{tree}) = ?$

OBS: If T_t is a tree w/ t vertices. Then any graph with n vertices and min degree $\geq t-1$ contains T_t.

greedy algorithm

Defn: A graph is <u>k-critical</u> if $\chi(G) = k$, and deleting any vertex results in a strictly smaller chromatic number.

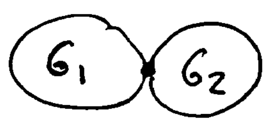
Eg: Odd cycles are ~~k-critical~~ 3-critical.

Theorem: Let G be a k-critical graph with n vertices. Then G has (i) a path of length $\geq c_1 \frac{\log n}{\log k}$

(ii) a cycle of length $\geq c_2 \sqrt{\frac{\log n}{\log k}}$ where $k \geq 3$.

Prop: if G is k-critical & k, then G is k-connected.

Proof: Suppose not, then delete a vertex.



G_1 and G_2 can be $k-1$ colored, which can extend to a $k-1$ coloring of G. ∎

Prop: If G is k-critical, then G is $(k-1)$-edge connected.

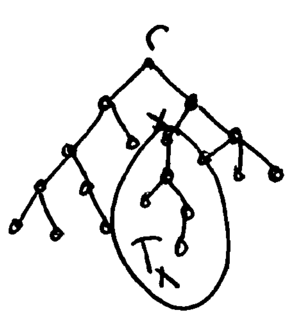
Proof of (i): uses DFS.

Notation: If T is a tree, $x \in T$, then T_x is subtree w/ root x

Path in the tree from a to b is aT_b.

Prop: If e is a non-DFS-tree edge in G, e has to join a vertex x to one of its ancestors.

$\Rightarrow$ all edges of G between T and $T \setminus T_x$ are between T_x and xT_r



Proof of (i):

We find DFS tree T with root r.

Let $d(u)$ be the depth of u, the length of uTr.

For an edge $e \in E(T)$, the depth of $e=(u,v)$ is $\min(d(u), d(v))$

Claim: T has at most $(k-1)^j$ many edges of depth j.

Next we show claim $\Rightarrow$ (i)

$$n-1 = e(T) \leq \sum_{j=0}^{R} \#\text{ of edges of type } j \leq \sum_{j=0}^{R} (k-1)^j \leq (k-1)^{R+1}$$

$$\Longrightarrow \quad R+1 \geq \frac{\log(n-1)}{\log(k-1)}$$

$$\Longrightarrow R \geq c_1 \frac{\log n}{\log k}$$

So $\underset{\text{length in } G}{\text{longest path}} \geq R \geq c_1 \frac{\log n}{\log k}$.

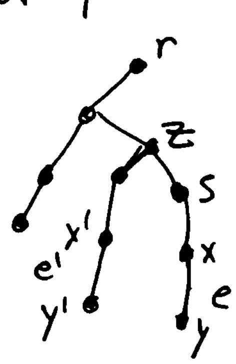
Proof of claim:

For each $e \in E(T)$, $G - \{e\}$ has a $(k-1)$-coloring since G is k-critical. Let f_e be its proper $(k-1)$-coloring. If the path rTy is $r v_2 v_3 \cdots y$, where $e=(x,y)$, $v_1 = r$, $v_{d-1} = x$, $v_d = y$.

Let $F(e)$ be the sequence $(f_e(v_1), f_e(v_2), \ldots, f_e(v_{d-1}))$ of colors $\{1, 2, \ldots, k-1\}$.

To prove the claim, it suffices to show different edges of depth j has a distinct sequence, $F(e_j)$.

To see this, consider distinct edges e, e' of the same depth j, but w/ same sequence $F(e) = F(e')$



Let z be the first vertex where the paths meet, going up from bottom. Let s be the vertex in the path rTy after z.

We show: colorings f_e and $f_{e'}$ will give a proper $(k-1)$-coloring f of the graph G.

Color vertices in $T - T_s$ using f_e } call it f.
Color vertices in T_s with $f_{e'}$ }

Each of f_e, $f_{e'}$ use colors $\{1, 2, \dots k-1\}$, so this is a $k-1$ coloring of G $*$.

↑

coloring is proper b/c all edges between T_x and $T - T_x$ are between T_x and xT_r.

Proof of (ii)

Follows by the fact that G is 2-connected and the following Lemma:

Lemma: If G is 2-connected and has a path of length L, then G has a cycle of length $\sqrt{L}$.

Proof: Exercise.
Menger's Theorem

2/14/14

Extremal Number

$ex(n, T_t)$ $\quad$ T_t is a fixed tree on t vertices
max # edges in n-vertex, T_t-free graph

<u>Use</u>: $\delta \geq t-1 \Rightarrow$ graph contains T_t.

<u>Lemma</u>: If G has average degree d, then G contains a subgraph with min-degree $> \frac{d}{2}$.

<u>Proof</u>: While there is a vertex of degree $\leq d/2$, remove it.
Show that this stops ~~before~~ graph is 0 vertices.

# vertices	n	$n-1$	$n-2$	$\cdots$	1
# edges	$\frac{nd}{2}$	$\geq (n-1)\frac{d}{2}$	$\geq (n-2)\frac{d}{2}$		$\geq d/2$

↑ must stop somewhere here

$\divideontimes$

<u>Lemma</u>: If G has avg. degree d, then G contains a subgraph w/ min-degree $\geq$ ~~d/2~~ $0.3d$ with at least vertices.

<u>Proof</u>: Repeat argument from before

# vertex	n	$n-1$	$n-2$	$\cdots$	$n-k$
# edges	$\frac{nd}{2}$	$\geq \frac{nd}{2} - 0.3d$	$\geq \frac{nd}{2} - 0.6d$	$\cdots$	$\geq d/2\,(n-0.6k)$

Must stop before $\frac{d}{2}(n-0.6k) > \binom{n-k}{2}$

↓ about

$$\frac{d}{2}(n-0.6k) \sim \frac{(n-k)^2}{2}$$

Let $k = tn$

$$\frac{d}{2}(0.4t) \sim \frac{(1-t)^2}{2}n$$

$$d\,t\,(0.4) \sim (1-t)^2 n$$

$\frac{d}{n}$ is original edge density.

Corollary: if # edges $> n(t-2)$, then graph contains T_t (any tree on t vertices).

Pf: $\Longrightarrow$ avg degree $\geq 2(t-2)$

$\Longrightarrow$ $\exists$ subgraph with minimum degree $> (t-2)$

t integer, min degree integer

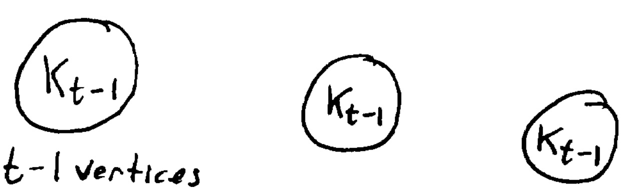
$\Longrightarrow$ min-degree of subgraph $\geq t-1$

So use fact from Zihao Ma.

Hence, $ex(n, T_t) < n(t-2)$.

Lower Bound: Many edges, avoids T_t.

Construction:



$t-1$ vertices

Suppose $(t-1) \mid n$

$$\# \text{ edges} = \binom{t-1}{2} \frac{n}{(t-1)} = n \frac{(t-2)}{2}.$$

Hence $\frac{n(t-2)}{2} \leq ex(n, T_t)$.

Erdős–Sós Conjecture: If $> \frac{n(t-2)}{2}$ edges, then $\exists T_t$.

Prop: $\chi(G) \geq t \Longrightarrow G$ contains T_t.

Consequences of big chromatic number?

Ex: If $\chi \geq 50$, then graph has...

a vertex of degree ≥ 49. $\Longrightarrow \Delta \geq 49$

Consequences of large χ:

There is a subgraph with minimum degree $\geq$ ~~48~~ 49. ($\Rightarrow$ has a T_t)

Otherwise, if every subgraph has a vertex of degree ≤ 48, gives ordering of greedy algorithm.

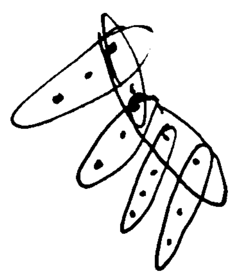
Strip vertex of degree ≤ 48, continue in this manner.

49 colors suffice to color graph.

Hypergraph Version: $\chi > e$, then there is a copy of any e-edge hypertree, in an r-uniform hypergraph.

Defn: χ of a hypergraph is min # of colors required to color vertices w/ no hyperedge monochromatic.

Defn: Hypertree has hyperedges that only ever overlap on one vertex. Start w/ one edge, repeatedly add $(r-1)$ vertices, create new edge w/ those and one old vertex.



Observation: Result is tight. Can find ~~(r-1) uniform~~ r-uniform hypergraph w/ $\chi = e$ and no hypertree w/e edges.

Construction: $(r-1)e$ vertices
complete r-uniform hypergraph has $\chi = e$, but no tree.

Proof of Hypergraph Version:

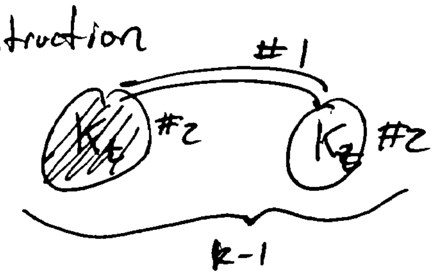
Due to Erdős, in any hypergraph w/ $\chi > 2$, $\exists$ two edges overlapping on 1 vertex.

Last time: $\chi > t \implies$ (hyper)graph has any t-edge tree you want

Why interesting?

Revisit: $\text{Ramsey}(K_k, t\text{-edge tree}) = (k-1)t+1$

Construction



Proof that $> (k-1)t$ vertices
$\implies$ 1-colored K_k or 2-colored t-edge tree

Look at G = color 2 graph (same vtxs, only color 2 edges)

$\alpha(G) \le k-1$ else color-1 K_k

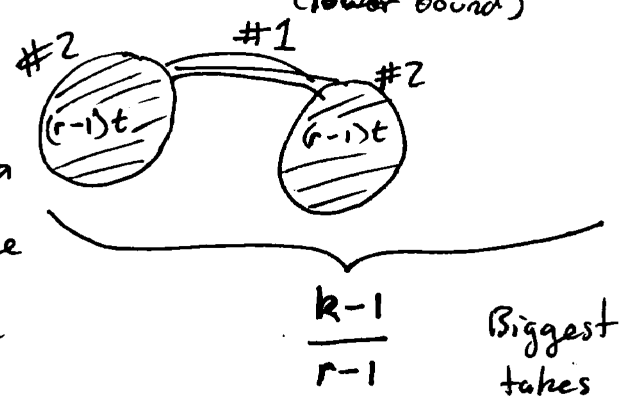
↑ max indep set

$$\chi(G) \ge \frac{v(G)}{\alpha(G)} > \frac{(k-1)t}{k-1} = t$$

Version for r-uniform hypergraphs.

$\text{Ramsey}(K_k^{(r)}, t\text{-edge } r\text{-uniform hypertree})$

Construction: (lower bound)



no t-edge r-uniform hypertree fits

Biggest #1-colored set takes $r-1$ vertices from each blob.

$\implies R(K_k^{(r)}, t\text{ edge } r\text{-unif hypertree}) \ge (k-1)t$

(upper bound)

Proof works the same as before.

Goal:

$$ex(n, C_4) = ex(n, K_{2,2}) \leq cn^{3/2}.$$

Prove: that if we have no , then # edges $\leq cn^{3/2}$

Observe that for any two vertices, # common neighbors is ≤ 1.

For all pairs (x,y), codegree $(x,y) \leq 1$.

If d_v = degree of $v \in V(G)$, gives $\binom{d_v}{2}$ pairs w/ common neighbor v.

As you go over all $v \in V$, never see same pair of neighbors twice:

$$\sum_{v \in V(G)} \binom{d_v}{2} \leq \binom{n}{2}$$

Since $\binom{x}{2}$ convex, get $n\binom{\bar{d}}{2} \leq \sum_{v \in V(G)} \binom{d_v}{2} \leq \binom{n}{2}$

$$\bar{d} = \frac{2m}{n}$$

$$\Longrightarrow n \frac{\left(\frac{2m}{n}\right)^2}{2} \lesssim \frac{n^2}{2} \approx \binom{n}{2}$$

$\approx \binom{\bar{d}}{2}$

$\Longrightarrow m^2 \leq n^3$ asymptotically

so $ex(n, C_4) \leq cn^{3/2}$.

$\binom{a}{b} \sim \frac{a^b}{b!}$ w/ multiplicative error $1+o(1)$

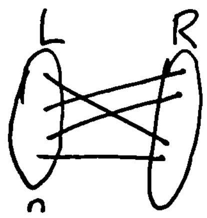
(Kövari-Sós-Turán)

Theorem: $ex(n, K_{s,t}) \leq c_t n^{2-1/s}$ c_t is constant depending on t.

Proof: Assume we have H as a subgraph of $K_{n,n}$, with no $K_{s,t}$.

Show not too many edges. $\Longleftrightarrow$ $p \leq$ ___.

p = edge density



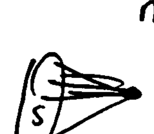
$= pn^2$ edges

Any s things have at most $(t-1)$ common neighbors

Count: $\#\{$  $\} = \sum_{x \in R} \binom{d_x}{s} \geq n\binom{\bar{d}_R}{s}$

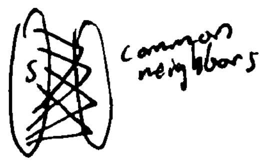
$\binom{x}{s}$ convex if we set equal to 0 when $x \leq s-1$.

$$\bar{d}_R = \frac{pn^2}{n} = pn$$

$\#$  $= n\binom{pn}{s} \sim n\frac{(pn)^s}{s!} \sim p^s n \binom{n}{s}$

common neighbors of S

<u>Proof continued:</u> $\#K_{s,t} = \sum_{\substack{S \subseteq L \\ \#S = s}} \binom{d_S}{t} \geq \binom{n}{s}\binom{\text{average } d_S}{t}$



$$\underset{\substack{S\subseteq L\\ \#S=s}}{\text{avg}}\ d_S = \frac{\text{total\#}}{\binom{n}{s}} \geq \frac{n\cancel{\binom{n}{s}}p^s}{\cancel{\binom{n}{s}}} = np^s$$

Combine: $\#K_{s,t} = \binom{n}{s}\binom{np^s}{t} \sim \frac{n^s}{s!}\frac{(np^s)^t}{t!} = \frac{n^{s+t}p^{st}}{s!\,t!} \sim \binom{n}{s}\binom{n}{t}p^{st}$

When is it ~~even~~ >1?

$$\frac{n^{s+t}p^{st}}{s!\,t!} > 1 \implies p \sim n^{-\frac{s+t}{st}} \quad \text{What's wrong?}$$

Estimate $\binom{n}{s}\binom{np^s}{t} \sim \frac{n^s}{s!}\frac{(np^s)^t}{t!}$ is wrong. requires $np^s \gg 1$ ~~and~~
$\iff p \gg n^{-1/s}$.

$$\boxed{x \gg y \iff \lim_{x,y\to\infty} \frac{x}{y} = \infty}$$

<u>Shown:</u> $p \gg n^{-1/s}$, then $\#K_{s,t} \gtrsim \binom{n}{s}\binom{n}{t}p^{st}$ ← random graph

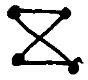
Now deduce $p \geq c_t n^{-1/s}$, then $\exists K_{s,t}$.

Conjecture (SIDORENKO)

For Any bipartite graph H, in any graph G with edge density p,

$\#$ copies of $H \geq \underbrace{\binom{n}{v(H)} p^{e(H)}}_{\text{what it is in } G_{n,p} \text{ random graph.}}$

Other side

$K_{2,2}$: 

want graph with no $K_{2,2}$ yet $\geq n^{3/2}$ edges

Defn: finite projective plane

finite set of points together w/ collection of lines, each a set of points.

- Any two lines cross at exactly one point.
- Every pair of points is on exactly one line.

Eg: 

points $= 7$
lines $= 7$

If q is prime power, can get finite projective plane with q^2+q+1 points.

Live in $\mathbb{F}_q^3$

"point" is 1D subspace of $\mathbb{F}_q^3$ point on line $\Leftrightarrow$ "point" $\subseteq$ "line".
"line" is 2D subspace of $\mathbb{F}_q^3$

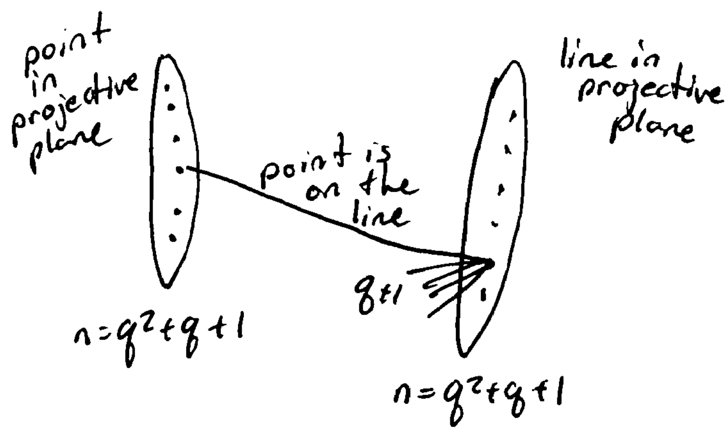
q^3-1 nonzero vectors in $\mathbb{F}_q^3$, Each 1D subspace uses $q-1$ distinct vectors, so there are $\dfrac{q^3-1}{q-1} = q^2+q+1$. such points

number 2D subspaces

Any line contains $\dfrac{q^2-1}{q-1} = q+1$ points.

~~points taken~~ ~~by q-1~~ taken by each 1D subspace inside.

Construction of graph with no $K_{2,2}$ (Incidence graph of projective ~~graph~~ plane.)



if $K_{2,2}$, both points are on two lines $\nrightarrow$, so no $K_{2,2}$.

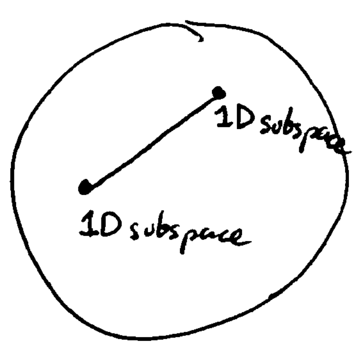
$$\#\text{edges} = \underset{\substack{\uparrow \\ \text{each line} \\ \text{has } q+1 \\ \text{points}}}{(q+1)}\underset{\substack{\uparrow \\ \text{number} \\ \text{of lines.}}}{(q^2+q+1)} \approx n^{3/2}$$ where $n = \#\text{vertices}$

Optimal for bipartite graphs, yet off by factor of 2.

<u>Issue:</u> Yes, the graph was regular, so convexity argument cannot be improved, but lots of pairs have codegree < 1, namely ones on opposite sides.

<u>Construction 2:</u> Polarity graph of projective plane.

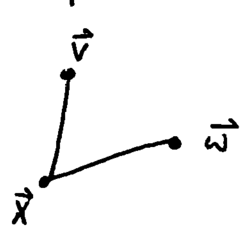
vertex set is 1D subspaces of $\mathbb{F}_q^3$



edge if and only if two subspaces are orthogonal.

<u>Observe:</u> 1D subspace $\langle \vec{v} \rangle$ has a 2D orthogonal complement

No C_4:



$\left.\begin{matrix} v \perp x \\ w \perp x \end{matrix}\right\} \Longrightarrow$ only one dimension left for $\vec{x}$ means there cannot be two $\vec{x}, \vec{y}$ both adjacent to both of $\vec{v}, \vec{w}$.

All degrees are $\dfrac{q^2-1}{q-1} = q+1 - \text{error}$

$\#$ 1D subspace inside 2D subspace

error: self-orthogonal

$\#\text{edges} \approx (q+1)(q^2+q+1)$ not off by factor of 2.

Lemma: If start with $2n$ vertex graph, there is a bisection with at least $\geq 1/2$ # edges.

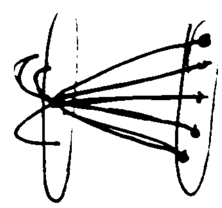
Proof: Either flip a coin to put vertices on left/right, or uniformly sample a random bisection.

$$\mathbb{E}[\#\text{edges in bisection}] = \sum_{e \in E(G)} \mathbb{P}[\text{pick edge } e]$$

Observe: in a bisection of K_{2n}, gets n^2 edges out of $\binom{2n}{2}$ total.

So $\mathbb{E}[\#\text{edges}] = e(G) \dfrac{n^2}{\binom{2n}{2}} = \dfrac{n}{2n-1} > 1/2$. ∎

Lemma: Every graph has a bipartition where each vertex has $\geq 1/2$ of it's neighbors across.



Proof: Start w/ any bipartition, as long as $\exists\, v \in V(G)$ with more neighbors on it's own side, move it to other side.

Then total # crossing edges increases.

Q: Does every graph have a bisection with this property?

No, but counterexamples are off by 1.

Q: $\exists$ Bisection s.t. $\forall v \in V(G)$, #neighbors across $\geq$ #vertices on same side -1

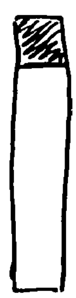
Open!

Theorem: Weighted version of Bollobás-Scott. for regular graphs

Possible to Assign each vertex a weight such that ~~all weights sum to n~~ unnecessary
and (a) sum of weights on left = sum of weights on right
(b) For all v $$\sum_{\substack{\text{neighbors}\\\text{across}}} \text{weight} \geq \sum_{\substack{\text{vertices}\\\text{on same side}}} \text{weights}$$

Proof:

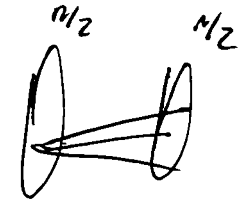
Let A be the adjacency matrix of the Graph.
Associate bipartition w/ vector x of ± 1, $+1$ means right, -1 left.

$A\vec{x} =$  $\longleftarrow \sum_{\substack{\text{vertices}\\\text{adjacent to}\\\text{first one}}} x_i$ = # neighbors on right − # neighbors on left

To be a good partition, ~~want~~ $A\vec{x}$ differs in sigh everywhere from $\vec{x}$.
To ensure sum of weights same on both sides: $\mathbb{1}^T\vec{x} = 0$.
If $\vec{x}$ is eigenvector of A w/ e-val <0, then get $A\vec{x} = \lambda\vec{x}$ and condition (b)
$\lambda < 0$

To ensure negative e-val, note that $\mathbb{1}$ is e-val of $\vec{x}$ with e-val d, since graph is d-regular, $tr(A) = 0$, guarantees $(d + \sum \lambda_i = 0) \implies \exists$ negative e-val. Since A symmetric, guaranteed $\vec{x} \perp \mathbb{1}$.
the entry of $\vec{x}$ tells you which side and the weights. ∎

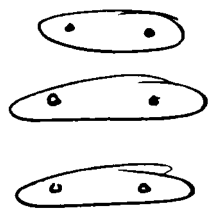
Generalized Bollobás-Scott:



Can you always find a bisection of G s.t. ~~all cross degree~~ for all $v \in V(G)$, cross deg $\geq$ same side deg $- \sqrt{n \log n}$?

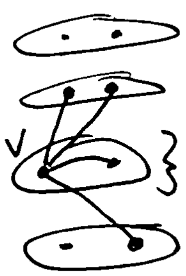
Proof: Randomly.

Arbitrarily pair up vertices. For each pair, ~~flip a coin~~ flip a coin and split either-or $\left\{ \begin{array}{c|c} L & R \\ x & y \\ y & x \end{array} \right.$



For each vertex v, define random variable $X_v =$ (# nbhrs across) $-$ (# nbhrs same side).

Want all $X_v \geq$ $-2\sqrt{N \log N}$



} contribute nothing to X_v

} if edge to partner, always have $+1$
else $+0$

} ± 1 w/ equal probability, and independent.

So $X_v =$ (optional $+1$) $+$ (sum of independent $\pm$ random variables at most n many)

Thm: (HOEFDING) If $Y = \sum_{k=1}^{N} Y_k$, Y_k is sum of ~~N indep~~ N many indep, ± 1 random variables,

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \leq 2e^{-\frac{t^2}{2N}}.$$

So:

$$\mathbb{P}[X < -2\sqrt{n \log n}] \leq 2e^{-2\log n} = \frac{2}{n^2}$$

Since only n-many X_v's, union bound $\Longrightarrow$ this works w/ probability $\to 1$.

Improving Bollobás-Scott:

cross deg $\geq$ same side deg $-(6\sqrt{n})\times 2$

<u>Discrepancy:</u> Given a hypergraph H

Given a coloring χ: vertices of $H \longrightarrow \{\pm 1\}$

define $\mathrm{disc}(H,\chi) = \max\limits_{\text{edges of } H} |\text{sum of } \chi \text{ on edge}|$

$\mathrm{disc}(H) = \min\limits_{\text{colorings } \chi} \mathrm{disc}(H,\chi)$ "best allocation of ± 1 you can do" trying to balance everything.

Use to solve Bollobás-Scott.

Consider the hypergraph H with vertices of H = vertices of G.

for all vertices of G, put neighbors of vertex in same edge in H.

So if $\mathrm{disc}(H)$ was $\leq B$ $\sum B$, to guarantee bisection, add one edge that is $V(G)$.

then there is χ: verts of $H \longrightarrow \pm 1$ s.t. all edges have $|\text{total sum}| \leq B$.

Move verts to balance into a bisection, this changes the balance by:

all edges have $|\text{cross deg} - \text{same deg}| \leq L + \frac{L}{2} 2$

<u>Theorem:</u> (SPENCER) n vertex hypergraph, n edges $\mathrm{disc}(\text{hypergraph}) \leq 6\sqrt{n}$

Gives cross deg $\geq$ same side deg $- 12\sqrt{n}$.

<u>Theorem:</u> For planar graphs, there is a bisection with all edges

\# cross neighbors $\geq$ \# same side neighbors $- 26$

<u>Fact:</u> Every planar graph has vertex deg ≤ 5.

<u>Pf:</u> If deg ≥ 6 $\forall$ verts, $\Rightarrow$ \# edges $\geq \frac{6n}{2} = 3n$

$V - E + F = 2 \Longrightarrow$ \# edges $\leq 3n-6$ $\divideontimes$.

Theorem: If G is planar, $\exists$ bisection such that
$\#$nbhrs across $\geq \#$nbhrs same side $- 26$

Theorem: (Beck-Fiala)
Let H be a hypergraph s.t each vertex is in $\leq t$ edges
(equivalent: all vertices degree $\leq t$)
Then $\mathrm{disc}(H) \leq 2t-1$.

Proof: Assign one variable to each vertex
$x_1, x_2, \ldots, x_n$.

They will store reals $\in [-1,1]$, all start at $x_i = 0$.
Introduce a constraint for each edge by

$$\sum_{i \in \text{edge}} x_i = 0.$$

can ignore 1 vertex edges, since doesn't contribute much to the discrepancy

Each variable is in at most t equations, by degree condition.
Each equation has ≥ 2 variables since we ignored eqns w/ 1 variable.

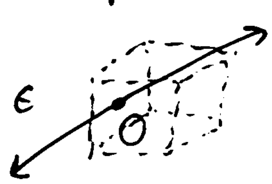
$$\Longrightarrow \quad \#\text{eqns} \leq \left(\frac{t}{2}\right) \#\text{vars}.$$

Actually, we can ignore edges with $\leq t$ vertices, since it doesn't matter, its contribution to discrepancy is $\leq 2t-1$.

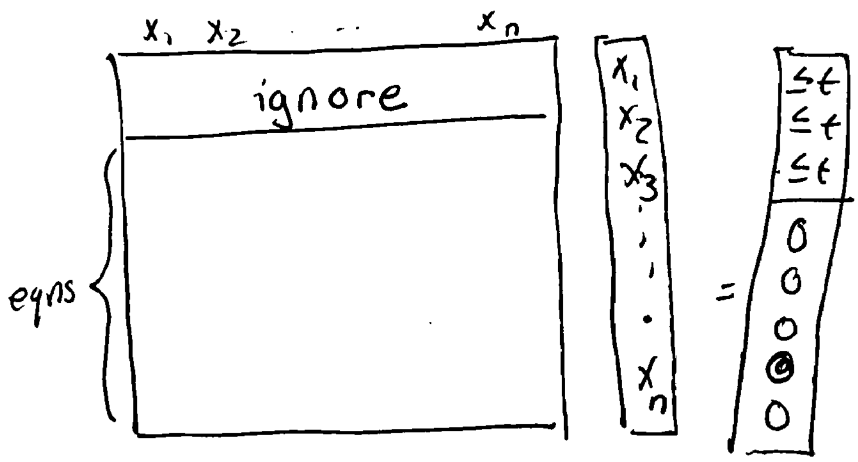
Hence, each eqn has $\geq t+1$ variables, since we ignore smaller ones.

So $\quad \#\text{eqns} \leq \left(\frac{t}{t+1}\right) \#\text{vars}. \Longrightarrow \infty$ # of solutions

Focus on subspace of solutions in 1-dim subspace

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in$ 

Pick spot where subspace hits $\partial[-1,1]^n$?



not all zero, at least one is ± 1

When a variable becomes ± 1, get another system with fewer variables, and freeze the variable at ± 1.

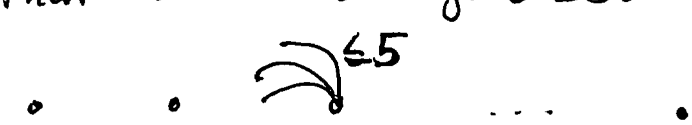
If an equation has $\le t$ active variables, ignore it. $x_1+x_2+1+x_7+x_9=0$

For rest of process, can't guarantee it will be $=0$, but X's will only be varying in $[-1,1]$ during that time.

In the end $\sum x_i$ moves from zero to at most a distance $< 2t$, so $2t-1$ from 0.

Proof (of # nbhr across $\ge$ # same side -26)

Since the planar graphs are 5-degenerate, can order vertices so that backwards degree ≤ 5.



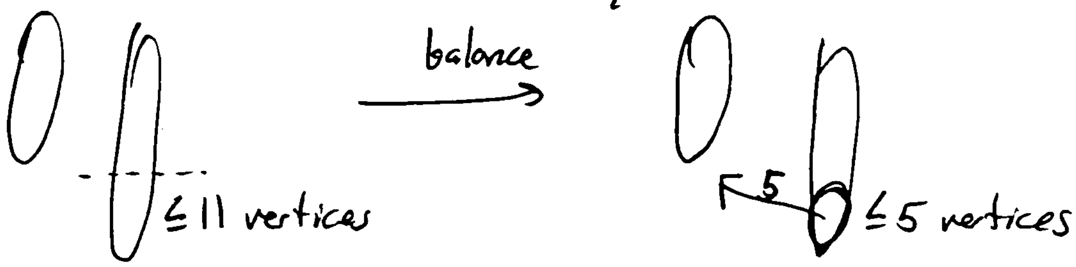
Let H be the hypergraph have one edge for each vertex: take all forward neighbors
+ 1 edge w/ all vertices.

Each vertex is in ≤ 6 edges.

Beck-Fiala $\Longrightarrow$ $\mathrm{disc}(H) \le 11$

So there is an assignment of ± 1 to vertices such that $\left|\sum_{\text{edge}} \chi(v_i)\right| \le 11$.

So the nbhd of v is unbalanced by at most 16.

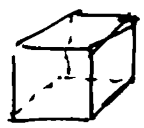


$\Longrightarrow$ ultimate inbalance $\leq 16 + 2(5) = 26$. ∎

Erdős Unit-Distance Problem:

Prove: Any n points in the plane, # pairs of points at distance $= 1$ from each other is $\leq n^{1+o(1)}$.

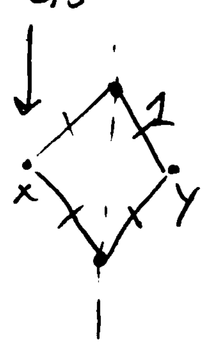
Construction: $\Delta\ \Delta\ \Delta \cdots \Delta$ n pairs



2D projection of hypercube
all degrees $\sim \log_2(n)$
edges = # unit distance pairs is $\frac{n \log_2(n)}{2}$

Upper Bound:
$\leq cn^{3/2}$.

Proof: Consider unit-distance graph
Has edge when distance $= 1$.
Show has #edges $\leq cn^{3/2}$ by Kővari-Sós-Turán, because
Can have $K_{2,2}$, but no $K_{2,3}$



only 2 points distance 1 from both x and y.

$\boxed{\text{Known} \leq cn^{4/3} \text{ (1984)}}$

<u>Question</u>: Can you decompose edges of K_n into K_t's.

Any n, $t=2$ possible

<u>Need</u>: $\binom{t}{2} \mid \binom{n}{2}$

$t \leq n/2$ or $t=n$ $\quad (t-1) \mid (n-1)$

<u>Theorem (Wilson)</u>: Given any graph H, $\exists N$ s.t. as long as $n \geq N$, and

(1) edges in $H \mid \binom{n}{2}$

(2) $\gcd(\text{degrees of } H) \mid (n-1)$

K_n can be decomposed into H's.

<u>Prop</u>: With particular $t \sim \sqrt{n}$, can decompose K_n's edes into K_t's.

<u>Proof</u>: Divide n vertices into $\approx t$ $\underbrace{\text{groups}}_{K_t}$, each with $=t$ vertices

Vertices are points in some projective plane with q^2+q+1 points.

K_t is line in that plane, all of its points as a clique.

Projective plane is $\mathbb{F}_q^3$ where points are 1D subspaces.

lines are 2D subspaces.

<u>Another Construction</u>:

Affine plane. A lattice of $\mathbb{F}_q^2$

Lines $\begin{cases} y=ax+b: & a,b \in \mathbb{F}_q \\ x=c \end{cases}$

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

q^2 points

q^2+q lines, each w/ q points

$n=q^2$, decompose K_n into q^2+q-many K_q's.

<u>Check</u>: $\binom{q^2}{2} = q(q+1)\binom{q}{2}$. ✓

$ex(n, C_{2t})$

<u>Theorem</u> (Bondy-Simonovits)

$$ex(n, C_{2t}) \le c_t n^{1+1/t}$$

<u>Know</u>: $ex(n, C_{2(2)}) \sim c n^{3/2}$

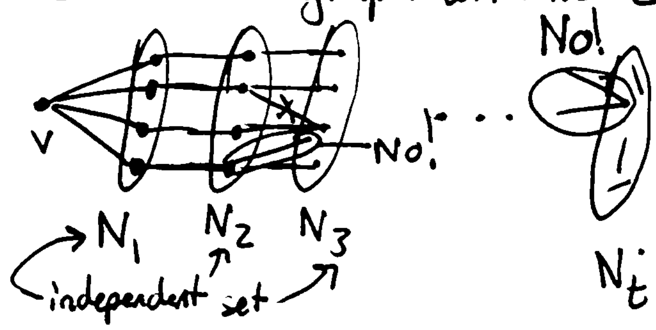
$ex(n, C_6) \sim c n^{1+1/3}$

$ex(n, C_{10}) \sim c n^{1+1/5}$

<u>Prop</u>: $ex(n, \{C_{2t}, C_{2t-1}, \dots C_3\}) \le c_t n^{1+1/t}$

"max number of edges with girth $> 2t$"

<u>Proof</u>: Consider a graph with no $C_3, \dots, C_{2t}$.

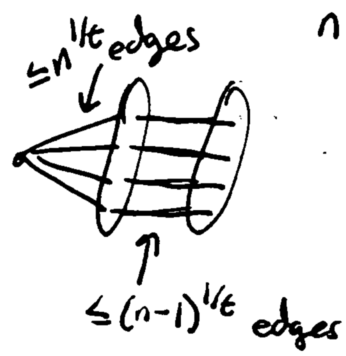


Out to distance t, is a tree except at distance t

Suppose no $C_3, \dots, C_{2t}$ and min-degree $= \delta$.

Then $n \ge 1 + \delta + \delta(\delta-1) + \dots + \delta(\delta-1)^{t-1}$

$n \gtrsim \delta^t \implies \delta \lesssim n^{1/t}$



Pull off vertices one at a time: total edges lost $\le n^{1/t} + (n-1)^{1/t} + \dots + 1^{1/t}$

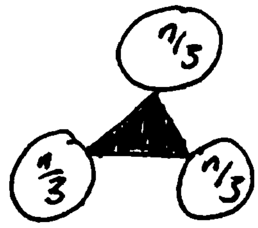
$\le n\, n^{1/t} = n^{1+1/t}$. ∎

Turán Hypergraph

$$ex(n, \boxtimes) = ?$$
$$\boxtimes = K_4^{(3)}$$

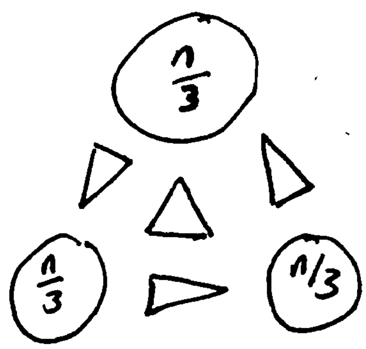
A construction:

3-partite



$\left(\frac{n}{3}\right)^3$ edges

Can do better: (but not unique)



edges $\left(\frac{n}{3}\right)^3 + 3\left(\frac{n}{3}\right)\binom{n/3}{2}$

$$= n^3\left(\frac{1}{27} + 3\,\frac{1}{3}\,\frac{1}{3\cdot 3\cdot 2}\right)$$

$$= n^3\left(\frac{1}{27} + \frac{1}{18}\right) = n^3\left(\frac{5}{54}\right) = \left(\frac{n^3}{6}\right)\left(\frac{5}{9}\right)$$

$$\sim \frac{5}{9}\binom{n}{3}$$

This construction has 5/9 density.

Chung (and Lo?): $ex(n, K_4^{(3)}) \leq \binom{n}{3}(0.6n)$

Rasborov: $\leq \binom{n}{3}(0.565\cdots)$

Turned it into semidefinite program.

Question: R-uniform Hypergraph on n vertices, average degree d.

Show there is always an $\underbrace{\text{independent set}}_{\text{no full edge inside}}$ of size of at least $C(r)\,n/d$

Construction: no big independent sets

Random: n-vertices, $\frac{nd}{r}$ edges

$$\text{edge probability} \sim \frac{\left(\frac{nd}{r}\right)}{\binom{n}{r}} \sim \frac{\left(\frac{nd}{r}\right)}{\left(\frac{n^r}{r!}\right)} \sim \frac{(r-1)!\,d}{n^{r-1}} = p$$

Show that this construction has all independent sets of size $\leq T$.

Union Bound: $\binom{n}{T}$ ways to pick a set of T vertices

And such a set is independent with probability

$$(1-p)^{\binom{T}{r}} \underset{1+x\leq e^x}{\leq} e^{-p\binom{T}{r}} = e^{-\frac{(r-1)!\,d}{n^{r-1}}\frac{T^r}{r!}} = e^{-\frac{dT^r}{rn^{r-1}}}$$

$$\binom{n}{T} e^{\frac{-dT^r}{rn^{r-1}}} \quad \text{choose } T \text{ so that } < 1.$$

$$\leq \left(\frac{en}{T}\right)^T e^{\frac{-dT^r}{rn^{r-1}}} = \left[\frac{en}{T} e^{\frac{-dT^{r-1}}{rn^{r-1}}}\right]^T \Longrightarrow T > \frac{n\left(r^{\frac{1}{r-1}}\right)(\log n)^{1/r-1}}{d^{1/r-1}}$$

So random construction has no independent sets bigger that

$$(\text{constant}) \cdot \frac{n}{d^{1/r-1}} (\log n)^{1/r-1}$$

lower bound: Randomly order vertices, pick vertex that isn't last in any of its ~~hypergraph~~ hyperedges. Hard to analyze.

Pick vertex that is first in <u>all</u> of its hyperedges.

$$\mathbb{P}[\text{pick vertex } v] \geq \frac{1}{\#\{v + \text{vertices nbhrs}\}} \geq \frac{1}{1+(r-1)d_v}$$

$$\mathbb{E}[\text{vertices picked}] \geq \sum \frac{1}{1+(r-1)d_v} \geq n\, \frac{1}{1+(r-1)d}$$

r=2 tight construction

$(d+1)$ $(d+1)$ $\qquad \bar{d} = d_v = d \;\; \forall v \in V(G)$

Independent set is $\frac{n}{d+1}$.

<u>Theorem:</u> There is always an independent set of size $\geq C(r) \frac{n}{d^{1/r-1}}$

<u>Proof:</u> Pick vertices w/ probability p, use alterations to throw out one vertex per edge that survives.

Extremal set theory.

<u>Question:</u> (Littlewood-Offord)

Studying random sums of n Real numbers, each multiplied by ± 1 with equal probability, independently.

each ≥ 1 in absolute value

We have concentration inequalities, w/ good probability of being close to the mean.

<u>Show:</u> $\forall z \in \mathbb{R}$, $\mathbb{P}[\text{random sum} \in (z-1, z+1)] \longrightarrow 0$ as $n \to \infty$

<u>Thm</u> (Littlewood, Offord): True

What if all same, all $= \pm 1$? $\sum_{i=0}^{n} \pm 1$

Essentially a binomial random variable.

$$\mathbb{P}[\text{random sum in } (z-1, z+1)] \leq \text{const.} \ 1/\sqrt{n}$$

<u>Thm:</u> (Katona - Kleitman)

<u>Thm:</u> (Sperner): $\mathcal{F}$ of subsets of $\{1, 2, \dots, n\}$ is an <u>antichain</u> if there is no $A \in \mathcal{F}$, $B \in \mathcal{F}$, $A \neq B$, $A \subseteq B$ or $B \subseteq A$.

Then if $\mathcal{F}$ is an antichain, $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

<u>Proof:</u> Show $\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1$. Consider random permutation of $\{1, 2, \dots, n\}$

For each A, E_A = first $|A|$ many elts of permutation $= A$

$$\sum_{A \in \mathcal{F}} \mathbb{P}[E_A] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1$$

Since E_B, E_A mutually exclusive in Antichain, so

We're interested in real roots of random polynomials w/ ± 1 coefficients:

$$P(x) = C_n z^n + C_{n-1} z^{n-1} + \dots + C_0.$$

Question: How many real roots can we expect?

Prop: With high probability, # real roots $\lesssim O(\sqrt{n})$

Proof: Descartes' rule of signs:

number of positive real roots $\leq$ # sign changes.

$$\mathbb{E}[\text{\# of sign changes}] = n \cdot 1/2 = n/2$$

number of negative real roots $\leq$ # sign changes in $P(-z)$

$$\mathbb{E}[\text{\# of sign changes in } P(-z)] = n/2$$

$$\mathbb{E}[\text{\# real roots}] = \mathbb{E}[\text{real roots} > 0] + \mathbb{E}[\text{\# roots} < 0] = n \quad !!$$

Let $Q(z) = P(z)(1 + z + z^2 + z^3 + \dots + z^n)$

pos $\mathbb{R}$ roots of P = # pos $\mathbb{R}$ roots of Q since $z > 0 \Rightarrow \sum z^i \neq 0$

Descartes rule of signs on Q.

$$Q(z) = C_0 + z(C_0 + C_1) + z^2(C_0 + C_1 + C_2) + \dots + z^n(C_0 + C_1 + \dots + C_n) + z^{n+1}(C_1 + \dots + C_n) + \dots + z^{2n} C_n$$

$$Q(z) = \underbrace{\sum_{i=0}^{n} \left(z^i \sum_{j=0}^{i} c_j \right)}_{\substack{\mathbb{E}[\text{\# sign change here}] \\ \sim \sqrt{n}}} + \underbrace{\sum_{i=n+1}^{2n} \left(z^i \sum_{j=i-n}^{n} c_j \right)}_{\substack{\mathbb{E}[\text{\# sign change here}] \\ \sim \sqrt{n}}}$$

Gives $4\sqrt{n}$.

Theorem: (SAUER-SHELAH)

Defn: If $\mathcal{F}$ is a family of subsets of A, then S is shattered by $\mathcal{F}$ if every subset $T \subseteq S$ appears as $T = F \cap S$ where $F \in \mathcal{F}$.

Defn: VC-dimension of $\mathcal{F}$ is the size of the largest set shattered by $\mathcal{F}$

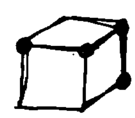
Theorem (SAUER-SHELAH): If $\mathcal{F}$ has $\geq \sum_{i=0}^{k-1} \binom{n}{i} + 1$ sets, then there is a shattered set of size at least k. ($\mathcal{F}$ is subsets of $\{1,2,3,\dots,n\}$).

Construction? $\mathcal{F}$ = all subsets of size $\leq k-1$.

Hence $|\mathcal{F}| = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k-1}$

Proof: Represent family of subsets nicely

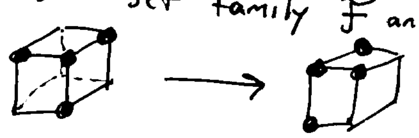
1. Binary vectors of length n
2. Hypercube



$\{1,3\}$ is shattered iff every vector $\perp$ $\{1,3\}$-plane is occupied. ← has a vector in the vertex

Defn: Compression in dimension i:

Takes a set family $\mathcal{F}$ and produces another set family $\mathcal{F}'$ with same # sets.



$S \in \mathcal{F} \longmapsto S \setminus \{1\}$ if $1 \in \{S\}$ and $S \setminus \{1\}$ wasn't already there.

Clear that compression preserves $\#\mathcal{F}$.

Need: if S is shattered by $\mathcal{F}' \Longrightarrow S$ shattered by $\mathcal{F}$. ✓

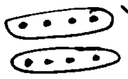
Can compress sequentially in all directions until nothing moves anymore, get $\mathcal{F}^*$

In particular, if $S \in \mathcal{F}^*$, all subsets of S are also in $\mathcal{F}^*$.

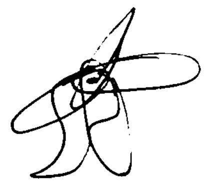
$|\mathcal{F}^*| >$ # of sets of size $0,\dots,k-1$

there is a set of size k in $\mathcal{F}^*$, which means all subsets of it are in $\mathcal{F}^*$.

■

ex(n, ) = max # hyperedges in r-uniform hypergraph with <u>no</u> non-intersecting edges.

<u>Construction:</u>



Pick one vertex, take all edges to have this vertex. $\binom{n-1}{r-1}$ ways

<u>Theorem:</u> (Erdős-Ko-Rado): If have a collection of sets of size $= r$, all in $\{1,2,3,\ldots,n\}$, then max # of sets can have is $\binom{n-1}{r-1}$. w/ nontrivial intersection.

Unless $r > n/2$, then can take $\binom{n}{r}$ many

<u>Proof:</u> Let $\mathcal{F}$ be an intersecting family of r-sets from $\{1,2,\ldots,n\}$ and $r \le n/2$.

Suffices to prove

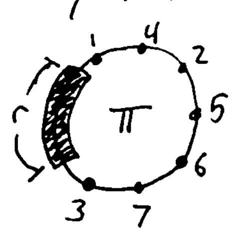
it sample an r-set from all $\binom{n}{r}$ at random, then

$$\mathbb{P}[\text{selected } r\text{-set} \in \mathcal{F}] \le \frac{\binom{n-1}{r-1}}{\binom{n}{r}} = r/n .$$

Sample the r-set by:

(1) choose random cyclic permutation of $\{1,2,\ldots,n\}$, call it π.

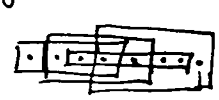
(2) pick uniformly random contiguous block of length $= r$.



(3) Bound $\mathbb{P}[\text{selected set} \in \mathcal{F} \mid \pi] \le r/n$

What is max # contiguous blocks in intersecting family?

To get maximal intersecting, take r, with overlap $r-1$.



(4) True for any π cyclic, so therefore $\mathbb{P}[\text{selected } r\text{-set} \in \mathcal{F}] \le r/n$ ∎

"Katona Cycle Trick"

Theorem (Bollobás): A (k,ℓ)-system is a family of pairs (A_i, B_i) where ~~$\#A_i = k$ for~~

(1) $|A_i| = k$ for all i

(2) $|B_i| = \ell$ for all i

(3) $A_i \cap B_i = \emptyset$ for all i

(4) $A_i \cap B_j \neq \emptyset$ for all i, j

Then ~~the~~ biggest (k,ℓ)-system (most # of i's) is of size

Construction:

If $k=\ell$, use $n=2k$ and each (A_i, B_i) is a partition of $\{1,2,\dots n\}$ into a k-set and compliment.

Get $\binom{2k}{k}$.

If $k\neq\ell$, do the same thing $\longrightarrow$ $\binom{k+\ell}{k}$ by taking both (A_i, B_i) and (B_i, A_i).

Proof: Ground set has size n. Let m be size of ~~family~~ system.

Suffices to prove for any finite n.

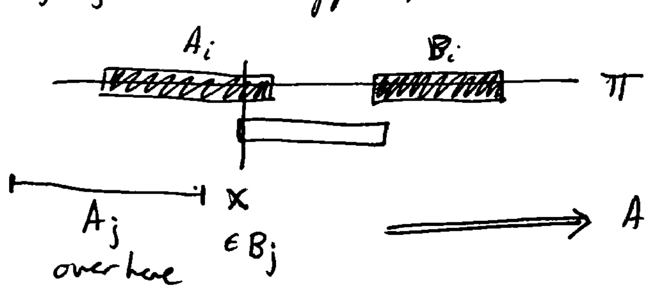
Generate uniformly random $\pi \in S_n$.

Let E_i = event that all of A_i elements precede B_i elements in π.

$$\mathbb{P}[E_i] = \frac{k!\,\ell!}{(k+\ell)!} = \frac{1}{\binom{k+\ell}{k}}$$

Observation: all E_i are mutually exclusive

pf: If E_i, E_j both happen.



$\Longrightarrow$ A_j cannot meet B_i ※.

So $\mathbb{P}[E_i \text{ happens for any } i] = \sum_{i=1}^{m} \mathbb{P}[E_i] = \frac{m}{\binom{k+\ell}{k}}$ but probability ≤ 1

So $m \le \binom{k+\ell}{k}$. ∎

Defn: Let F be a graph. Another graph is called *F-saturated* if G doesn't contain F as a subgraph and for any missing edge of G, adding it creates a copy of F.

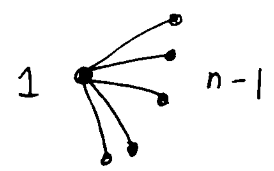
$ex(n,F)$ = max # of edges in any F-saturated graph.

Defn: *Saturation number* of F

$$sat(n,F) = \text{min \# of edges in any } F\text{-saturated graph.}$$

Eg: $Sat(n,K_3)$ with few edges, yet saturated

Star:

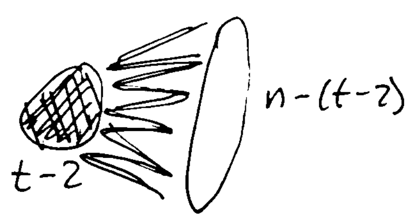


has $n-1$ edges $\qquad sat(n,K_3) = n-1$

Since $\leq n-2$ edges $\Rightarrow$ not connected, we cannot be K_3-saturated.

What about $sat(n,K_t)$?

Construction:



gives

$$sat(n,K_t) \leq \binom{n}{2} - \binom{n-t+2}{2}$$

~~Theorem (Erdős)~~

Theorem (Erdős-Hajnal-Moon): this is the best

Proof: Show if K_t saturated graph, then # missing edges is $\leq \binom{n-t+2}{2}$

For each missing edge, let B_i = set of two vertices endpoints.

let A_i = set of $n-t$ vertices not in K_t created if edge added.

Gives a $(n-t,2)$ system, as in Bollobás thm.

$\Rightarrow$ # missing edges $\leq \binom{n-t+2}{2}$.

Question: 6 buckets, each with a penny

~~Move by taking a ben~~ move by taking a penny out, add 2 to bin on right.

How many can you get?

$\boxed{2^6 - 1}$

IMO: 2010 #5: 6 bins, two moves.

Take one out, +2 to bin on right

Take one out, swap next two bins on right

~~Ex: 1 1 1 1 1 1~~ Eg: $1\underset{A}{1}1111 \mapsto 10\underset{A}{3}111$

$\downarrow$

$102\underset{AAA}{3}11 \longmapsto 10\underset{B}{2}071 \mapsto 101701$

How much can you make?

At most finite, because we can impose lexicographic ordering on $(\mathbb{Z}^+)^6$.

Simple Cases:

(1) only one bin, x coins worth x.

(2) two bins $x, 0$ worth $2x$.

(3) three bins $x, 0, 0$ can get to $0, 2^x, 0$ worth at least 2^{x+1}

$x, 0, 0$

$x-1, 2, 0$

$x-1, 0, 4$

$x-2, 4, 0$

$x-2, 0, 8$

$\downarrow$

$x-3, 8, 0$

$x-3, 0, 16$

$x-4, 16, 0$

$x-x, 2^x, 0$

(4) four bins $x, 0, 0, 0$

Can get $0, 2^x, 0, 0$ by applying (3)

$0\ 0\ 2^{2^x}\ 0$

But we can do better!

$0 \quad \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\} x \text{ times!} \quad 0 \quad 0$

Tower Function $T(x)$

$T(4) = 2^{16} = 65536 \qquad T(5) = 2^{65536}$.

Szemerédi Regularity Lemma:

Approximate graphs by nicer graphs.

What do we mean by nicer?

Study $G(n,p)$ random graph # n nodes, each edge w/ probability p.

OBS: $\mathbb{E}[\text{degree of vertex}] = (n-1)p \approx np$

$\forall \varepsilon > 0$, with high probability $(1-o(1))$, all degrees in $G_{n,p}$ are $(1 \pm \varepsilon)np$
as long as $p > \frac{C \log n}{n}$ for some C.

Proof: Union bound over n vertices.

expected

For each vertex v, $\mathbb{P}[\deg v \text{ outside } (1 \pm \varepsilon) np] \underset{\text{Chernoff}}{\leq} e^{-C_\varepsilon \mathbb{E}} \leq e^{-C(\varepsilon)(np)}$

$\leq e^{-C_\varepsilon C \log(n)}$

$\leq 1/n^2$ by choosing C large enough.

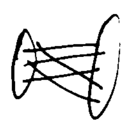
tends to 1 with n

OBS: In $G_{n,p}$, with high probability, every pair of disjoint sets of size $\geq C\sqrt{n}$ has #edges between them $= (1 \pm \varepsilon)|A||B|p$

A, B $\quad$ p, ε are constants, determine C.

Proof: Union bound over all ways to choose sets A, B.

#ways to choose $(A,B) \leq 2^n \times 2^n$.

For fixed A  B, # edges is $Bin(|A||B|, p)$

$\mathbb{P}[\text{this number off by more than } (1 \pm \varepsilon)|A||B|p] \leq e^{-C_\varepsilon |A||B| p} \leq e^{-C_\varepsilon C^2 n p}$

$\leq e^{-2n}$

by choosing C, can make small.

Defn: Pair of disjoint sets A, B is called ε-regular iff
for any $A' \subseteq A$, $B' \subseteq B$, and $|A'| \geq \varepsilon |A|$, $|B'| \geq \varepsilon |B|$
#edges between A', B' is $(1 \pm \varepsilon)$ WISB

WISB = what it should be $= (1 \pm \varepsilon)|A'||B'|p$

equivalently, $|d(A', B') - d(A,B)| < \varepsilon$

where $d(A', B') = \frac{\#\text{edges } A' \leftrightarrow B'}{|A||B|}$

edge probability $= \frac{\#\text{edges } A \leftrightarrow B}{|A||B|}$.

More Szemeredi Regularity Lemma:

ε-regular pair $\quad \forall\, A' \subseteq A,\ B' \subseteq B$ with $|A'| \geq \varepsilon |A|$, $|B'| \geq \varepsilon |B|$ $\quad$ edge density $A' \to B' \underset{\pm\varepsilon}{\approx}$ edge density $A \to B$ $\quad \dfrac{\#\text{edges } A \text{ to } B}{|A|\,|B|}$

Observation: In random graph, all pairs are ε-regular.

(degree is degree to B, not total degree)

Lemma: $\left(\#\text{ vertices in } A \text{ with degree } < (d-\varepsilon)|B|\right) < \varepsilon |A|$ $\quad d =$ density of edges $A \to B$.

equivalently $\left(\text{fraction of vertices in } A \text{ with degree to } B < (d-\varepsilon)\right) < \varepsilon$.

Proof: Suppose not. There is a subset of vertices in A, each of which have density to B less than $d-\varepsilon$. Put all such low degree vertices in a set A'.

By assumption for contradiction $|A'| \geq \varepsilon |A|$. Let $B' = B$. Clearly $|B'| > \varepsilon |B|$

Condition $\Longrightarrow d(A', B') \geq d - \varepsilon$

Violated because each vertex of A' has density $< (d-\varepsilon)$ to B and $d(A', B') =$ average of individual densities.

Lemma 1′: For any B' of size $\geq \varepsilon |B|$, then (fraction of vertices in A w/ ~~degree~~ density to $B' < (d-\varepsilon)$) $< \varepsilon$ ■

Lemma 2: (fraction of vertices in A with density $> \varepsilon + d$) $< \varepsilon$

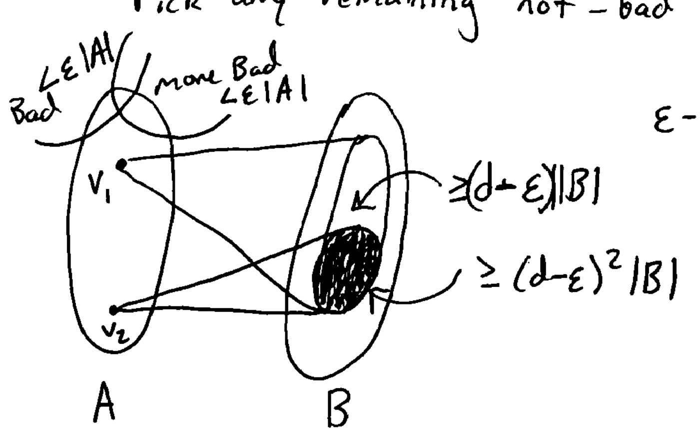
Corollary: (fraction of vertices in A with density outside $d \pm \varepsilon$) $< 2\varepsilon$

Lemma: $\# K_{2,2}$ is $\geq \left((1-\varepsilon)|A|\right)^2 \left((d-\varepsilon)^2 |B|\right)^2$ long as $d \geq \varepsilon \cdot 2$

Proof: By Lemma 1, $\geq (1-\varepsilon)$ fraction of vertices in A have density $\geq (d+\varepsilon)$ to B

Pick one, call it v_1. Apply Lemma 1′ using nbhd of v_1 in B as B'. Find that $< \varepsilon |A|$ vertices in A $< \varepsilon |A|$ vertices in A with density $< d - \varepsilon$ to B'.

Pick any remaining not-bad vertex v_2 in A (avoid those w/ density $< d-\varepsilon$ to B').



ε-regular A, B, density d.

If $d \gg \varepsilon$, then

$\#$ways $\geq$ ~~[illegible]~~

$\geq |A|^2 |B|^2 d^4 (1 - \varepsilon^*)$

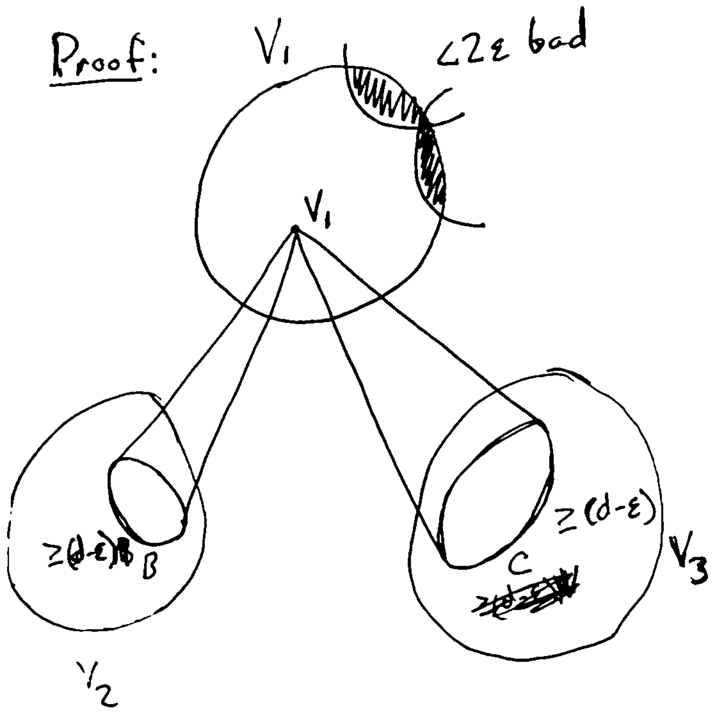
where $\varepsilon^* \approx 2\varepsilon + 4\varepsilon/d$

what you expect in a random graph

Lemma: Given some d, ε and three sets of size n, and all 3 pairs have density d and are ε-regular. Assume $d \geq 2\varepsilon$

Then $\#\Delta \geq n^3 d^3 (1-\varepsilon^*)$, $\varepsilon^* \approx 2\varepsilon + 3\varepsilon/d$.

Proof:



Lemma 1 $\Longrightarrow$ $<\varepsilon$-fraction of V_1 has density $< d-\varepsilon$ into V_2

Lemma 1 $\Longrightarrow$ $<\varepsilon$-fraction of V_1 has density $< d-\varepsilon$ into V_3

$\boxed{\text{Pick any vertex}}$ not bad in V_1, get a K_3 $\boxed{\text{for each edge}}$ B to C.

Note (V_2, V_3) is ε-regular, so so long as

$$\left.\begin{array}{l} |B| > \varepsilon |V_2| \\ |C| > \varepsilon |V_3| \end{array}\right\} \Longrightarrow \text{density}(B,C) \underset{\pm\varepsilon}{\approx} d(V_2, V_3)$$

$$\Longrightarrow \#\text{edges}(B,C) \geq (d-\varepsilon)|B||C|.$$

$\#$ways to get K_3 is

$$\geq ((1-2\varepsilon)n)\,\Big((d-\varepsilon)\underbrace{((d-\varepsilon)n)}_{|B|}\underbrace{((d-\varepsilon)n)}_{|C|}\Big)$$

$\geq n^3 d^3 (1-\varepsilon^*)$ with $\varepsilon^* \approx 2\varepsilon + 3\varepsilon/d$, so $(1-\varepsilon^*)$ is $(1-o(1))$ when $\varepsilon \ll d$. ∎

Same method lets you find right number of copies of any fixed size graph, in a picture as long as you could actually make that picture.

3/21/14

Theorem (Szemeredi Regularity Lemma): For any $\varepsilon > 0$, there is an $M < \infty$, such that for any graph G, the vertices of G can be partitioned into vertex sets $V_1, V_2, \dots, V_t$ such that

$$\left.\begin{array}{l} \text{① } |V_i| \text{ are the same up to } \pm 1 \\ \text{② All but } \leq \varepsilon t^2 \text{ of pairs } (V_i, V_j) \text{ are } \varepsilon\text{-regular.} \\ \text{③ } \frac{1}{\varepsilon} < t < M \end{array}\right\} \text{"Regularity Partition"}$$

Proof gives $M = \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\}\frac{1}{\varepsilon^5}$ or something. Mostly used for asymptotics

Given a regularity partition, summarize by $t \times t$ symmetric matrix with (i,j)-entry density(V_i, V_j) or "?" if (V_i, V_j) not ε-regular.

<u>Theorem (Gowers)</u>: There are graphs G that require $\geq \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\} 1/\varepsilon$ many parts for the decomposition.

<u>Theorem: (Triangle Removal Lemma)</u>: $\forall \varepsilon^* > 0\ \exists \delta^* > 0$ such that (in any graph G that requires $\geq \varepsilon^* n^2$ edge deletions to clear all triangles, the number of triangles is $\geq \delta^* n^3$.) $\Longleftrightarrow$ (if number of Δ's $< \delta^* n^3$, then can destroy all of them by deleting only $< \varepsilon^* n^2$-many edges.)

Proof will give $\delta^* \approx 1 / \left.2^{2^{\cdot^{\cdot^{2}}}}\right\} 1/\varepsilon^{*5}$.

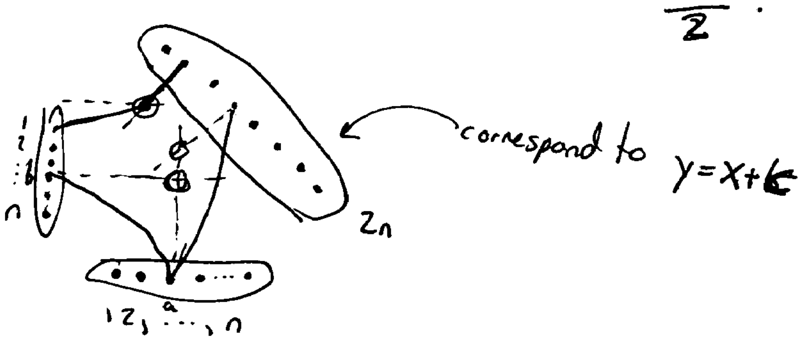
<u>Theorem (Roth)</u>: $\forall \varepsilon > 0\ \exists N$ so that any subset of $\{1, 2, \ldots, n\}$ of density $\geq \varepsilon$ contains 3-term arithmetic progresssion, when $n \geq N$.

<u>Proof:</u> (Due to Solymosi).

Suppose we have a subset A of density $\geq \varepsilon$. Take the $n \times n$ grid $[1, n]^2 \cap \mathbb{Z}^2$. Mark points for which $x + y \in A$.

Will circle entire diagonals, # circle points is $\geq \frac{\varepsilon^2 n^2}{2}$.

Create a graph



Draw edge if point (a, b) is circled.

Draw edge if line $x = a$, $y = x + c$ has intersection, and intersection is circled.

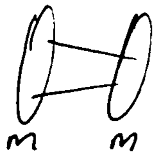
Each circle gives Δ, which has at most one edge in common w/ another.

Edge disjoint Δ's, so need $\geq \frac{\varepsilon^2 n^2}{8}$ edge deletions, to remove Δs.

Apply Triangle removal $\Longrightarrow$ #Δs is $\frac{1}{T(\varepsilon)} N^3$ where $N = n/4$.

3/26/14

<u>Lemma:</u> $\forall \varepsilon > 0$, if $\delta \geq 2\varepsilon$ in any ε-reg



$m \quad m$

density $\geq \delta$

there is a

.

<u>Proof:</u> (1) <u>Pick</u> vertex on left with density $\geq \delta - \varepsilon$

$\uparrow$ # <u>choices</u> $\geq (1-\varepsilon)m$.

(2) <u>Pick</u> vertex on left that $\begin{cases} \text{sees} \geq (\delta-\varepsilon)^2 m \text{ vertices in nbhd of } 1 \\ \text{and} \geq (\delta-\varepsilon)m \text{ vertices on right in total} \end{cases}$

$\nwarrow$ # <u>choices</u> $\geq (1-2\varepsilon)m$

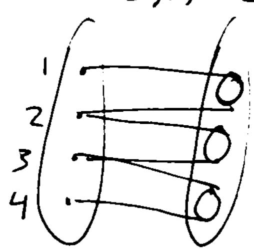
(3) <u>Pick</u> vertex on left that $\begin{cases} \text{sees} \geq \text{～～～} \text{ in nbhd of } 2 \\ \text{and} \geq \text{～～～} \text{ on right in total} \end{cases}$

$\nwarrow$ # <u>choices</u> $\geq (1-2\varepsilon)m - 1$

(4) <u>Pick</u> vertex on left that sees $\geq (\delta-\varepsilon)^2 m$ vertices in nbhd of (3)

$\uparrow$ # <u>choices</u> $\geq (1-\varepsilon)m - 2$

gives picture



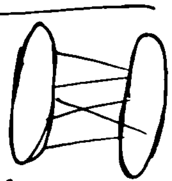
<u>Pick</u> one vertex in each common nbhd on right.

choices $\geq \cancel{[(\delta-\varepsilon)^2 m]^3}$.

$((\delta-\varepsilon)^2 m)((\delta-\varepsilon)^2 m - 1)((\delta-\varepsilon)^2 m - 2)$

■

<u>Stronger Lemma:</u> $\forall \varepsilon > 0$, if $\delta \gg \varepsilon$, then in the graph

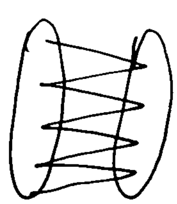


$m \quad m$

density $\geq \delta$

ε-regular

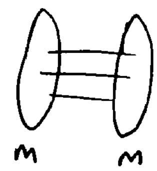
there is



where each side has $\geq (\delta-\varepsilon)^2 m - 1$ vertices. And as long as $\delta \gg \varepsilon$, then this is $\geq \frac{1}{2}\delta^2 m$.

~~Lemma: $\forall \varepsilon > 0, \forall H$, if $\delta \gg \varepsilon$, then in given the graph~~

Lemma: $\forall \delta > 0$, $\forall \Delta \in \mathbb{Z}^+$, $\exists \varepsilon > 0$ such that if H is a bipartite graph with maximum degree $\leq \Delta$ and each part of H has $\leq \frac{1}{2}\delta^{\Delta} m$ vertices then H is in any



ε-regular density $\geq \delta$

size m.

Lemma: For any $\delta \geq 0$, $\forall \Delta \in \mathbb{Z}^+$, $\exists \varepsilon > 0$: so that whenever $H \subseteq$ "filled up regularity graph w/ parameters $\varepsilon, \delta, \frac{\delta}{2}$"

and H as max degree $\leq \Delta$, then $H \subseteq$ "not filled up regularity graph with parameters ε, δ."

H has $\leq \frac{1}{2}\delta^{\Delta} m$ vertices

graph w/ all deletions inside each part, between not ε-regular parts, between parts with density $\leq \delta$

graph has same things as not-filled up version, but replace each ε-regular, density δ by complete bipartite graph.

Ramsey numbers of bounded degree graphs:

$R(H)$ = min # vertices in a graph s.t. any 2-coloring of edges of K_n, $\exists$ there is a monochromatic H.

Thm: $R(H) \leq 4^{\#\text{vertices of } H}$

Q: exponential always?

Theorem (CHVÁTAL, RÖDL, SZEMERÉDI–TROTTER)

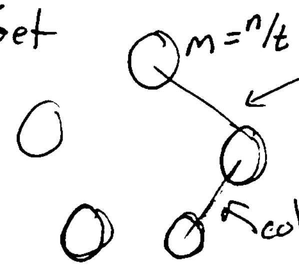
$\forall \Delta, \exists$ constant C so that $\forall H$ with max degree Δ, $R(H) \le C \,\#\text{vtxs}(H)$.

Proof: Given 2-coloring of edges of K_n.

Let G = n vertex graph w/ edges from first color.

Apply Szemeredi to G with $\varepsilon = \frac{1}{4^{\Delta+1}}$

Get



density $> 1/2$ color 1
$\le 1/2$ color 2

color all edges one or other

use $t \ge 4^{\Delta+1}$

By Ramsey's Theorem on Supergraph, $\exists$ monochromatic $K_{\Delta+1}$; $\Delta + 1 \ge \chi(H)$

... ?

03/31/14

Theorem (Szemeredi Regularity Lemma). $\forall \varepsilon > 0, \exists M$ such that every graph has a partition into $V_0, V_1, \ldots, V_t$ with properties:

"rubbish" $\rightarrow V_0$

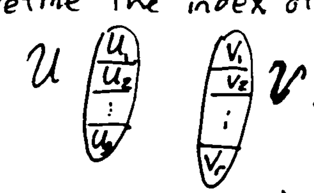
(0) $|V_0| \le \varepsilon n$

(1) All but at most εt^2 of the pairs (V_i, V_j), $i,j \ne 0$, are ε-regular

(2) $t < M$ and $1/\varepsilon < t$

(3) $|V_i| = |V_j|$ for $i \ne j$, $i,j \ne 0$.

Proof: Define the index of a partition as $q(\mathcal{U}, \mathcal{V}) = \sum_{\substack{U_i \in \mathcal{U} \\ V_j \in \mathcal{V}}} \underbrace{d^2(U_i, V_j)}_{\text{density squared}} \frac{|U_i||V_j|}{n^2}$



$\leftarrow$ bipartite case partition

n vertices <u>total</u>, <u>not</u> per side.

$q(U_i, V_j) = d^2(U_i, V_j) \frac{|U_i||V_j|}{n^2}$

Given a partition $\mathcal{P}$ of n vertices, $q(\mathcal{P}) = \sum_{\substack{U,V \in \mathcal{P} \\ \text{distinct, unordered}}} q(U,V) = \sum_{\substack{U,V \in \mathcal{P} \\ \text{distinct, unordered}}} d^2(U,V) \frac{|U||V|}{n^2}$

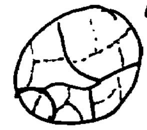
<u>Observe</u>: For any $\mathcal{P}$, $q(\mathcal{P}) \le 1/2$. ("That's why it's called q, b/c there's a q in the word index).

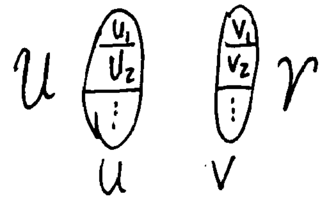
<u>Why?</u> $d(\ ) \le 1$, so $q(\mathcal{P}) \le \sum_{U \ne V \text{ in } \mathcal{P}} \frac{|U||V|}{n^2} = \frac{\#\text{edges in complete multipartite}}{n^2} \le \frac{\binom{n}{2}}{n^2} \le 1/2.$

$\longrightarrow$

$\mathcal{P}$

<u>Outline</u>: Start from a partition $\mathcal{P}$

If it's not (1), then we find a refined partition $\mathcal{P}'$ (a partition $\mathcal{P}'$ that looks like  compared to $\mathcal{P}$) such that $q(\mathcal{P}') \geq q(\mathcal{P}) + \varepsilon^5/2$ and the # of parts in $\mathcal{P}' \leq f(\text{\# parts in } \mathcal{P})$.

<u>Lemma:</u> 

$q(\mathcal{U},\mathcal{V}) \geq q(U,V)$

<u>Cor:</u> If $\mathcal{P}'$ refines $\mathcal{P}$, then $q(\mathcal{P}') \geq q(\mathcal{P})$.

<u>Proof:</u> Let z be a random variable:

pick uniformly random vertex $u \in U$ and uniformly random $v \in V$

Let $Z = d(U_i, V_j)$ such that $u \in U_i$ and $v \in V_j$.

$$\mathbb{E}[Z] = \sum_{U_i \in \mathcal{U}} \sum_{V_j \in \mathcal{V}} d(U_i,V_j)\, P(u \in U_i)\, P(v \in V_j) = \sum_{U_i \in \mathcal{U}} \sum_{V_j \in \mathcal{V}} d(U_i,V_j) \frac{|U_i|}{|U|} \frac{|V_j|}{|V|}$$

$$= \frac{1}{|U||V|} \sum_{i,j} \#\text{edges between } U_i, V_j = \frac{\#\text{edges } U \to V}{|U||V|} = d(U,V)$$

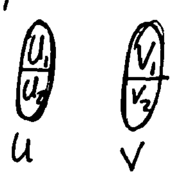
$$\mathbb{E}[Z^2] = \sum_{U_i \in \mathcal{U}} \sum_{V_j \in \mathcal{V}} d^2(U_i,V_j) \frac{|U_i||V_j|}{|U||V|} = q(\mathcal{U},\mathcal{V}) \frac{n^2}{|U||V|} \implies q(\mathcal{U},\mathcal{V}) = \frac{|U||V|}{n^2} \mathbb{E}[Z^2]$$

By Jensen's Inequality

$$q(\mathcal{U},\mathcal{V}) \geq \frac{|U||V|}{n^2} \left(\mathbb{E}[Z]\right)^2 = \frac{|U||V|}{n^2} d^2(U,V) = q(U,V) \quad (*)$$

■ Lemma.

<u>Lemma:</u> ~~Say $\mathcal{P}$ is a partition with k parts~~

Say 

is not ε-regular. Then $\exists$ split into $\mathcal{U} = \{U_1, U_2\}$ and $\mathcal{V} = \{V_1, V_2\}$ so that $q(\mathcal{U},\mathcal{V}) \geq q(U,V) + \frac{|U||V|}{n^2}\varepsilon^4$

<u>Proof:</u> By $(*)$, $q(\mathcal{U},\mathcal{V}) - q(U,V) = \frac{|U||V|}{n^2} \operatorname{Var}(Z)$

Since not ε-regular, $\exists\, U_1 \subseteq U$ with size $\geq \varepsilon|U|$ and $\exists\, V_1 \subseteq V$ with size $\geq \varepsilon|V|$, with $d(U_1,V_1)$ at least ε away from $d(U,V)$.

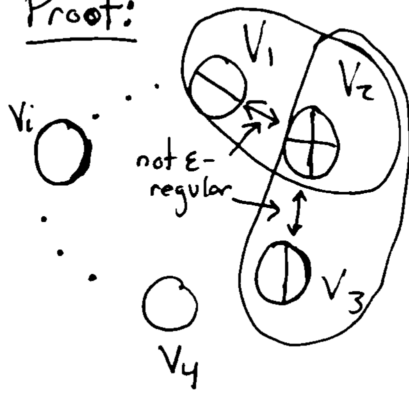
$\longrightarrow$

Proof continued:

$$\mathrm{Var}(Z) = \mathbb{E}\left[\left(Z-\mathbb{E}[Z]\right)^2\right] \geq \underset{\text{land in } U_1}{\varepsilon} \cdot \underset{\text{land in } V_1}{\varepsilon} \cdot \underset{\text{squared deviation}}{\varepsilon^2}$$ ■

Lemma: Say $\mathcal{P}$ has t parts of size m and $\geq \varepsilon t^2$ of those pairs are not ε-regular. Then $\exists$ a refinement $\mathcal{P}'$ with $q(\mathcal{P}') \geq q(\mathcal{P}) +$
and # parts $\leq$

Proof:



use Lemma!

For each pair (at least εt^2 many) that are not ε-regular, introduce bipartitions to get $\mathcal{P}^*$ with

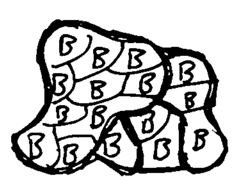
$$q(\mathcal{P}^*) \geq q(\mathcal{P}) + \varepsilon t^2 \frac{m^2}{n^2} \varepsilon^4 > q(\mathcal{P}) + \varepsilon^5$$

Note that $t^2 \frac{m^2}{n^2} > 1/2$ since $tm + |V_0| = n \implies tm < n$.

But also $\mathcal{P}^*$ has $\leq 2^{t-1} t$ many parts

But we want parts of $\mathcal{P}^*$ to have equal sizes.

Let $B = \frac{m}{4t}$. $\mathcal{P}^*$ looks like:



⟵ Break into blocks of size B greedily, remainder goes into V_0

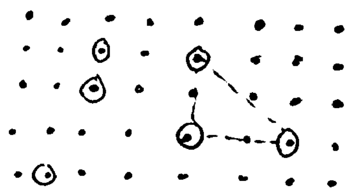
V_0 gained $\leq B|\mathcal{P}^*| \leq \frac{m}{4t}(t2^{t-1}) \leq \frac{n}{2^t}$ vertices: basically nothing.

V_0 grows with each successor partition as $0 + \frac{n}{2^{\#\mathcal{P}_1}} + \frac{n}{2^{\#\mathcal{P}_2}} + \dots \approx 0$. ■

Combining these gives the proof of the Szemeredi regularity lemma. ■

(AJTAI-SZEMEREDI "Corners Theorem")

<u>Theorem:</u> For any $\delta>0$, there is some N so that ~~[illegible]~~ any grid of size $\geq N \times N$ with $\geq \delta$-fraction of its points selected contains an isosceles right triangle.



<u>Theorem (Szemeredi's Theorem):</u> For all $\delta>0$, there is N so that any subset $\{1,2,\dots,N\}$ of density $\geq \delta$ contains an arithmetic progression of length L.

<u>Proof of Corners Theorem:</u> Uses density increment argument.

Key Element: $\forall$ density $\delta>0$, $\forall L \in \mathbb{Z}^+$, there is N s.t. any grid w/ $\geq N \times N$ with $\geq \delta$-fraction nodes selected, it either contains isosceles right triangle or has a subgrid $\geq L \times \geq L$ with density $\geq \delta + \frac{\delta^3}{100}$.

Why this finishes proof: Start w/ original input δ. Never increment $\geq \overset{T}{\overbrace{\frac{100}{\delta^3}}}$ times.

Study a sequence $L_T = 2$

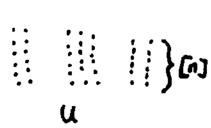
L_{T-1} = the N for δ_T and L_T.

$\vdots$

$L_0 \rightarrow$ sets N in original statement.

Wowzer function $\longrightarrow$ repeated towering.

<u>Theorem:</u> $\forall \epsilon \forall \delta \forall L \exists N$ s.t. for any $n \geq N$, if given an array which is a subset of the grid $\geq \delta n$ rows cols in array, then $\exists$ partition of array into $L \times L$ grids and a rubbish pile of $\leq \epsilon n^2$ points. Array is $U \times V$.

<u>Lemma:</u> Prove for $U \times \{1,2,\dots,n\}$, e.g. 

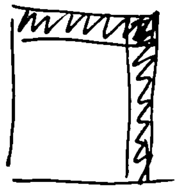
<u>Proof:</u> Szemeredi's Theorem <u>twice</u>!

(Szemeredi's: $\forall \delta\ \forall L\ \exists N$: when $n \geq N$, any subset of $[n]$ w/ density $\geq \delta$ has L-AP.)

w/ density parameter =

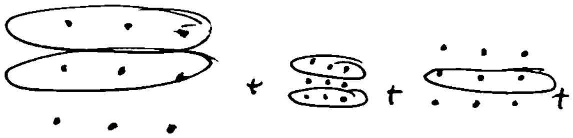
and length parameter = L/ϵ, break U = $1/\epsilon$ many L-APs.

Find partition of $U \times [n]$ into $L \times L$ grids covering all but:



miss:
$\le 2\varepsilon n$ rows of $[n]$
$\le \varepsilon n$ cols of n
Total: miss $\le 3\varepsilon n^2$ points. ■

Proof for $U \times V$: Use Lemma to partition $U \times [n]$ into $M \times M$ grids w/ some leftover. Select rows belonging to V.



$+ \dots +$ error

Density of $V \ge \delta$. We only worry about $M \times M$ grids which have some density. Szemerédi greedy partitioning again can break selected rows into L/ε-APs. Tile the grids across again, only small error. ■

Theorem (Erdős-Stone-Simonovits): $\forall \varepsilon > 0\ \exists N$ such that for any $n \ge N$ ($\forall r, \forall t$), if a graph has $\ge \frac{n^2}{2}(1 - 1/r + \varepsilon)$ edges, then it contains a $K_{r+1}(t)$.

Corollary: If H has chromatic number χ, then $ex(n, H) = (1+o(1))\left(\frac{n^2}{2}\right)\left(1 - \frac{1}{\chi - 1}\right)$.

Theorem (Erdős-Simonovits Stability):
~~For any r, t, ε ∃N. ∀r, ∀t, ∀δ, ∃N, ∃ε so that if n≥N, and an n-vertex graph G, G has ≥ n²/2 (1−1/r−ε) edg~~es

$\forall H, \forall \delta\ \exists N, \exists \varepsilon$ so that $\forall n \ge N$, n-vertex graph G has $\ge \frac{n^2}{2}\left(1 - \frac{1}{\chi(H)-1} - \varepsilon\right)$ edges and no H, then G can be made $(\chi(H)-1)$ partite by deleting only $\le \delta n^2$ edges.

Corollary: $\forall \delta, \exists \varepsilon$ s.t. every sufficiently large Δ-free graph with $\ge n^2\left(\frac{1}{4} - \varepsilon\right)$ edges is δ-close to being bipartite (within δn^2 edges).

04/07/14

Theorem: Max # edges in F-free graph is $\frac{n^2}{2}\left(1-\frac{1}{\chi-1}+o(1)\right)$ — no F as a subgraph

Theorem: F-free graphs with close to max # edges are almost $(\chi-1)$-partite.

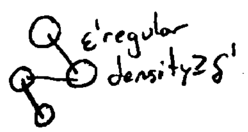
$\forall F\ \forall \varepsilon>0, \exists \delta$ s.t. if F-free graph has #edges $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}-\delta\right)$ then it can be made $(\chi-1)$ partite by deleting $\leq \varepsilon n^2$ edges. (For sufficiently large N).

Proof: Apply Szemeredi Regularity Lemma with $\varepsilon' =$, gives partition $V_1,\dots,V_t$ almost all same size, $V_i \approx \frac{n}{t}$, all but $\varepsilon' t^2$ ~~are~~ ε-regular pairs, and $\frac{1}{\varepsilon'} < t < M$.

Then: (1) Delete all edges in each V_i
(2) Delete edges between not ε'-regular pairs
(3) Delete edges between pairs w/ density $< \delta' =$

Total deletions:
(1) $t\binom{n/t}{2} \leq t\,\frac{(n/t)^2}{2} = n^2/2t < \frac{\varepsilon'}{2}n^2$
(2) $\varepsilon' t^2\left(\frac{n}{t}\right)^2 = \varepsilon' n^2$
(3) $\delta'\binom{n}{2} \leq \frac{\delta'}{2}n^2$

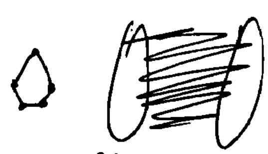
Got: graph w/ #edges $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}-\delta-\left(\frac{3}{2}\varepsilon'+\frac{\delta'}{2}\right)\right)$ and F-free

$\Longrightarrow$ Regularity graph 

has no K_χ, (or else graph would have K_χ blown up by $|F|$ times)

Also, graph of all edges post-deletion has no K_χ, for same reason.
Does this mean $\chi-1$ partite? No! odd cycle is not 2-colorable, for K_3.
But we know more!

Q: if graph has no K_3 and #edges $\geq 0.249n^2$ is it 2-partite?

A: No! 

$n/2-2$ $\quad$ $n/2-3$

To finish the proof, need

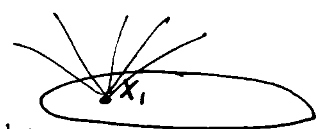
<u>Define:</u> $T_r(n)$ as the complete r-partite graph w/ $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ vertices per part. Max # edges in K_{r+1} free graph.

<u>Lemma:</u> Graph with no K_{r+1} and #edges $\geq e(T_r(n)) - k$. Then can make it r-partite by deleting at most k more edges.

<u>Proof:</u> Goal: Find a <u>good</u> partition of vertices into <u>r</u> parts.

total # edges inside parts $\leq k$.
(then delete them)

(1) Let x_1 = max degree vertex. First part contains x_1 and all of its non-neighbors



V_1

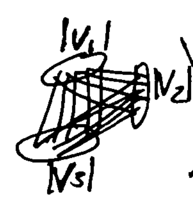
(2) Let x_2 = max degree vertex in graph $\setminus \{x_1\}$ $\longrightarrow$ induced subgraph of the rest.
Second part (V_2) contains x_2 and its non-neighbors (who are not in part 1)

(3) Repeat w/ $x_3, x_4, \dots x_r$. Note that $\{x_1, x_2, \dots, x_{last}\}$ is a clique.
$\Longrightarrow$ last $\leq r$ since no K_{r+1}.

<u>Goal:</u> Upper bound # edges in these parts. Let d_v^+ = number of neighbors that v has in its part or any future part. Then

$$\sum_v d_v^+ = e(G) + e(\text{inside parts})$$

$$\sum_v d_v^+ \leq \sum_i |V_i| d_{x_i}^+ = e\left(\text{complete multipartite}\right) \leq e\left(T_r(n)\right)$$



max degree among Graph of future parts

$\Longrightarrow$ $e(G) + e(\text{inside parts}) \leq e(T_r(n)) \Longrightarrow$ $e(\text{edges inside parts}) \leq e(T_r(n) - e(G))$
$\leq K$. ∎

So by lemma, since #edges $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}-\left(\delta+\frac{3}{2}\varepsilon'+\frac{\delta'}{2}\right)\right)$,

$\Longrightarrow$ get $(\chi-1)$ partite by deleting in total $\leq 2\left(\delta+\frac{3}{2}\varepsilon'+\frac{\delta'}{2}\right)\frac{n^2}{2}$ edges

Choose ε', δ' so that $\left(\delta+\frac{3}{2}\varepsilon'+\frac{\delta'}{2}\right) < \varepsilon$. ■

04/09/14

<u>Theorem</u> (Stability): If F-free graph has almost max # edges (i.e. #edges $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}-o(1)\right)$) then it's $o(n^2)$ edges away from being $(\chi-1)$-partite.

<u>Theorem (Hungarians)</u>: If F has chromatic number χ (barely), i.e. $\exists$ edge e whose deletion guarantees $\chi(F-e)=\chi(F)-1$, then for large enough n, the unique F-free graph with maximum # edges is $T_{\chi-1}(n)$.

<u>E.g.</u> Edge-color critical graphs
- odd cycle
- K_n

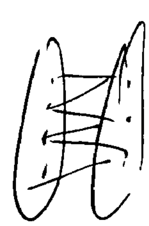
<u>Proof</u>: Using Stability Method.

Show for large n, if # edges in F-free graph $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}\right) = e\left(T_{\chi-1}(n)\right)$ (already F-free.)

then graph $= T_{\chi-1}(n)$.

<u>Apply Stability Theorem</u>: $\forall \varepsilon>0\ \exists \delta>0$ s.t. if #edges $\geq \frac{n^2}{2}\left(1-\frac{1}{\chi-1}-\delta\right)$, then deleting at most εn^2 edges makes me $(\chi-1)$-partite.

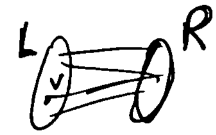
Do the case $\chi=3$.

Know graph is  w/ some edges within parts, but total internal edges $\leq \varepsilon n^2$.

Take a max cut bipartition $\longrightarrow$ maximize edges crossing.

internal edges in this bipartition $\leq$ internal edges before $\leq \varepsilon n^2$.

$\longrightarrow$



Proof continued:

Property of max cut: $\forall v \in V(G)$ degree across $\ge$ degree inside.

$e(G) \le LR + \varepsilon n^2$

Also, given that $e(G) \ge \frac{n^2}{2}\left(1 - \frac{1}{\chi - 1}\right)$ but $\chi = 3$, so $e(G) \ge n^2/4$.

$\Longrightarrow \quad L, R \in \left[n/2 - \varepsilon, \frac{n}{2} + \varepsilon\right]$

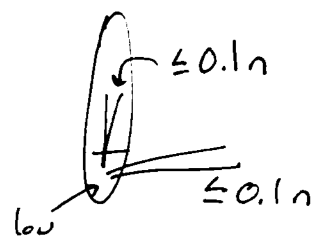
L

high: vtxs w/ R-degree $\ge R - \frac{n}{100f}$ $\qquad$ f = # vertices in F.

mid: rest of them.

low: vtxs w/ deg-to-R $\le n/10$

Analize the vertices in low first



Vertices in high cannot have edges between them, else not F-free.

~~Vertices in mid cannot hav~~

Not too many vertices in mid

Count edges:

a high, b mid, c low

$[\text{\# edges } L-R] + 2(\text{\# edges in } L) \le LR$

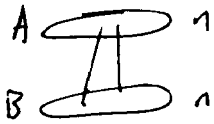
same for right

$\Longrightarrow \quad \text{\# edges} \le LR \le e(T_2(n))$ ∎

04/14/14

Quasirandomness

Thm: Given a bipartite graph with ~~$\not p$~~ proportion p of edges across where $p = \frac{\#\text{edges across}}{n^2}$



Then $\# \bowtie \geq n^4 p^4$ ⟵ "counting homomorphisms from $\bowtie \longrightarrow G$".

(vertices 1, 2, 3, 4)

Proof: $\# \bigvee = \sum_{v \in B} d_v^2 \geq n(\bar{d})^2 = n(pn)^2 = n^3 p^2$,

($\bar{d}$ = avg degree of vertex in B)

$\# \bowtie = \sum_{(3,4) \in A^2} (\#\text{ of common neighbors of both 3 and 4})^2$.

$\geq n^2 (\text{avg } \# \ldots\ldots)^2$

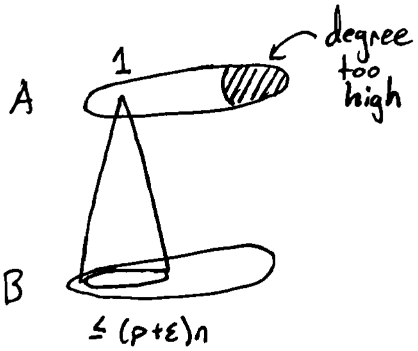
$\geq n^2 \left(\frac{\# \bigvee}{n^2} \right)^2 \geq n^2 \left(\frac{(n^3 p^2)}{n^2} \right)^2 = n^4 p^4$. ∎

- Now suppose (A, B) as above is also ε-regular. Can we improve the bound?

Theorem: Then $\# \bowtie \leq n^4 p^4 \cancel{(1+\frac{\varepsilon}{p})^4} \to (1+3\varepsilon^{1/5}+4\varepsilon^{4/5})$, as long as $\varepsilon \leq p^5$

Proof: By ε-regularity, at most ε of the fraction of the vertices of A have degree $\geq (p+\varepsilon)n$. So $\# \bowtie$ with 1 in good degree part ...

(and $<\varepsilon$-fraction have degree $\leq (p-\varepsilon)n$)



And $\# \bowtie$ w/ 1 in bad degree part $\leq$

(# ways to put 1 in bad degree part) $\times\, n^3 \leq 2\varepsilon n^4$

(tiny since $\varepsilon \ll p$)

Since nbhd(1) has size $\geq \varepsilon$-frac of B, ε-regularity $\Longrightarrow \leq \varepsilon$-fraction of vertices of A w/ degree into nbhd(1) too big! $\geq (p+\varepsilon)^2 n$ ⟵ $\varepsilon \lll p$

- $\# K_4$ in good degree part ⟶ $\# \bowtie$ w/ 2 in bad part $\leq (\varepsilon n) n^3 = \varepsilon n^4$

↳ $\# \bowtie$ w/ 1,2 in good parts $\leq \underset{\text{pick } 1,2}{n^2} \times (\underbrace{(p+\varepsilon)^2 n}_{\text{common nbhd size}})^2 \approx p^4 n^4 (1+\frac{\varepsilon}{p})^4$.

Theorem:

If $\#\overset{3\ \ 4}{\underset{1\ \ 2}{\bowtie}} \le n^4p^4(1+\varepsilon)$ then it's $\varepsilon^{1/12}$-regular, in a bipartite graph with proportion $p = \frac{\#\text{edges}}{n^2}$. (Intuition: close to regular)

Proof:

Claim 1: $\#\overset{3}{\underset{1\ \ 2}{\Lambda}} \le n^3p^2(1+\varepsilon)$

Proof: As before, $\#\overset{3\ \ 4}{\underset{1\ \ 2}{\bowtie}} \ge n^2\left(\frac{\#\overset{3}{\underset{1\ \ 2}{\Lambda}}}{n^2}\right)^2$ by pinning down two vertices on one side and counting codegree squared.

$$\Longrightarrow n^2\left(\frac{\#\overset{3}{\underset{1\ \ 2}{\Lambda}}}{n^2}\right)^2 \le n^4p^4(1+\varepsilon)$$

$$\Longrightarrow \left(\#\overset{3}{\underset{1\ \ 2}{\Lambda}}\right)^2 \le n^6p^4(1+\varepsilon) \Longrightarrow \#\overset{3}{\underset{1\ \ 2}{\Lambda}} \le n^3p^2(1+\varepsilon) \quad \blacksquare$$

Claim 2: Let $\delta = 2\varepsilon^{1/3}$. Then for any $X \subseteq A$, with $|X| \ge \delta|A|$, the number of edges between X and $B = (1\pm\delta)\underset{\uparrow |X|=x.}{x}np$.

Proof: Let Z be a random variable = choose uniformly random vertex of A, return its degree.

$\mathbb{E}[Z] = pn$ (avg degree of vtx in A)

$\mathbb{E}[Z^2] = \frac{\#\Lambda}{n} \underset{\text{claim 1}}{\le} n^2p^2(1+\varepsilon)$

$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 \le \varepsilon n^2p^2$

But also $\mathrm{Var}(Z) = \mathbb{E}[(Z-\bar{Z})^2]$

for contradiction

Suppose

has # edges $> (1+\delta)xnp$. ~~Then (avg value of Z over vtxs in X) $> (1+\delta)xnp$~~

Then (avg value of Z over vertices in X) $> (1+\delta)np$.

$$\mathbb{E}\left[(Z-\bar{Z})^2 \mid \text{vtx} \in X\right] \ge \left[\mathbb{E}[(Z-\bar{Z}) \mid \text{vtx} \in X\right]^2 \ge (\delta np)^2$$

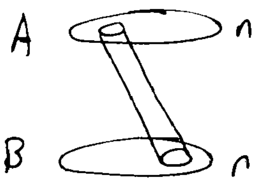
$$\Longrightarrow \mathbb{E}\left[(Z-\bar{Z})^2\right] \ge \underset{\text{land in } X.}{\delta}(\delta np)^2$$

So $\delta^3n^2p^2 \le \mathrm{Var}(Z) \le \varepsilon n^2p^2 \Longrightarrow \delta^3 \le \varepsilon$ $\divideontimes$.

04/16/14

Quasirandomness:

Theorem: $\# \bowtie \le (1+\varepsilon) n^4 p^4$



Then $\forall X \subseteq A \quad |X| \ge \quad |A|$

$\forall Y \subseteq B \quad |Y| \ge \quad |B|$

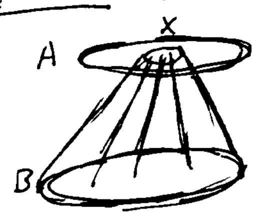
$p = \frac{\#\text{ edges between } A, B}{n^2}$

the # edges between X and Y is $(1 \pm \quad) |X||Y| p$

Last Time:

Claim 1: $\# \Lambda \le n^3 p^2 (1+\varepsilon)$

Claim 2: $\forall X \subseteq A$ with $|X| \ge 2\varepsilon^{1/3} n$



$\#\text{edges}(X,B) = (1 \pm 2\varepsilon^{1/3}) |X||B| p$

For the rest of the proof, fix an arbitrary $X \subseteq A$ with ~~$|X| \ge 2\varepsilon^{1/4} n$~~. $|X| \ge \delta n$

Let $x = |X|$

Claim 3: $\# \Lambda^{X}_{B} \le x^2 n p^2 (1+\delta)$, $\delta = 2\varepsilon^{1/4}$.

Proof: Let Z = pick two random vertices in A, return codegree. (Random variable)

$$\mathbb{E}(Z) = \frac{\# \Lambda^{A}_{B}}{n^2} \ge \frac{\sum_{v \in B} d_v^2}{n^2} \ge \frac{n(\bar{d})^2}{n^2} = \frac{n(np)^2}{n^2} = np^2$$

n^2 ← ways to pick two vertices in A

$$\mathbb{E}(Z^2) = \frac{\# \bowtie}{n^2} \le \frac{(1+\varepsilon) p^4 n^4}{n^2} \le (1+\varepsilon) n^2 p^4$$

$$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 \le \varepsilon n^2 p^4$$

Suppose for contradiction that $\exists X \subseteq A$ with $|X| \ge \delta$-fraction of A, and on this set, $\# \Lambda^{X} > x^2 n p^2 (1+\delta)$

In the restricted probability space, where we pick 2 vtxs in X

the $\mathbb{E}[Z \mid \text{both vtxs in } A \text{ are in } X] = \frac{\# \Lambda^{X}_{B}}{x^2} > np^2(1+\delta)$ ⊛

$\mathbb{E}[(Z-\bar{Z})^2 \mid \text{both vtxs in } X] \ge (\mathbb{E}(Z-\bar{Z} \mid \text{both vtxs in } X)^2)$ ← by convexity

By claim 1, $\underbrace{\bar{Z} \le np^2(1+\epsilon)}$. Hence

$\mathbb{E}[Z-\bar{Z} \mid \text{both vtxs in } X] = \mathbb{E}[Z \mid \text{both vtxs in } X] - \bar{Z} > np^2(\delta-\epsilon)$ by ⊛

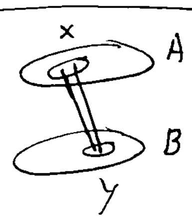
$\Longrightarrow \left(\mathbb{E}[Z-\bar{Z} \mid \text{both vtxs in } X]\right)^2 \ge \left(np^2(\delta-\epsilon)\right)^2$

$Var(Z) \ge \left(\frac{x}{n}\right)^2 n^2p^4(\delta-\epsilon)^2 \ge n^2p^4\delta^2(\delta-\epsilon)^2$

But from before, $Var(Z) \le \epsilon n^2p^4$.

Get the contradiction by picking $\delta = \epsilon^{1/4}\cdot 2$. ■

Claim 4 (Proof of Theorem): Let $\delta' =$

if 

$|X|, |Y| \ge \delta n$, then #edges$(X,Y) = (1\pm\delta)xyp$

Remember, $X \subseteq A$ is fixed

Proof: Let Z = choose uniformly random vertex in B, return its x-degree.

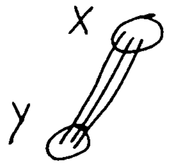
$\bar{Z} = \mathbb{E}(Z) = \frac{\#\text{edges}(X,B)}{|B|=n} \underset{\text{claim 2}}{\ge} \frac{(1-2\epsilon^{1/3})xnp}{n} = (1-2\epsilon^{1/3})xp$. but also, $\bar{Z} \le (1+2\epsilon^{1/3})xp$ again by claim 2

$\mathbb{E}(Z^2) =$ # 

$/\, n \underset{\text{claim 3}}{\le} \frac{x^2np^2(1+2\epsilon^{1/4})}{n} = x^2p^2(1+2\epsilon^{1/4})$

So $Var(Z) \le \mathbb{E}[Z^2] - (\mathbb{E}(Z))^2 \le x^2p^2(3\epsilon^{1/4})$

Assume for contradiction that $\exists\, Y \subseteq B$ of size $y = |Y| \ge \delta|B| = \delta n$ and



#edges$(X,Y) > (1+\delta)(xyp)$.

$\mathbb{E}[Z \mid \text{vtx in } Y] = \frac{\#\text{edges}(X,Y)}{y} > (1+\delta)xp$

$\Longrightarrow \mathbb{E}[Z-\bar{Z} \mid \text{vtx in } Y] > \left((1+\delta)-(1+2\epsilon^{1/3})\right)xp = (\delta-\epsilon^{1/3})xp$

Jensen:

$\mathbb{E}\left[(Z-\bar{Z})^2 \mid \text{vtx in } Y\right] > (\delta-\epsilon^{1/3})x^2p^2 \Longrightarrow Var\, Z \ge \delta(\delta-\epsilon^{1/3})^2x^2p^2$

Choose δ for contradiction, $\delta = 2\epsilon^{1/12}$. ■

Pseudorandomness

in abs. value

Defn: (n,d,λ)-graph has n vertices, d-regular, and the second-largest eigenvalue of G is $\le \lambda$

Observation: not quite random, but good approximation.

Theorem: Given an (n,d,λ) graph, every pair of sets B, C has $e(B,C) = \frac{d}{n}|B||C| \pm \lambda\sqrt{|B||C|}$

$$e(B,C) = \frac{d}{n}|B||C| \pm \lambda\sqrt{|B||C|}. \qquad b = \frac{|B|}{n} \quad c = \frac{|C|}{n}$$

If $\lambda = o(d)$ then the error above is of smaller order than main term

For any d-regular graph, d is largest e-value.

Lemma: For all sets $B \subseteq G$, almost all vertices have the right # of edges into B.

$$\sum_v (d_B(v) - bd)^2 \le \lambda^2 b(1-b)n$$

Proof: Consider $\mathbb{1}_B = \begin{pmatrix} \vdots \end{pmatrix}$ the vector which has 1's where B has ~~vectors~~ vertices

$$\vec{x} = \mathbb{1}_B - b\mathbb{1} \qquad (A\vec{x})_v = d_B(v) - bd$$

vth component of vector

hence, $\sum_v (d_B(v) - bd)^2 = \langle A\vec{x} | A\vec{x}\rangle \le \langle A^2 x | x\rangle$

Note that $\vec{x} \perp \mathbb{1}$, so $\vec{x}$ is a candidate for the second eigenvalues: therefore

$\|A\vec{x}\| \le \lambda\|\vec{x}\|$, because eigenvector of $\vec{x} \le \lambda$.

$$\langle A^2 x | x\rangle = \lambda^2\langle x, x\rangle = \lambda^2\langle \mathbb{1}_B - b\mathbb{1} | \mathbb{1}_B - b\mathbb{1}\rangle = \lambda^2\left((1-b)^2 bn + b^2(1-b)n\right) = \lambda^2 b(1-b)n. \quad \blacksquare$$

Proof of Theorem:

$$\left|e(B,C) - \frac{d}{n}bncn\right| = \left|e(B,C) - dnbc\right| \le \sum_{v\in C}|d_B(v) - bd|$$

$$\Longrightarrow \left|e(B,C) - \frac{d}{n}bncn\right| \le \sqrt{\sum_{v\in C}(d_B(v) - bd)^2} \le \sqrt{\lambda^2 b(1-b)n}\sqrt{cn} = \lambda\sqrt{bc(1-b)}\,n = \lambda\sqrt{|B||C|}$$

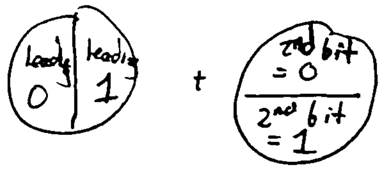
Cauchy Schwarz

04/21/14

<u>Question:</u> How few bipartite graphs can you use to cover edges of K_n?
(overlap in vertices and edges)

<u>Answer:</u> $\lceil \log_2 n \rceil$ is enough: split K_n in half and cover w/ complete bipartite

Relabel vertices in binary:



+ . . . .

$\#$ bits $= \lceil \log_2 n \rceil$.

Why covers edges of K_n? Any two vertices are vectors which differ somewhere.

<u>Prop:</u> Need $\sim \log_2 n$ bipartite graphs. ($\geq \lceil \log_2 n \rceil$)

If you are a vertex and there were K cuts (defining bipartition), each w/ a 0-side and a 1-side. So I observed K many 0/1 cuts telling me which side I was in (I am a vertex), for each cut.

If two vtxs have same sequence of 0 and 1, they are always on same side of each partition, the edge between them is always on same side of all cuts. $\Longrightarrow K \geq \lceil \log_2 n \rceil$ since need n distinct K-vectors. ∎

<u>Question:</u> How few complete bipartite graphs can I use to partition the edges of K_n?

cover each edge exactly once.

<u>Answer:</u> Take largest clique and split in half, recurse $\longrightarrow$ $n-1$ for n a power of two.
Number the vertices and connect i^{th} vertex to j^{th} for all $j > i$ $\longrightarrow$ $n-1$ complete bipartite.

<u>Theorem:</u> (Graham-Pollack)
Fewest number of complete bipartite possible is $n-1$.

<u>Remark:</u>
- number theoretic thing about n = power of 2
- no stability of extremal graph for n = power of 2.

Resists induction.

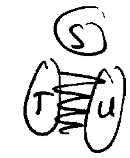
All known proofs are linear algebraic.

Proof: Adjacency Matrix of K_n: $J - I$ $\qquad$ 7^2

Suppose we have a partition of K_n into $\bigcup B_i$.

This means $J - I = \sum_{i=1}^{t} B_i$ $\qquad$ B_i is similar to $\begin{array}{c|ccc} & S & T & U \\ \hline S & 0 & 0 & 0 \\ T & 0 & 0 & 1 \\ U & 0 & 1 & 0 \end{array}$

$\uparrow$ adjacency matrix



Claim: if there are too few B_i, then this is impossible.

Lemma: $\operatorname{rank}(A+B) \le \operatorname{rank}(A) + \operatorname{rank}(B)$

Proof: rank = dim column space

~~Suffices to s~~

pick some columns $v_1, \ldots, v_a$ a basis for $\operatorname{col}(A)$.

pick some columns $W_1, \ldots, W_b$ a basis for $\operatorname{col}(B)$.

Then $\{v_1, \ldots, v_a, W_1, \ldots, W_b\}$ spans $\operatorname{col}(A+B)$. ■

Observation: Rank $B_i = 2$

$\operatorname{Rank}(\sum B_i) \le 2t$

$\operatorname{Rank}(J - I) = n \longrightarrow$ sum of cols $= (n-1)\mathbb{1} \in \operatorname{col}(J-I) \Rightarrow \mathbb{1} \in \operatorname{col}(J-I)$

so $\mathbb{1} - (i\text{th column}) = e_i \in \operatorname{col}(J-I)$.

Hence, we need $\ge \frac{n}{2}$ of the B_i.

Try instead $C_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. $\operatorname{Rank}(C_i) = 1$ and $\sum_{i=1}^{t} C_i = C$ has rank $\le n$,

and $C + C^T = J - I$.

Suppose we only have $n-2$ of the $C_i \Rightarrow$ rank $C \le n-2 \Rightarrow \dim(\operatorname{Null}(C)) \ge 2$.

Also rank $J = 1 \Rightarrow \dim(\operatorname{null}(J)) = n - 1$

nullspace of C and nullspace of J have nontrivial intersection $\langle \vec{x} \rangle$.

$$C + C^T = J - I \Rightarrow C\vec{x} + C^T\vec{x} = J\vec{x} - I\vec{x}$$

$$\Rightarrow C^T\vec{x} = -\vec{x}$$

$$\Rightarrow x^T C^T \vec{x} = -\vec{x}^T x \Rightarrow 0 = -\|\vec{x}\| < 0 \quad \divideontimes .$$ ■

04/23/14

Theorem: If a d-regular graph w/ second e-val λ, then for any $B, C \subseteq G$, $|B| = bn$, $|C| = cn$

$$e(B,C) = \frac{d}{n}(bn)(cn) \pm \lambda\sqrt{bc}\, n$$

Question: Why does the second eigenvalue have anything to do with random-like behavior?

Show: If 2^{nd} eval small, then random walks converge to uniform distribution quickly. After k steps, position is essentially random.
Uniform means you're in each position w/ probability $1/n$.

Suppose your current position is the probability vector v where $\sum v_i = 1$.
Then your position at the next step is $\frac{1}{d}Av$.

If $\vec{v}_0$ is initial position, after k steps you are at $\underbrace{\frac{1}{d^k}A^k v_0}_{\bar{A}^k \vec{v}_0}$. Let $\bar{A} = \frac{1}{d}A$

Since $\bar{A}$ is real symmetric, has ON basis of evecs, $\vec{w}_1, \dots, \vec{w}_n$, real e-vals $\lambda_1, \dots, \lambda_n$

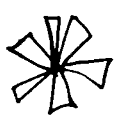
$\lambda_1 = 1$

$\frac{1}{\sqrt{n}}\mathbb{1} = \vec{w}_1, \dots, \vec{w}_n$

Write $\vec{v}_0 = \sum c_i \vec{w}_i$

$$\bar{A}^k v_0 = c_1 1^k w_1 + c_2 \lambda_2^k w_2 + \dots + c_n w_n$$
$$= 1/n\, \mathbb{1} + \left(\begin{array}{c}\text{stuff bounded by} \\ \lambda^k \cdot \text{constants}\end{array}\right)$$

Theorem (Friendship Theorem): If a graph has the property that every pair of vertices has exactly one common neighbor, then the graph is



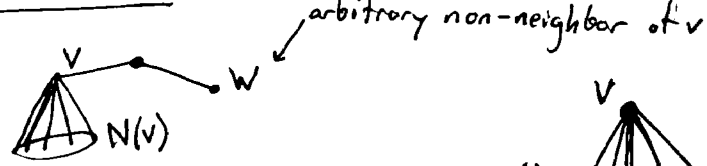
Proof: Suppose there is a vertex of degree $n-1$ $\longrightarrow$ forces the shape to be the flower/petal shape above.

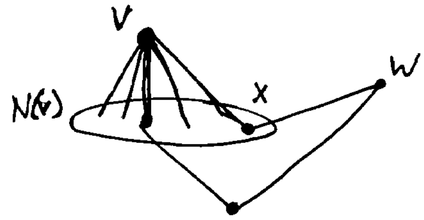
Remains to prove that there is a vertex of degree $n-1$. Suppose not.

Claim: graph is regular

$\longrightarrow$

Proof of claim:



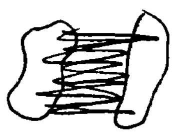


All neighbors of w are also neighbors of something in $N(v)$, and distinct $\Longrightarrow \deg(w) \geq \deg(v)$

$\exists$ a neighbor of y and w, $\neq v, w$

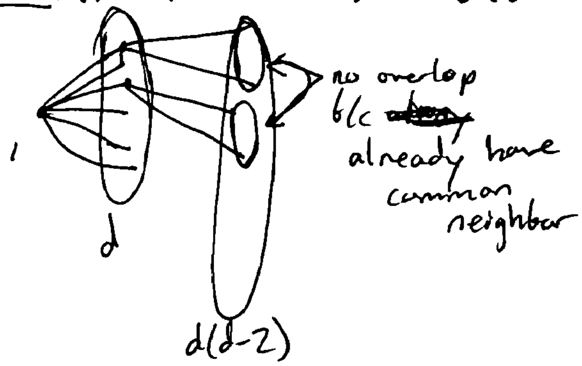
Symmetry $\Longrightarrow \deg(w) = \deg(v)$ for any non adjacent vertices v, w

Suffices to show: compliment of graphs is connected.

Why? If not, then  very wrong.

So our graph is regular, w/ degree d say.

Claim: $n = 1 + d + d(d-2) = 1 + -d + d^2$



$(A^2)_{ij} =$ walks of length 2 from i to j

$\Longrightarrow A^2 = J + (d-1)I$

e-vectors of J w/ e-vals $\lambda + (d-1)$

e-vectors of J include $\mathbb{1}$

So $(J + (d-1)I)\mathbb{1} = n + (d-1)\mathbb{1}$
$= d^2 \mathbb{1}$

but $A\mathbb{1} = d\mathbb{1}$
$\Longrightarrow A^2\mathbb{1} = d^2\mathbb{1}$

or $(J + (d-1)I)v = (d-1)v$ if $v \perp \mathbb{1}$

but $tr(A) = 0$

$\Longrightarrow$ e-vals of A are $d, \sqrt{d-1} \leftarrow n-1$

$\Longrightarrow \sum d + (n-1)\sqrt{d-1} \neq 0$

but $tr(A) = 0$.

04/25/14

Frankl-Wilson Theorem:

<u>Defn:</u> A family of sets (subsets of $[n]$) is L-intersecting if every pair of sets has intersection size $\in L$

<u>Q:</u> In terms of $s = |L|$, how big can an L-intersecting family be?

<u>A:</u> $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{s}$ all sets of size $\le s$ and $L = \{0,1,\dots,s-1\}$.

<u>Q:</u> In terms of $s=|L|$ and n, how big can an L-intersecting family of sets be if all sets are same size.

<u>A:</u> $\binom{n}{s}$ All subsets of size $= s$.

<u>Theorem:</u> If $\mathcal{F}$ is a family of subsets of $[n]$ s.t. every pair intersects w/ even size, but every ~~pair~~ set is odd size, then $|\mathcal{F}| \le n$.

<u>Theorem:</u> for p prime, if L is a set of s residues mod p, and $\mathcal{F}$ is L-intersecting mod p, then if $\forall F \in \mathcal{F}$ has $|F| \bmod p \notin L$, then

$$\#\text{sets in } \mathcal{F} \le \binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{s}.$$

<u>Proof review for mod p version:</u>

(1) Associate each set to a vector
(2) Show vectors independent
(3) Show they're in a space w/ dimension $\le$ thing
(4) Then also # sets $\le$ same thing

(1) $F \in \mathcal{F} \longmapsto \prod_{\ell \in L} (\vec{x}\cdot \mathbb{1}_F - \ell) = p_F(\vec{x}) \in \mathbb{F}_p[\vec{x}]$

(2) Take a linear combination $\sum_F c_F p_F = 0$, evaluate at $\mathbb{1}_E = \vec{x}$, for $E \in \mathcal{F}$

$$p_F(\mathbb{1}_E) = \begin{cases} \neq 0 & E=F \\ 0 & E \neq F \end{cases}, \text{ since } \mathbb{1}_E \cdot \mathbb{1}_F = \ell \text{ for some } \ell \in L, \text{ and so } p_F(\mathbb{1}) = \prod_{\ell \in L}(|E\cap F| - \ell)$$

So $\sum c_F p_F(\mathbb{1}_E) = c_E p_E(\mathbb{1}_E) = 0 \Rightarrow c_E = 0.$

(3) What is the dimension of $\{p_F \mid F \in \mathcal{F}\}$?

Don't use the p_F's! use the q_F's, which come from linearizing the polynomials $\rightarrow$ (set all exponents to 1). Doesn't matter b/c we only ever plug in $x_i \in \{0,1\} \Longrightarrow$ $q_F(\mathbb{1}_E) = p_F(\mathbb{1}_E)$.

$q_F \in \bigoplus_{i=1}^{s} \mathbb{F}_p[x_1 \ldots x_n]_{\deg \leq i}$, and the dimension is $\sum_{i=1}^{s} \binom{n}{i}$ ■

<u>Proof of Frankl-Wilson:</u> (<u>not</u> mod p)

Suppose $\mathcal{F} = \{F_1, \ldots, F_m\}$ are ordered by size, $|F_i| \leq |F_j|$ for $i<j$.

For $F \in \mathcal{F} \longmapsto p_F(\vec{x}) = \prod_{\substack{\ell \in L \\ \ell < |F|}} (\vec{x} \cdot \mathbb{1}_F - \ell)$

immediately gives $p_F(\mathbb{1}_F) \neq 0$.

~~What about~~ $p_F(\mathbb{1}_E) = \begin{cases} 0 & \text{if } |E \cap F| < |F| \\ \text{nonzero} & \text{else} \end{cases}$

in other words $p_{F_i}(\mathbb{1}_{F_j}) = \begin{cases} 0 & \text{if } j<i \\ \text{nonzero} & \text{else} \end{cases}$

<u>Claim</u>: $\{p_F\}$ linearly independent.

Create a matrix $A = \left(p_{F_i}(\mathbb{1}_{F_j})\right)_{1 \leq i,j \leq m}$ $a_{ij} = p_{F_j}(\mathbb{1}_{F_i})$

Since $p_F(\mathbb{1}_F) \neq 0$, then diagonal is nonzero

since $p_{F_i}(\mathbb{1}_{F_j}) = 0$ for $j<i$, the above diagonal is zero.

Suppose p_{F_i} are dependent over $\mathbb{Q}[x_1, \ldots, x_n] \Longrightarrow$ columns are dependent.

But if columns of A are dependent, then A is singular, but $\det(A) \neq 0$, so A is nonsingular $\divideontimes$

So p_{F_i} are independent $\longrightarrow$ linearize, and you're done. ■

Theorem: $\mathcal{F}$ collection of sets, all intersection sizes $\in L$, then if $s = |L|$, and if also all sets are the same size, then
$$\#\mathcal{F} \le \binom{n}{s}$$

Proof: For each $F \in \mathcal{F}$, define $p_F(\vec{x}) = \prod_{\substack{\ell \in L \\ \ell < |F|}} (\vec{x} \cdot \mathbb{1}_F - \ell)$

Order $\mathcal{F}$ such that $|F_1| \le |F_2| \le \cdots \le |F_n|$.

$$\left.\begin{array}{ll} \text{if } i<j & p_{F_j}(\mathbb{1}_i) = 0 \\ \text{if } i=j & p_{F_j}(\mathbb{1}_j) \ne 0 \end{array}\right\} \Longrightarrow \text{Matrix } A = \left(p_{F_j}(\mathbb{1}_i)\right)_{1 \le i,j \le n} \text{ is upper triangular, nonzero on diagonal.}$$

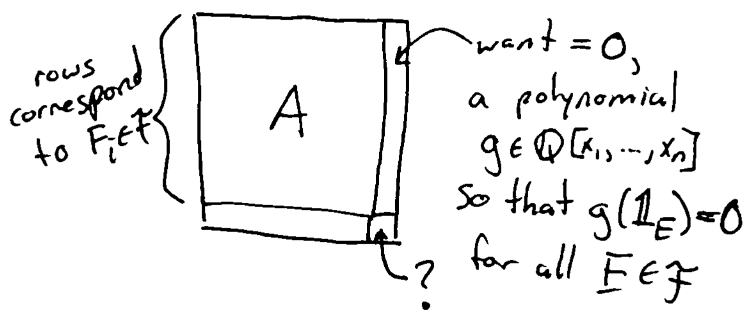
$\det(A) \ne 0 \Longleftarrow$ upper triangular, nonzero on diagonal.

So we conclude A is full rank $\Rightarrow \{p_F \mid F \in \mathcal{F}\}$ linearly independent.

Multilinearize $p_F \mapsto q_F$, also independent.

This proves it for the F not all the same size. Suppose $|F| = K \;\forall F \in \mathcal{F}$.

Strategy: Find more polynomials to add to the linearly independent set. A will be a minor of the new matrix

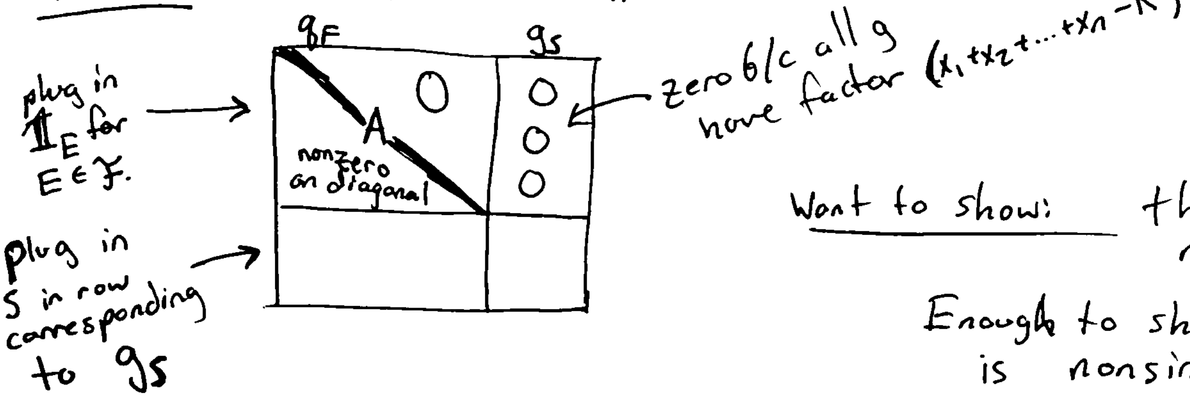


Define
$$\left.\begin{array}{l} g_\emptyset(\vec{x}) = (x_1 + x_2 + \cdots + x_n - K) \\ g_{\{1\}}(\vec{x}) = (x_1 + x_2 + \cdots + x_n - K)x_1 \\ g_{\{2\}}(\vec{x}) = (x_1 + x_2 + \cdots + x_n - K)x_2 \\ g_{\{n\}}(\vec{x}) = (x_1 + \cdots + x_n - K)x_n \\ \vdots \\ g_S(\vec{x}) = (x_1 + \cdots + x_n - K)\prod_{s \in S} x_s \\ \quad \text{for } S \subseteq [n],\ S \ne [n] \end{array}\right\} \text{multilinearize the } g_S.$$

We added $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{s-1}$ polynomials to our set.

Claim: $\{q_F : F \in \mathcal{F}\} \cup \{g_S : S \subseteq [n], |S| < n\}$ are linearly independent.
(well, multilinearized versions thereof)

Proof: Construct a matrix



Want to show: this matrix is nonsingular.

Enough to show bottom right block is nonsingular.

Proof Continued: Suffices to show that the g_S are linearly independent. (and so their multilinear ~~parts~~ versions) g_T^* are multilinearized g_T

Assume
$$\sum_{T \subseteq [n-1]} c_T g_T^* = 0$$ and c_T are not all zero. Pick T_0 the smallest set with $c_{T_0} \neq 0$. Plug in $\mathbb{1}_{T_0}$.

$$\underbrace{c_{T_0} g_{T_0}^*(\mathbb{1}_{T_0})}_{=(|T_0|-K)} + \underbrace{\sum_{T \not\subseteq T_0} c_T g_T^*(\mathbb{1}_{T_0})}_{=0} = 0 \implies c_{T_0}(|T_0|-K)=0 \implies c_{T_0}=0.$$

$|T_0| < s$, so $|T_0| - K < s - K$ but $s \leq K$ by ~~assumption~~.

$=0$ since $T \cap T_0 \subsetneq T$, there is an elt in $T \setminus T_0$, so $g_T^*(\mathbb{1}_{T_0}) = 0$

$\blacksquare$

~~We may assume $s \leq K$, but if $s > K$, then max number of sets of same size K is $\binom{n}{K}$.~~

In terms of K, $L \subseteq \{0,1,2,\dots,K-1\} \implies |L| \leq K \implies s \leq K$.

4/30/14

Chromatic number of unit distance graph

Defn: Unit distance graph has vertex set $\mathbb{R}^n$, and edge between every pair of vertices at distance 1.

Observe: for $n=1$, $\chi(\mathbb{R}^1) = 2$ $\quad \dots \underset{\text{red}}{[0,1)} \underset{\text{blue}}{[1,2)} \underset{\text{red}}{[2,3)} \dots$

Question: $\chi(\mathbb{R}^2) = ?$

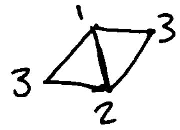
Bounds: $\chi(\mathbb{R}^2) \geq 3$ b/c two colors are insufficient, as it contains a Δ

Fact: $\chi(\mathbb{R}^2) \geq 4$

Proof:

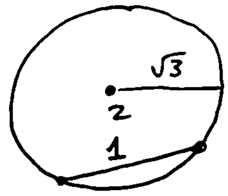
Proof: suppose we could color w/ 3 ~~triangles~~. colors.

Has a triangle:



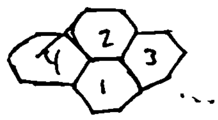
$\Longrightarrow$ for all points at distance $\sqrt{3}$, equal color

So if a point has color 2; all points have ~~the~~ the same color at distance $\sqrt{3}$, but there's a chord on this circle of length 1, contradicts 3-colorability. ■



Fact: $\chi(\mathbb{R}^2) \leq 7$

Proof: Break up into hexagons of diameter < 1, pick a pattern.

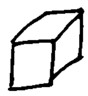


... etc.

Question: $\chi(\mathbb{R}^d)$ asymptotics
(Hadwiger-Nelson)

Upper Bounds?

Decompose into $\frac{1}{\sqrt{d}}$ hypercubes, tile $\mathbb{R}^d$



Let's control χ of a cube-based coloring by studying max # of other cubes in conflict w/ a given one.

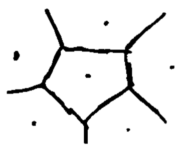
Bounded $\leq (2\sqrt{d})^d \approx d^{d/2}$.

Also, $\#\text{cubes} \leq \dfrac{\text{vol of ball of radius } 2}{\text{vol of cube}}$

Another upper bound:

Pick a maximal collection of points in $\mathbb{R}^d$ such that all pairs are at least $1/2$ distance apart. Call it C.

Let each point in C dictate the color of all points in $\mathbb{R}^d$ that have P as their closest C-point.

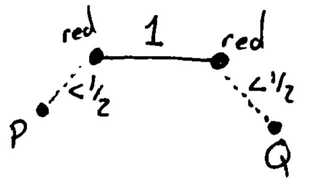


Construct a graph G: put edge between points of C if distance ≤ 2.

~~Why this~~ Why coloring G corresponds to coloring $\mathbb{R}^d$:

Suppose properly colored G, but still embedded in $\mathbb{R}^d$.

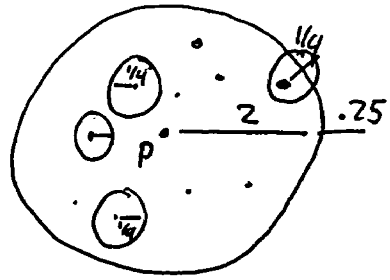
Have:



$d(P,Q) < 2$, PQ same color, yet PQ is edge in G.

What is $\chi(G)$? Bound via $\Delta(G)$. Let $P \in C$

$$\text{Ball radius 2 contains} \le \frac{\text{vol Ball radius } 2.25}{\text{vol Ball radius } .25} \le \left(\frac{5/4}{1/4}\right)^d = 9^d.$$



So $\chi(\mathbb{R}^d) \le 9^d$.

Lower Bounds?

Observe: $\chi(\mathbb{R}^d) \ge d+1$ simplex K_{d+1}

Theorem: (Larman-Rogers): $\chi(\mathbb{R}^d) \ge \text{constant} \cdot d^2$

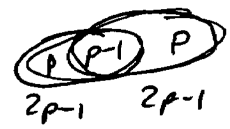
Theorem: (Frankl-Wilson): $\chi(\mathbb{R}^d) \ge 1.1^d$.

Proof: Consider a set of points in $\mathbb{R}^d$, a subset of $\{0,1\}^d$ x scale factor.
Talk about the hypercube in $\mathbb{R}^d$, ~~the~~ but ~~not~~ the unit distance graph but the $\frac{1}{\text{scale factor}}$ distance graph. Distance between points is $\sqrt{\text{Hamming Distance}}$!

Vertices of hypercube correspond to subsets of $[d]$ as ~~inclusion/exclusion~~ characteristic vectors. Take all sets of size $= 2p-1$ (let $d = 4p$).

Choose $L = \{0, 1, 2, \ldots, p-1\}$.

What it means to have intersection size $\equiv (p-1) \bmod p$:



equivalent to

Hamming distance between $v, u = 2p$.

Collection of points avoiding hamming distance $2p$ is L-intersecting family.

So by Frankl-Wilson theorem, size is $\le \binom{d}{0} + \binom{d}{1} + \dots + \binom{d}{p-1}$

In graph w/ edges at distance $\sqrt{2p}$, collection of vectors in $\{0,1\}^d$ which avoids distance $\sqrt{2p}$ is independent set,

independence # $\le p\binom{d}{p-1}$

$$\Longrightarrow \chi \ge \frac{\#\text{vtxs}}{\text{max indep set}} = \binom{d}{2p-1} \Big/ p\binom{d}{p-1} \approx \frac{2^{4p}}{2^{4pH(1/4)}} \approx c^p = c^d$$

$H(x) = -x\log_2 x - (1-x)\log_2(1-x)$