Group Theory + Physics

Equivalence Classes and Cosets

$\phi: \frac{\mathbb{Z}}{4\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}}$ HM

$\phi(0)=0$
$\phi(1)=1$
$\phi(2)=0$
$\phi(3)=1$

$D_3 = \langle a,b \mid a^2 = b^3 = e,\ ab = b^2 a \rangle$

$D_n = \langle a,b \mid a^2 = b^n = e,\ aba^{-1} = b^{-1} \rangle$

Groups of Order 8:

$\frac{\mathbb{Z}}{8\mathbb{Z}},\ \frac{\mathbb{Z}}{4\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}},\ D_4,\ \mathbb{Z}/2\mathbb{Z} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

$Q := \langle a,b \mid a^4 = e,\ a^2 = b^2,\ bab^{-1} = a^{-1} \rangle$ quaternion group

$q = a + bi + cj + dk,\quad i^2 = j^2 = k^2 = ijk = -1$

$a: q \longmapsto iq$
$b: q \longmapsto jq$

Groups of Order 12:

Symmetries of Tetrahedron

$A_4 = \langle a,b \mid a^2 = b^3 = (ab)^3 = e \rangle$ Tetrahedral group

dicyclic $Q_{2n} = \langle a,b \mid a^{2n} = e,\ b^2 = a^{-1},\ bab^{-1} = a^{-1} \rangle$

$\Bigg\}$ nonabelian

Cayley's Theorem

Every finite group G is ~~the~~ a subgroup of a symmetric group.

Proof: Find a map $G \xrightarrow{P} S_n$, injective, where $n = |G|$.

Let $G = \{g_1, \dots, g_n\}$

Let $P(g) = \begin{pmatrix} g_1, & \dots, & g_n \\ gg_1, & \dots, & gg_n \end{pmatrix}$ $\quad \left[P(g) = g_i \mapsto gg_i \right]$

P is a homomorphism, because

$P(e) = id$, clearly

$P(ab) = \begin{pmatrix} g_1, & \dots, & g_n \\ abg_1, & \dots, & abg_n \end{pmatrix} = \begin{pmatrix} g_1, & \dots, & g_n \\ & & \end{pmatrix}$

$$
\begin{aligned}
P(a)P(b) = P(a) \circ (g_i \mapsto bg_i) &= (g_i \mapsto ag_i) \circ (g_i \mapsto bg_i) \\
&= g_i \mapsto abg_i \\
&= P(ab)
\end{aligned}
$$

P is injective, b/c if $g_i = gg_i$, then $g = e$. ■

Defn: Regular Representation: the representation of a group G by ~~the~~ viewing it as permutations as a subgroup of S_n and permuting basis vectors.

Defn: If G is a group, $I(G)$ is the set of inner automorphisms of G given by $I_a(g) = aga^{-1}$ $\forall g \in G$.

$$I(G) = \{I_g : g \in G\}$$

$I(G) \leq Aut(G)$

Is $I(G)$ normal in $Aut(G)$?

$$F \circ I_a \circ F^{-1}(g) = F(I_a(F^{-1}(g))) = F(aF^{-1}(g)a^{-1})$$
$$= F(a)\, g\, F(a)^{-1} = I_{F(a)}(g)$$

So $I(G) \trianglelefteq Aut(G)$.

Consider the map $i : g \mapsto I_g : G \to I(G)$. Clearly i is an HM.

What is $\ker(i)$?

$$\ker(i) = \{g : I_g = \mathbb{1}\}$$

$g \in \ker(i) \Leftrightarrow I_g(h) = h \Leftrightarrow$ ~~$g^{-1}hg$~~ $ghg^{-1} = h \Longleftrightarrow gh = hg$.

$$\ker(i) = \{g \in G \mid g \text{ commutes with every element in } G\}$$
$$= Z(G) \text{ ("center of the group")}$$

Inner Automorphisms can be reflected physically as a change of basis.

Representation Theory of Groups

Defn: A group representation is a function
$\mu : G \to \mathcal{L}^*(V,V)$ where V is a vector space.
V is the representation space, $\mathcal{L}^*(V,V) =$ invertible linear maps $V \to V$.

$$\mu(g_1 g_2) = \mu(g_1) \circ \mu(g_2)$$
$$\mu(e) = id_V$$
$$\mu(g^{-1}) = \mu(g)^{-1}$$

Defn: The dimension of the representation is the dimension of the vector space V.

If two representations μ, μ' have the same dimension, there is $S \in \mathcal{L}(V_\mu, V_{\mu'})$ such that

$$\mu'(g) = S \mu(g) S^{-1} \quad \forall g \in G$$

In this case, we consider the representations μ and μ' the same.

If $\mu = \mu_1 \oplus \mu_2$, then it is reducible, for some representations μ_1 and μ_2.

Defn: A representation ϕ is irreducible if the only invariant subspaces under $\mu(g)$ for each $g \in G$ are 0 and V

Defn: Let V be a vector space, V_1, V_2 subspaces with $V_1 \cap V_2 = \{0\}$. Then if every element $\vec{v} \in V$ can be written as $\vec{v} = \vec{v}_1 + \vec{v}_2$, then $V = V_1 \oplus V_2$

Alternatively, $V_1 \oplus V_2 = \{(\vec{v}_1, \vec{v}_2) \mid v_1 \in V_1, V_2 \in V_2\}$ w/ obvious operations.

and V_i invariant under T

If $T: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$, then there exist T_1, T_2

$T_i : V_i \longrightarrow V_i$ such that $T = T_1 \oplus T_2$

~~[illegible]~~

Representations

Assume we represent G on an inner product space.

Lemma: Any representation of a finite group is equivalent to one in which each linear transformation is unitary.

Proof: Let $G = \{g_1, \dots, g_N\}$ and V be a vector space over $\mathbb{C}$ w/ inner product $\langle \cdot | \cdot \rangle$. $\Gamma : G \to GL(V)$ is the representation,

~~[illegible]~~ Let $\Gamma(g_i) = \Gamma_i$.

Define $H = \sum_{i=1}^{n} \Gamma_i \Gamma_i^*$ $\qquad D = U^{-1} H U$

Claim: Entries in D are positive? Yes:

$$v_j^T D v_j = \sum_{i=1}^{n} v_j^T \Gamma_i \Gamma_i^* v_j = \cancel{\sum_{i=1}^{n} (v_j \Gamma_i)^* (v_j \Gamma_i)}$$

$$= \sum_{i=1}^{n} (\Gamma_i^* v_j)^* (\Gamma_i^* v_j) = \sum_{i=1}^{n} \| \Gamma_i^* v_j \| > 0.$$

over $\longrightarrow$

Criterion for diagonalizability: commutes w/ hermitian adjoint

Proof continued:

Convert Γ_i into Γ_i'' where Γ_i'' is unitary

$$U^{-1}HU = \sum_{i=1}^{n} U^{-1}\Gamma_i\Gamma_i^*U = \sum_{i=1}^{n} U^{-1}\Gamma_i U U^{-1}\Gamma_i^* U$$

Define $\Gamma_i' = U^{-1}\Gamma_i U$. Note that since H is hermitian, we can take U to be Unitary.

So $D = U^{-1}HU = \sum_{i=1}^{n} \Gamma_i'\Gamma_i'^*$

makes sense since entries in D positive

$$\Longrightarrow \quad \sqrt{D}^{-1}\left(\sum_{i=1}^{n}\Gamma_i'\Gamma_i'^*\right)\sqrt{D}^{-1} = I$$

Define $\Gamma_i'' = D^{-1/2}\Gamma_i' D^{1/2}$

So $$\Gamma_i''\Gamma_i''^* = \underbrace{\left(D^{-1/2}\Gamma_i' D^{1/2}\right)}_{\Gamma_i''}\underbrace{\left(D^{-1/2}\left(\sum_{i=1}^{n}\Gamma_i'\Gamma_i'^*\right)D^{-1/2}\right)}_{I}\underbrace{\left(D^{+1/2}\Gamma_i'^* D^{-1/2}\right)}_{\Gamma_i''^*}$$

$$= D^{-1/2}\left(\sum_{j=1}^{n}(\Gamma_i'\Gamma_j')(\Gamma_j'^*\Gamma_i'^*)\right)D^{-1/2}$$

$$= D^{-1/2}\left(\sum_{k=1}^{n}\Gamma_k'\Gamma_k'^*\right)D^{-1/2}$$

$$= D^{-1/2}\,D\,D^{-1/2} = I$$

So Γ_i'' is unitary, and equivalent to Γ_i. ■

Schur's Lemmae:

(1): Let R be an irrep of G and suppose M commutes with Γ_i for all i, that is, $[M, \Gamma_i] = 0$. Then $M = \alpha I$ for some α.

Proof: By the previous lemma, we may assume all Γ_i are unitary. Then for all i

$$M\Gamma_i = \Gamma_i M \Rightarrow M^* \Gamma_i^* = \Gamma_i^* M^* \Rightarrow M^* \Gamma_i^{-1} = \Gamma_i^{-1} M^*$$

$$\Rightarrow [M^*, \Gamma_i] = 0.$$

Write $M = \left(\frac{M+M^*}{2}\right) + i\left(\frac{M+M^*}{2i}\right)$, and restrict to the case where M is hermitian. Diagonalize M via unitary matrices U, U^{-1}:

$$D = U^{-1} M U$$

$$0 = [M, \Gamma_i] = [U^{-1} D U, \Gamma_i] = U^{-1}[D, U\Gamma_i U^{-1}]U$$

Let $\Gamma_i' = U\Gamma_i U^{-1}$, which is an equivalent unitary ~~matrix~~ representation.

So $0 = [D, \Gamma_i']$ and D commutes with Γ_i' for all i.

Now remains to show that all diagonal elements of D are the same. In changing Γ_i to Γ_i', we also made M into D.

Look at the α, β entry of $[D, \Gamma_i']$.

$$0 = [D, \Gamma_i']_{\alpha\beta} = \sum_\gamma D_{\alpha\gamma}(\Gamma_i')_{\gamma\beta} - (\Gamma_i')_{\alpha\gamma} D_{\gamma\beta}$$

$$= \sum_\gamma D_{\alpha\alpha}\delta_{\alpha\gamma}(\Gamma_i')_{\gamma\beta} - D_{\beta\beta}(\Gamma_i')_{\alpha\gamma}\delta_{\gamma\beta}$$

$$= \sum_\gamma (\Gamma_i')_{\alpha\beta}(D_{\alpha\alpha} - D_{\beta\beta}) = 0$$

either $(\Gamma_i')_{\alpha\beta} = 0$ $\forall i$ in which case this representation is reducible, $\ast$

so $D_{\alpha\alpha} = D_{\beta\beta}$ and $D = \alpha I \Rightarrow M = \alpha I$ ∎

Let V be a complex vector space with basis $\{|i\rangle \mid i=1\ldots n\}$.

The inner product $\langle i|j\rangle$ isn't necessarily invariant under action by a group G: $\langle i(g)|j(g)\rangle \neq \langle i|j\rangle$.

Want to define a new inner product s.t. $(i(g)\,|\,j(g)) = (i\,|\,j)$.

Define $$(i\,|\,j) = \frac{1}{|G|}\sum_{g\in G}\langle i(g)\,|\,j(g)\rangle.$$ This satisfies the invariance we want.

<u>Claim:</u> Γ_i are unitary under the inner product $(\cdot\,|\,\cdot)$.

Definition of Hermitian: $\langle M^* u|v\rangle = \langle u|Mv\rangle$

So

$$(\Gamma^{-1}(g)\, i\,|\,j) = (\Gamma(g^{-1})\, i\,|\,j) = (\Gamma(g^{-1})\, i\,|\,\Gamma(g^{-1})\Gamma(g)\, j)$$
$$= (i\,|\,\Gamma(g)\, j) = (\Gamma^*(g)\, i\,|\,j)$$

invariant under group action

$\Longrightarrow \Gamma^{-1}(g) = \Gamma^*(g)$ since true for any vectors. ∎

Schur's Lemma 2: (Motivation):

Let $\rho: G \to GL(V)$ be ~~an~~ a representation for ρ such that $\rho = \alpha \oplus \beta$, where α is irreducible and β may or may not be irreducible. Let $M^{\rho}(g)$ be the matrix of $\rho(g)$, so

$$M^{\rho}(g) = \begin{pmatrix} M^{\alpha}(g) & 0 \\ 0 & M^{\beta}(g) \end{pmatrix}, \text{ also let } V = V_{\alpha} \oplus V_{\beta}.$$

Then if $\{|i\rangle\}_{i=1}^{N}$ is a basis for V, and $\{|a\rangle\}_{a=1}^{d_1}$ is a basis for V_{α}, then $|a\rangle = \sum_i S_{ai} |i\rangle$, and

d_1 ← $\dim(V_\alpha)$

$$|a(g)\rangle = \sum_b M^{\alpha}_{ab} |b\rangle = \sum_{b,j} M^{\alpha}_{ab} S_{bj} |j\rangle$$

Also $$|a(g)\rangle \to \sum_{i,j} S_{ai} M_{ij} |j\rangle$$

$$\Longrightarrow \sum_i S_{ai} M_{ij} = \sum_b M^{\alpha}_{ab} S_{bj}$$

(Statement)

Lemma: If the matrices of two different irreps of different dimensions are related by

ϕ, ψ ← irreps

$$\sum_i S_{ai} M^{\phi}_{ij} = \sum_b M^{\psi}_{ab} S_{bj},$$

then $S = 0$ as a matrix.

If the dimensions are the same, then either the irreps are the same, or $S = 0$.

Proof:

WLOG take the irreps (Φ, Ψ) to be unitary. As before, we know

$$S\Gamma_i = \Gamma_j S \implies \Gamma_i^* S^* = S^* \Gamma_j^*$$

$$\implies \Gamma_i S^* = S^* \Gamma_j$$

So then $\Gamma_i S^* S = S^* \Gamma_j S \implies \Gamma_i S^* S = S^* S \Gamma_i$

By Schur's First Lemma, $S^*S = \alpha I$ since S^*S commutes with Γ_i.

Case 1: $\dim \Phi = \dim \Psi = d$

$$\det(S^*S) = \alpha^d \implies |\det S|^2 = \alpha^d$$

if $\alpha \neq 0$, $\det(S) \neq 0$, so S invertible, and so unitary

$$S\Gamma_i = \Gamma_j S \implies S\Gamma_i S^{-1} = \Gamma_j$$

$$\implies \Phi \text{ equivalent to } \Psi.$$

if $\alpha = 0$, $\det S = 0$ and $S^*S = 0$

So $\langle v | S^* S | v \rangle = 0$

$$\implies \langle Sv | Sv \rangle = \|Sv\|^2 = 0 \implies \|Sv\| = 0 \ \forall v$$

$$\implies S = 0.$$

Case 2: $\dim \Phi = d_1 \neq d_2 = \dim \Psi$.

WLOG $d_1 < d_2$.

$$\begin{array}{cc} & d_1 \\ d_2 & \boxed{\;S\;} \end{array}$$

S is $d_2 \times d_1$ matrix

Let $P = \begin{array}{cc} & d_2 \\ d_2 & \begin{bmatrix} S & 0 \end{bmatrix} \end{array}$.

$$P^*P = \begin{bmatrix} S^* \\ 0 \end{bmatrix} \begin{bmatrix} S^* & 0 \end{bmatrix}$$

$$P^*P = \begin{bmatrix} S^*S \\ 0 \end{bmatrix} \begin{array}{l} \}d_1 \\ \}d_2 \end{array}$$

Repeating the argument from before, and filling out the block for Γ_i to be $d_2 \times d_2$, so $P^*P = \alpha I$ $\exists \alpha$.

Then $\det P^*P = \alpha^{d_2}$, but P^*P cannot be invertible, so $\det P^*P = 0 \Rightarrow \alpha^{d_2} = 0 \Rightarrow \alpha = 0$, so $P^*P = 0$ and as before, $S^*S = 0$ so $S = 0$. ∎

Great Orthogonality Theorem:

$$\frac{1}{|G|}\sum_{g\in G} M^{\alpha}_{ij}(g)\, M^{\beta}_{pq}(g^{-1}) = \frac{1}{\dim \alpha}\,\delta^{\alpha\beta}\,\delta_{iq}\,\delta_{jp}$$

Proof: Let $S = \sum_{g\in G} M^{\alpha}(g)\, N\, M^{\beta}(g^{-1})$. Then:

$$M^{\alpha}(h)S = \sum_{g\in G} M^{\alpha}(h) M^{\alpha}(g)\, N\, M^{\beta}(g^{-1})$$
$$= \sum_{g\in G} M^{\alpha}(hg)\, N\, M^{\beta}(g^{-1})$$

So sum over $\tilde{g} = hg \in G$, to get

$$M^{\alpha}(h)S = \left(\sum_{\tilde{g}\in G} M^{\alpha}(\tilde{g})\, N\, M^{\beta}(\tilde{g}^{-1})\right) M^{\beta}(h)$$
$$= S\, M^{\beta}(h)$$

Hence $\forall g \in G$ $M^{\alpha}(g)S = SM^{\beta}(g)$.

By Schur's Second Lemma, either $\alpha \neq \beta$ and $S = 0$, or $\alpha \hat{=} \beta$.

Case 1: $S = 0 \Rightarrow \sum_{g\in G} M^{\alpha}(g)\, N\, M^{\beta}(g^{-1}) = 0$

$$\Rightarrow \sum_{g\in G}\sum_{a,b} M^{\alpha}_{ia}(g)\, N_{ab}\, M^{\beta}_{bq}(g^{-1}) = 0$$

Case 2: $\alpha \cong \beta$.

Then $M^\alpha(g) S = S M^\alpha(g)$, so by Schur's first lemma,

$S = \mu I$, so $\sum_{g\in G} M^\alpha(g) N M^\alpha(g^{-1}) = \mu I$.

Choosing N appropriately, we get that

$$\sum_{g\in G} M_{ij}^\alpha(g) M_{pq}^\alpha(g^{-1}) = \mu \, \delta_{iq} \delta_{jp}$$

If $j=p$, sum LHS to get

$$\sum_{g\in G} M_{ij}^\alpha(g) M_{jq}^\alpha(g^{-1}) = \delta_{iq} |G|$$

sum RHS to get

$$\sum_{j=1}^{\dim\alpha} \mu \, \delta_{iq} = \mu (\dim \alpha) \, \delta_{iq}$$

$\Longrightarrow$ $\mu = \dfrac{|G|}{\dim(\alpha)}$, gives the result we want. $\blacksquare$

Characters of a Representation

Representation

$G \longrightarrow GL(V)$

we really want to think about equivalence classes of $G \to GL(V)$ under change of basis/similarity transform

Define $\chi : G \longrightarrow \mathbb{C}$

by $\chi^\alpha(g) = tr(\Gamma^\alpha(g))$

χ is a class function: constant on conjugacy classes.

Great Orthogonality Theorem for Characters:

$$\frac{1}{n_G} \sum_{g \in G} \chi^\alpha(g) \chi^\beta(g^{-1}) = \delta^{\alpha\beta}$$

Note that $tr(M^\beta(g^{-1})) = tr(M^\beta(g)^{-1}) = tr(M^\beta(g)^*) = tr(M^\beta(g))^*$ ← $\mathbb{C}$ conjugate

So then $\frac{1}{|G|} \sum_{g \in G} \chi^\alpha(g) \chi^\beta(g)^* = \delta^{\alpha\beta}$.

Next time:

(1) Number of irreps = number of conjugacy classes (finite groups)

(2) Decomposing the regular representation into irreps.

It contains each possible representation

$$|G| = \sum_{\alpha \text{ an irrep}} (\dim \alpha)^2$$

Defn: A class function is a function which is constant on conjugacy classes.

First Orthogonality Theorem (for characters):

Use relation in Great Orthogonality theorem, take trace

$$\frac{1}{n_G} \sum_{g \in G} tr(M^{[\alpha]}(g)) tr(M^{[\beta]}(g)^*) = \frac{1}{d_\alpha} \delta^{\alpha\beta}$$

$$\Rightarrow \sum_{g \in G} \chi^\alpha(g) \chi^\beta(g)^* = \frac{n_G}{d_\alpha} d_\alpha \delta^{\alpha\beta}$$

\# conj. classes → n_C

$$\Rightarrow \frac{1}{N_G} \sum_{i=1}^{n_C} |C_i| \chi^\alpha(c_i) \chi^\beta(c_i) = \delta^{\alpha\beta}$$

Define $\{\chi^\alpha, \chi^\beta\} = \frac{1}{|G|} \sum_{i=1}^{n_C} |C_i| \chi^\alpha(c_i) \chi^\beta(c_i)^*$

So we consider the vector space over $\mathbb{C}$ which has vectors

$$\left\{ \left(\chi^\alpha(c_1) \ldots \chi^\alpha(c_{n_C}) \right) \right\}_{\alpha=1}^{n_R}$$

where c_i is an element of i^{th} conjugacy class

Hence $n_R \leq n_C$. ∎

Now, consider ~~th~~ any representation R, and let $R_1, \ldots, R_{n_R}$ be the irreps of G. Suppose

$$R = R_1 \oplus R_2 \oplus \cdots \oplus R_p = \bigoplus_{\alpha=1}^{n_R} r_\alpha R_\alpha .$$

Then $\chi^R(g) = \sum_{\alpha=1}^{n_R} r_\alpha \chi^\alpha(g)$. To determine r_α, take inner product:

$$\{\chi^\alpha, \chi^R\} = \{\chi^\alpha, \sum_{\sigma=1}^{n_R} r_\sigma \chi^\sigma\} = \sum_{\sigma=1}^{n_R} r_\sigma \{\chi^\alpha, \chi^\sigma\} = r_\alpha$$

So for each α, $\{\chi^\alpha, \chi^R\} = r_\alpha$, and so

$$\{\chi^R, \chi^R\} = \sum_{\alpha=1}^{n_R} r_\alpha^2 .$$

Fact: If $\{\chi^R, \chi^R\} = 1$, then χ^R is irreducible.

Regular Representation:
n_G-dimensional representation.

Let $G = \{g_1, \ldots, g_n\}$, V an n_G-dimensional vector space, given basis $\{e_{g_1}, \ldots, e_{g_n}\}$. Let G act on V by

$$g \circ e_{g_i} = e_{gg_i} .$$

Then $\chi^{reg}(e) = n_G$

$\chi^{reg}(g) = 0$ for $g \neq e$

Furthermore, the regular representation is faithful.

~~Say $\phi: G \to \mathrm{Aut}(GE$~~ Say $\phi: G \to GL(V)$.

Then if $\phi(h) = I$, then $\forall\, e_g$ in basis for V,

$$e_{gh} = e_g \implies gh = g \implies h = e. \quad \blacksquare$$

Claim 3: Regular representation is reducible.

Claim: regular $= \bigoplus_{\alpha=1}^{n_R} d_\alpha R^\alpha$, and $r_\alpha = d_\alpha$

$$r_\alpha = \{\chi^{reg}, \chi^\alpha\} = \frac{1}{n_G} \sum_{i=1}^{n_c} |[C_i]|\, \chi^{reg}(c_i)\, \chi^\alpha(c_i)^*$$

since $\chi^{reg}(g) = 0$ for $g \neq e$, then

$$\{\chi^{reg}, \chi^\alpha\} = \frac{1}{n_G} \chi^{reg}(e)\, |[e]|\, \chi^\alpha(e)$$

$$= \frac{1}{n_G} \cdot n_G \cdot 1 \cdot d_\alpha = d_\alpha \implies r_\alpha = d_\alpha. \quad \blacksquare$$

Now to prove: $n_G = \sum_{\text{irreps}} \dim(\text{irrep})^2$

$\{\chi^{reg}, \chi^{reg}\} = \sum_{\alpha=1}^{n_R} d_\alpha^2$ but also $n_G = \{\chi^{reg}, \chi^{reg}\}$,

because $\{\chi^{reg}, \chi^{reg}\} = \frac{1}{n_G} \sum_{i=1}^{n_c} |[c_i]|\, \chi^{reg}(c_i)\, \chi^{reg}(c_i)^*$

$$= \frac{1}{n_G}\, 1\, n_G^2 = n_G. \quad \blacksquare$$

Finally, we know $n_R \leq n_c$, and now we want to get $n_R \geq n_c$.

~~To prove $n_R \le n_C$~~

$$\sum_{\alpha=1}^{n_R} \chi^\alpha(c_i)\,\chi^{\alpha}(c_j)^* = \frac{n_G}{|C_i|}\,\delta_{ij}$$

c_i, c_j are representatives of conjugacy classes C_i, C_j.

<u>Character Table:</u>

n_C = number of conjugacy classes

		$1 \cdot C_1$	$N_2 \cdot C_2$	$\cdots$	$N_{n_C} C_{n_C}$
trivial	$D^{(1)}$	1	1	$\cdots$	1
	$\vdots$	d_2			
		$\vdots$			
	$D^{(n_R)}$	d_{n_R}			

n_R is number of irreps

If a conjugacy class contains both an element g and its inverse g^{-1}, then the character of that conjugacy class is real.

$$\chi(g) = \chi(g^{-1}) \Rightarrow \operatorname{tr}(M^\alpha(g)) = \operatorname{tr}(M^\alpha(g)^{-1})$$
$$= \operatorname{tr}(M^\alpha(g)^*)$$
$$= \operatorname{tr}(M^\alpha(g))^*$$

So $\chi(g) = \chi(g^{-1}) \in \mathbb{R}$.

Example: Character table of D_3

	C_1 $\{e\}$	C_2 $\{R, R^2\}$	C_3 $\{P, PR, PR^2\}$
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

Frobenious Algebra of a Group:

Multiply conjugacy classes $C_i C_j = \{c_i c_j : c_i \in C_i,\ c_j \in C_j\}$ multiset

add conjugacy classes $C_i + C_j = C_i \cup C_j$ (multisets)

In D_3, $C_2 C_3 = 2C_3$

$C_3 C_3 = 3C_1 + 3C_2$

$$C_2 C_2 = \{R^2, R^3, R^2, R^3\} = \{e, e, R^2, R^3\} = 2C_1 + C_2 .$$

Fact: Say $C_i C_j = \sum_k \lambda_{ijk} C_k$. Then

$$|C_i|\,\chi^\alpha(C_i)\,|C_j|\,\chi^\alpha(C_j) = \sum_{k=1}^{n_C} |C_k|\,\chi^\alpha(C_k)\,\lambda_{ijk} .$$

Now to prove

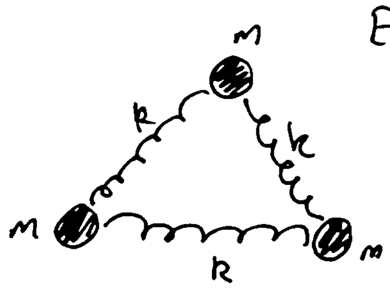
$$\sum_{\alpha=1}^{n_R} \chi^\alpha(C_i)\,\chi^\alpha(C_j)^* = \frac{n_G}{|C_i|}\,\delta_{ij}$$ (sum cols in character table)

$$\sum_{\alpha=1}^{n_R} \sum_{i,j=1}^{d_\alpha} \mu^\alpha(g)_{ij}\,\mu^\alpha(g')_{ij} = n_R\,\delta_{gg'}$$

Since
$$\sum_{\alpha=1}^{n_R} \chi^\alpha(C_k)\,\chi^\alpha(C_m)^* = \frac{n_G}{N_m}\,\delta_{mk}$$

Then this gives $n_C \leq n_R$, b/c we have an n_R-dimensional space and the above gives a nontrivial linear dependence among n_R things.

Normal Modes of Spring Systems



Equal Masses connected by springs in equilateral triangle.

position of masses $L = (x_1, y_1, x_2, y_2, x_3, y_3) = \eta$

kinetic energy $T = \frac{1}{2} m \dot{\vec{\eta}}^T \dot{\vec{\eta}}$ $\quad k$ = kinetic energy

$$V = \frac{1}{2} k\, \eta^T U \eta$$

Normal Modes

$$m\ddot{\vec{\eta}} = -kU\vec{\eta} \qquad \eta = \vec{A}e^{i\omega t}$$ ← everyone has common frequency

$$\left(\frac{m\omega^2}{k}\right)\vec{A} = U\vec{A}$$

Hamiltonian: $H = \dfrac{\vec{p}^T\vec{p}}{2m} + \frac{1}{2} k\, \eta^T U \eta$

What if we rotate the system? $\vec{\eta} \mapsto R\vec{\eta}$

$$m(R\ddot{\vec{\eta}}) = -kU(R\vec{\eta})$$
$$m\ddot{\vec{\eta}} = -k\underbrace{(R^{-1}UR)}_{U}\vec{\eta}$$

since invariant under rotation

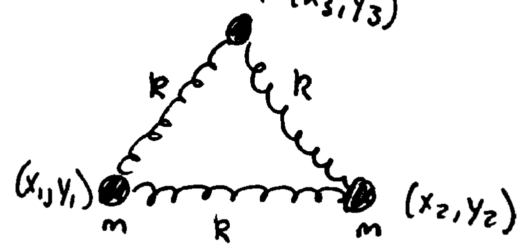
So $RHR^{-1} = H$.

So to solve this eigenvalue problem, U is going to diagonalize in blocks of irreps of D_3.

Normal Modes from physics:

rotation
translation

$D_3 = \langle R, P \mid R^3 = P^2 = 1, PRP^{-1} = R^{-1} \rangle$



Conjugacy Classes:

$C_1 = \{1\}$

$C_2 = \{R, R^2\}$

$C_3 = \{P, PR, PR^2\}$

Kinetic energy

$\eta = (x_1, y_1, x_2, y_2, x_3, y_3)$

$$L = \frac{m}{2}\dot{\vec{\eta}}^T\dot{\vec{\eta}} - \frac{1}{2}k\vec{\eta}^T U \vec{\eta}$$

potential energy

Character Table:

	C_1	$2C_2$	$3C_3$
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

Replacing $\vec{\eta}$ by $\vec{\eta}' = O\vec{\eta}$, the physics shouldn't change. Enforce invariance of L under D_3 transformations. The reference frame shouldn't matter.

Can always choose O such that $O^T O = I$, by representation theory.

Hence L is invariant, $L' = \frac{m}{2}\dot{\vec{\eta}}^T\dot{\vec{\eta}} - \frac{1}{2}k\vec{\eta}^T U\vec{\eta} = L$.

Look at the potential: $\vec{\eta}'^T\vec{\eta}' = \vec{\eta}(O^T U O)\vec{\eta} = \vec{\eta}^T U\vec{\eta}$

Hence $U = O^T U O$, and U must commute with O.

So U is a reducible representation of D_3, and must be six-dimensional.

So U sits in the regular representation of D_3.

Can conclude that $$\mathcal{U} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_{31}\,(2\times 2) & \\ & & & \lambda_{32}\,(2\times 2) \end{pmatrix}$$

since regular representation

$$Reg = D^{(1)} \oplus D^{(2)} \oplus 2D^{(3)}$$

$$M^{(reg)}(e) = I$$

permute positions of masses

$$M^{(reg)}(R) = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos(\frac{2\pi}{3}) \end{pmatrix}$$

permute masses positions

local coordinates change

$$M^{(reg)}(P) = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \otimes \begin{pmatrix} -1&0\\0&1 \end{pmatrix}$$

$$\left.\begin{aligned} \chi^{(reg)}(R) &= 0 \\ \chi^{(reg)}(P) &= 0 \end{aligned}\right\} \text{unsurprisingly}$$

$$tr\left(M^{[reg]}(g)\,\mathcal{U}\right)$$

$$= tr\begin{pmatrix} M^1(g)\lambda_1 & & & \\ & M^2(g)\lambda_2 & & \\ & & M^3(g)\lambda_{31} & \\ & & & M^3(g)\lambda_{32} \end{pmatrix}$$

$$= \chi^1(g)\lambda_1 + \chi^2(g)\lambda_2 + \lambda_{31}\chi^3(g) + \lambda_{32}\chi^3(g)$$

For $g = e$,

$$Tr\left(M^{(reg)}(g)\,\mathcal{U}\right) = 6 = \lambda_1 + \lambda_2 + 2(\lambda_{31} + \lambda_{32})$$

For $g = R$

$$Tr\left(M^{(reg)}(g)\,\mathcal{U}\right) = \frac{3}{2} = \lambda_1 + \lambda_2 - (\lambda_{31} + \lambda_{32})$$

For $g = P$

$$Tr\left(M^{(reg)}(P)\,\mathcal{U}\right) = 3 = \lambda_1 - \lambda_2$$

Solve!

$\lambda_1 = 3$

$\lambda_2 = 0$

$\lambda_{31} + \lambda_{32} = 3/2$

From physics, we know $\mathcal{U}$ is symmetric, and

$$\mathcal{U} = \begin{bmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -3 \\ & 3 & 0 & 0 & -\sqrt{3} & -3 \\ & & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ & & & 3 & \sqrt{3} & -3 \\ & & & & 2 & 0 \\ & & & & & 6 \end{bmatrix}$$

Original Problem:

Solve the e-value problem $U\vec{a} = \frac{m\omega^2}{k}\vec{a}$ to find frequencies.

Given two ~~rotational~~ translational modes and a ~~translational~~ rotational mode, we know that the e-value 0 has multiplicity 3.

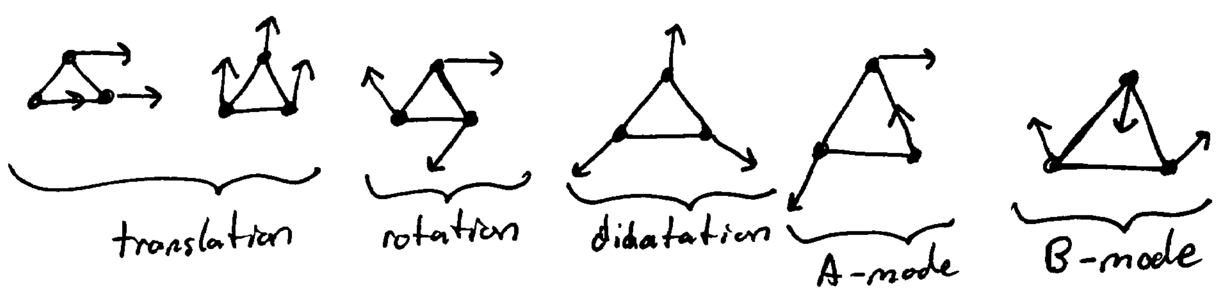
So $\lambda_1 = 3$

$\lambda_2 = 0$

$\lambda_{31} + \lambda_{32} = 3/2$, each of λ_{3i} has multiplicity 2.

So either $\lambda_{31} = 0$ or $\lambda_{32} = 0$.

Can find that the modes are:



Quantum Mechanics Problem:

Have a hamiltonian H and eigenbasis of wave functions, along with eigenvalues, $\{|\psi_n\rangle, E_n\}$.

Let G be a symmetry of H. For $g \in G$, the action preserves e-vals.

$$H\,(g \cdot |\psi_n\rangle) = E_n\,(g \cdot |\psi_n\rangle)$$

Fixing g and n,

$$Hg \cdot |\psi_n\rangle = E_n g \cdot |\psi_n\rangle = g \cdot E_n |\psi_n\rangle = g \cdot H|\psi_n\rangle$$

Since true for arbitrary n, then

$$Hg = gH \iff [H, g] = 0.$$

Hence,

<u>Claim 1</u>: each energy eigenspace forms a representation of g.

When is this an irrep?

Claim 2: If the action of G on the Hilbert space of states is transitive, then each eigenspace forms an irrep of G.

Defn: If G is a group acting on Hilbert space V, then the action is transitive iff $G|\psi_n\rangle = V$ for each $|\psi_n\rangle$ in the basis of V.

Proof of Claim 2: If an eigenspace H_{E_n} has a nontrivial invariant subspace V, then if $v \in V$, by the transitivity of the group action, $gv \in H_{E_n} \setminus V$ for some $g \in G$, $\divideontimes$.

Defn: "Normal degeneracies" are when the group action is transitive.

If E_n is the eigenspace, let $\{|\psi_{n,k}\rangle : k = 1 \dots \dim V_n\}$ be a basis for E_n

$$g \cdot |\psi_{n,k}\rangle = \sum_{l=1}^{\dim V_n} |\psi_{n,l}\rangle \underbrace{\Gamma^{(n)}(g)_{lk}}_{\text{matrix of } g \text{ in representation } \Gamma^{(n)} \text{ on } E_n}$$

Claim: The $\Gamma^{(n)}$ form a representation of G

Proof: WTS $\Gamma^{(n)}(gh) = \Gamma^{(n)}(g)\,\Gamma^{(n)}(h)$

$$(gh) \cdot |\psi_{n,k}\rangle = \sum_{l=1} |\psi_{n,l}\rangle \Gamma^{(n)}(gh)_{lk}$$

$$\parallel$$

$$g \cdot \left(\sum_{m=1} |\psi_{n,m}\rangle \Gamma^{(n)}(h)_{mk} \right) = \sum_{m=1} \sum_{r=1} |\psi_{n,r}\rangle \Gamma^{(n)}(g)_{rm} \Gamma^{(n)}(h)_{mk}$$

$$= \sum_{r=1} |\psi_{n,r}\rangle \left(\Gamma(g)\Gamma(h) \right)_{rk}$$

$$\Longrightarrow \quad \Gamma^{(n)}(gh) = \Gamma^{(n)}(g)\,\Gamma^{(n)}(h) \quad \blacksquare$$

Thanks to Mr. Gram-Schmidt, we may assume that $\{|\psi_{n,k}\rangle\}$ is an orthonormal basis. Then the Γ form a unitary representation of G

<u>Proof:</u> $\langle \psi_{n,l} | \psi_{n,k} \rangle = \delta_{lk}$. Without loss, the inner product is group invariant

$$\delta_{lk} = \langle g \cdot \psi_{n,l} | g \cdot \psi_{n,k} \rangle = \sum_{r,s} \langle \psi_{n,r} | \Gamma^{(n)}(g)^*_{rk} \, \Gamma^{(n)}(g)_{sl} | \psi_{n,s} \rangle$$

$$= \sum_{r,s} \Gamma^{(n)}(g)^*_{rk} \, \Gamma^{(n)}(g)_{sl} \, \delta_{rs}$$

$$= \sum_{s} \Gamma^{(n)}(g)^*_{sk} \, \Gamma^{(n)}(g)_{sl}$$

$$= \left(\Gamma^{(n)}(g)^* \, \Gamma^{(n)}(g) \right)_{kl} \implies \Gamma^{(n)}(g)^{-1} = \overbrace{\Gamma^{(n)}(g)^*}^{\text{Hermitian}}$$

Now assume G is a finite abelian group.

Eg. $G = \mathbb{Z}/n\mathbb{Z}$

Character Table

	C_1	C_2	C_3	$\cdots$	C_n
$D^{(1)}$	1	1	1	$\cdots$	1
$D^{(2)}$	1	ω	ω^2	$\cdots$	ω^n
$\vdots$	1	ω^2	ω^3	$\cdots$	ω
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	ω^{n-1}
$D^{(n)}$	1	ω^n	ω	$\cdots$	

<u>Coproduct:</u>

$$\begin{tikzcd} & C \arrow[dl] \arrow[d] \arrow[dr] & \\ A \arrow[r, leftrightarrow] & A \times B \arrow[r, leftrightarrow] & B \end{tikzcd}$$

$\oplus$

Bloch's Theorem:

Period l for translations

$x \mapsto x+a \qquad L = la$

$$(T_a(f))(x) = f(x+a) \qquad T_a^{l} = \underbrace{T_a \circ T_a \circ \cdots \circ T_a}_{l \text{ times}} = 1$$

Energy eigenspaces are all l-dimensional, representation matrix $e^{\frac{2\pi i}{l} q}$

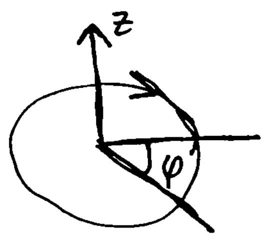
$$T_a(f_q)(x) = e^{\frac{2\pi i q}{l}} f_q(x) = e^{\frac{2\pi i}{la}(qa)} f_q(x) = e^{i\frac{2\pi q}{L}a} f_q(x)$$

Let $k = \frac{2\pi q}{L} \qquad T_a(f_k)(x) = e^{ika} f_k(x)$

Want to solve the functional equation: $f_k(x+a) = e^{ika} f_k(x)$

Has general solution $e^{ikx} u_k(x)$ where u_k is periodic with period a.

Constant magnetic field in $+z$ direction, symmetries are radial around z-axis. Rotations form a group.



The representation satisfies

$$\Gamma(\varphi_1)\Gamma(\varphi_2) = \Gamma(\varphi_1 + \varphi_2)$$

Solutions to this functional equation are only the exponential.

$$\Gamma(\varphi) = e^{ia\varphi}$$

The representation of this infinite cyclic group means that the wavefunctions here take the form as in Bloch's Theorem $e^{im\varphi}u(r)$.

Say G is a finite group of symmetries, $\mathcal{L}$ a lagrangian, $\psi_1 \dots \psi_n$ the operators on the system, associated with $V^{(1)} \dots V^{(n)}$.

$$\psi_i \longrightarrow \left(M^{[\alpha i]}(g)\,\psi_i\right)$$

Tensor Products of Representations

If V is a vector space, $V^* = \{f : V \to \mathbb{C},\ f \text{ linear}\}$.
If $\{e_i\}$ is a basis for V, $\{f^j\}$ is a basis for V^* where $f^j(e_i) = \delta_{ij}$

Defn: A map is multilinear $T : V_1 \times \dots \times V_n \longrightarrow \mathbb{C}$ if it is linear in each argument.

Let $T : V^* \times \dots \times V^* \longrightarrow \mathbb{C}$; take n fixed vectors in V, $\vec{v}_1, \dots, \vec{v}_n$

$$T(f_1, \dots, f_n) = \prod_{i=1}^{n} f_i(v_i) \qquad \text{Observe } T \text{ is multilinear.}$$

Define $(f_1 \otimes f_2 \otimes \dots \otimes f_n)(v_1, \dots, v_n) = T(f_1, \dots, f_n) = \prod_{i=1}^{n} f_i(v_i)$.

$G \in \mathrm{Bilin}(V^* \times V^*; \mathbb{C})$

$$G(\vec{v}, \vec{w}) = \sum_{i,j} G(v_i \vec{e}_i, w_j \vec{e}_j)$$

$$= \sum_{i,j} G(e_i, e_j)\, v_i w_j$$

$$= \sum_{i,j} G(e_i, e_j)\, f_i(v) f_j(w)$$

$$= \sum_{i,j} G(e_i, e_j)\,(f_i \otimes f_j)(v, w)$$

$$\begin{tikzcd}
V_1^* \times V_2^* \times \cdots \times V_n^* \arrow[rr, "\otimes"] \arrow[dr, "F"'] & & V_1^* \otimes V_2^* \otimes \cdots \otimes V_n^* \arrow[dl, "G"] \\
& W &
\end{tikzcd}$$

F is multilinear, W another vector space.
Then there is a unique linear $G : \bigotimes_{i=1}^{n} V_i^* \longrightarrow W$
such that $\forall\, (\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n) \in V_1^* \times V_2^* \times \cdots \times V_n^*$,

$$F(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = G(\vec{v}_1 \otimes \vec{v}_2 \otimes \cdots \otimes \vec{v}_n).$$

<u>Theorem:</u> The tensor product exists and is unique up to isomorphism.

<u>Proof:</u> <u>Existence</u>: already done $\checkmark$.

<u>Uniqueness</u>: Suppose that there were two tensor products.

$$\begin{tikzcd}
& V^n \arrow[dl, "\otimes"'] \arrow[dr, "\otimes'"] & \\
V^{\otimes n} \arrow[rr, bend right, "\phi"'] & & V^{\otimes' n} \arrow[ll, "\psi"']
\end{tikzcd}$$

$\psi \circ \otimes' = \otimes$
$\phi \circ \otimes = \otimes'$

$\psi \circ (\phi \circ \otimes) = \otimes$
$\phi \circ (\psi \circ \otimes') = \otimes'$

$\Longrightarrow \psi, \phi$ inverses
hence ψ, ϕ IM. ∎

Given two representations, a finite group G

$$\Gamma^1 : G \to \cancel{V_1}\, GL(V_1, V_1)$$
$$\Gamma^2 : G \to \cancel{V_2}\, GL(V_2, V_2)$$

Is there a representation $\Gamma^1 \otimes \Gamma^2 : G \to \cancel{V_1 \otimes V_2}$.
$GL(V_1, V_1) \otimes GL(V_2, V_2)$

Say we have maps
$$A_i : V_i \to W_i$$

Make $A_1 \otimes \cdots \otimes A_n : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_n$, as follows:

$$\begin{tikzcd}
V_1 \times \cdots \times V_n \arrow[rr, "(A_1, \dots, A_n)"'] \arrow[d, "\otimes"'] \arrow[rrrd, dashed, bend left, "\otimes \circ (A_1, \dots, A_n)"] & & W_1 \times \cdots \times W_n \arrow[rd, "\otimes"] & \\
V_1 \otimes \cdots \otimes V_n \arrow[rru, "G"] \arrow[rrr, "A_1 \otimes \cdots \otimes A_n"] & & & W_1 \otimes \cdots \otimes W_n
\end{tikzcd}$$

$$G \circ \otimes = (A_1, \dots, A_n)$$

$$\bigotimes_{i=1}^{n} A_i \, (v_1 \otimes \cdots \otimes v_n) = (A_1 v_1) \otimes (A_2 v_2) \otimes \cdots \otimes (A_n v_n).$$

So therefore,
$$\Gamma^1 \otimes \Gamma^2 (g) = \Gamma^1(g) \otimes \Gamma^2(g)$$

And if χ_1, χ_2 are the characters of Γ^1, Γ^2 respectively, then

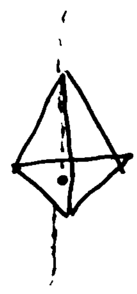
$$\chi_1 \otimes \chi_2 (g) = \chi_1(g)\, \chi_2(g).$$

$$\left(\Gamma^1 \otimes \Gamma^2(g)\right)^{ij}_{kl} = \Gamma^1(g)^i{}_k \, \Gamma^2(g)^j{}_l$$

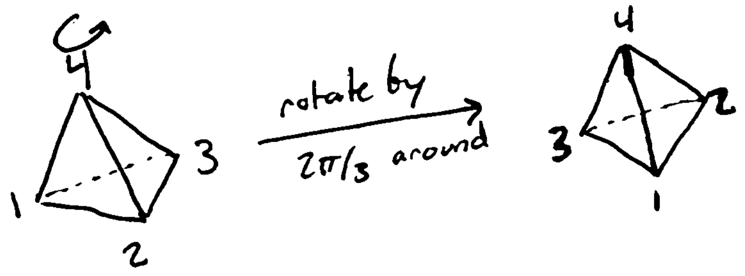
The Tetrahedral Group

Symmetries of the tetrahedron, that send it to itself

rotation around a vertex
by multiples of 120°.



$T \cong S_4$



Corresponds to permutation: $(123)(4)$

Wigner - Eckart Theorem:

H is the hamitonian of some system, $H = H_0 + H_{breaking}$

H_0 is invariant under a symmetry G

$\langle x | H_{breaking} | \psi \rangle$

Part (1): States belonging to different inequivalent irreps of a symmetry group are orthogonal. Within the same irrep, different states are also orthogonal.

$$\langle \chi_j^{(\alpha)} | \phi_a^{(\beta)} \rangle = \delta^{\alpha\beta} \delta_{aj} \underbrace{\langle \chi \| \phi \rangle}_{\text{reduced matrix element}}$$

vectors in the representation space

Proof:

$\Gamma : G \to GL(V)$ is representation, $\Gamma(g)$ has matrix $D(g)$

$$\Gamma(g) | \phi_a^{(\beta)} \rangle = \sum_b | \phi_b^{(\beta)} \rangle D^{(\beta)}(g)_{ba}$$

$$\Gamma(g) | \chi_i^{(\alpha)} \rangle = \sum_j | \chi_j^{(\alpha)} \rangle D^{(\alpha)}(g)_{ji}$$

Let $\langle \chi_j^{(\alpha)} | \phi_a^{(\beta)} \rangle =: X^{\alpha\beta}_{ja}$.

$$\langle \chi_j^{(\alpha)} | \Gamma(g)\phi_a^{(\beta)} \rangle = \langle \Gamma(g^{-1})\chi_j^{(\alpha)} | \phi_a^{(\beta)} \rangle$$

since we may assume representation unitary.

$$\sum_b \langle \chi_j^{(\alpha)} | \phi_b^{(\beta)} \rangle D^{(\beta)}(g)_{ba} = \sum_i \underbrace{D^{(\alpha)}(g^{-1})^*_{ij}} \langle \chi_i^{(\alpha)} | \phi_a^{(\beta)} \rangle$$

$$D^{(\alpha)}(g^{-1})^*_{ij} = \left[D^{(\alpha)}(g)^{-1}\right]^*_{ij}$$

$$= D^{(\alpha)}(g)_{ij}$$ since unitary, $D^{(\alpha)}(g)^* = D^{(\alpha)}(g)^\dagger$

$$\Longrightarrow \sum_b X^{\alpha\beta}_{jb} D^{(\beta)}(g)_{ba} = \sum_i D^{(\alpha)}(g)_{ji} X^{\alpha\beta}_{ia}$$

$$\Longrightarrow \left(X^{\alpha\beta} D^{(\beta)}(g)\right)_{ja} = \left(D^{(\alpha)}(g)\, X^{\alpha\beta}\right)_{ja}$$

Schur 2: $\alpha = \beta$ gives $\delta^{\alpha\beta}$

Schur 1: $X^{\alpha\beta}_{ia} = \delta^{\alpha\beta}\delta_{ia}\langle X \| \phi \rangle$ since multiple of identity b/c $X^{\alpha\beta}$ commutes w/ matrices $D^{(\alpha)}(g)$

Part (2): $P(g)\,\mathcal{O}_i\,P(g)^{-1} = R_i{}^j(g)\,\mathcal{O}_j$

Is this a representation of G on the operators?

Given a representation on a vector space, form representation on operators on vector space.

$$P(gh)\,\mathcal{O}_i\,P(gh)^{-1} = R(gh)^j{}_i\,\mathcal{O}_j$$

$$= P(g)\underbrace{P(h)\,\mathcal{O}_i\,P(h)^{-1}}P(g)^{-1}$$

$$= \underbrace{\cancel{R_i^k \mathcal{O}_k}\ \cancel{R_i^k \mathcal{O}_k}\ R_i{}^k(h)\,\mathcal{O}_k}$$

$$= R^j{}_k(g) R^k{}_i(h)\mathcal{O}_j = \left(R(g)R(h)\right)^j{}_i\,\mathcal{O}_j$$

Yes.

$$P(g)\left(\mathcal{O}_i \,|\phi^{(\alpha)}_a\rangle\right) = \underbrace{P(g)\,\mathcal{O}_i\,P(g)^{-1}}_{=\;R^j{}_i\,\mathcal{O}_j}\;\underbrace{P(g)\,|\phi^{(\alpha)}{}_a\rangle}_{\sum_b |\phi^{(\alpha)}{}_b\rangle D^{(\alpha)}(g)_{ba}}$$

$$= \sum_j \sum_b \left(\mathcal{O}_j \,|\phi^{(\alpha)}_b\rangle\right)\underbrace{\left(R^j_i(g)\,D^{(\alpha)}(g)\right)_{ba}}_{\text{"Clebsch-Gordon coefficients"}}$$

$$R_{\mathcal{O}} \otimes R^{(\alpha)} = \bigoplus_{\beta \text{ irrep}} a^{(\beta)} R^{(\beta)}$$

representation of G on the operators $\mathcal{O}_i$ ("$R_{\mathcal{O}}$")

"Clebsch-Gordon-Series"

$$\langle \chi^{(\alpha)} |\, \mathcal{O}^{(\gamma)}_i \,\phi^{(\beta)}\rangle = \delta^{\alpha,\gamma\otimes\beta}\underbrace{\left(\quad\right)}_{\text{conversion from tensor matrices to irrep matrices}}\underbrace{\langle \chi \,\|\, \mathcal{O} \,\|\, \psi\rangle}_{\text{reduced matrix elements}}$$

Crystallographic Point Groups:

3D Lattice of points such that there exist $\{a_1, a_2, a_3\}$ such that any point in the lattice can be reached from any other point via a translation $n_1 a_1 + n_2 a_2 + n_3 a_3$ for $n_1, n_2, n_3 \in \mathbb{Z}$.

Which groups fix a point in the lattice but respect the translational symmetry?

Notation (Schönflies Notation):

$C_n \equiv$ n-fold rotations by $2\pi/n$ around a specified axis

reflections:
- $\sigma_h \equiv$ mirror perpindicular to rotation axis
- $\sigma_v \equiv$ mirror across axis of rotation

$S_2 \equiv$ inversion $\vec{x} \mapsto -\vec{x}$ $\qquad S_2 = C_2\,\sigma_h$

$S_n = C_n\,\sigma_h$

How many different groups?

For single-axis crystals, $n = 1, 2, 3, 4, 6$

<u>The space group:</u>

Group of rotation/translations of space.

$g(R, \vec{a})$ where R is an orthogonal matrix, $\vec{a}$ a vector

$$g(R, \vec{a})\vec{r} = R\vec{r} + \vec{a}.$$

$$S = \{g(R, \vec{a}) : R \in O(3), \vec{a} \in \mathbb{R}^3\}$$

is the space group. For any particular crystal, not every $(R, \vec{a})$ pair may yield a symmetry.

For a crystal with primitive vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, $R \in O(3)$ preserving crystal symmetry

$$g(R, \vec{\alpha}) = T(\vec{\ell})\, g(R, \vec{t})$$

$T(\vec{\ell})$ is translation

$$\vec{t} = \sum t_i \vec{a}_i \quad 0 \le t_i \le 1$$

This is unique:

remove integer component of a_i in α such that

$$\vec{\alpha} = \vec{t} + \vec{\ell}$$

non-integer components.

$$g(R, \vec{t})\, g(R, \vec{t}'^{-1})$$
$$= g(\mathbb{I}, R^{-1}(\vec{t}' - \vec{t})) = T(\underbrace{R^{-1}(\vec{t}' - \vec{t})}_{\ell \text{ a lattice vector}})$$

$\Rightarrow \vec{t}' - \vec{t} = R\vec{\ell} \leftarrow$ lattice vector

$\vec{t}' = \vec{t} + R\vec{\ell} \leftarrow$ already removed as many lattice vectors as possible.

$\Rightarrow \vec{t} = \vec{t}'$

(1) Space Group S $\quad g(R, \vec{\alpha})$

(2) G Point Group $\quad g(R, \vec{\alpha} = 0)$

(3) Translation Group T $\quad g(\mathbb{1}, \vec{\alpha})$

Is T normal in S? Yes

$$g(R, \vec{\alpha})\, T(\vec{\ell})\, g(R, \alpha)^{-1} = T(R\vec{\ell}) \in T.$$

So $\quad S/T \cong G$

What does the matrix of an element of the point group look like?

$$g(R,\vec{o})a_i = \sum_{j=1}^{3} R_{ji}a_j \quad \text{is a lattice vector, so } R_{ji}\in\mathbb{Z}\ \forall i,j$$

In this case, R may not be orthogonal with respect to the basis $\{a_1,a_2,a_3\}$, since it isn't an orthonormal basis.

But R will be orthogonal after a similarity transform by the change of basis matrix to the standard one.

$$\vec{a}_i = \sum_a X_{ai}\vec{e}_a \qquad e_a = \sum_i (X^{-1})_{ia}\, a_i \qquad \bar{R} = X R X^{-1}$$

$\bar{R}$ — R in standard basis; R — R in a_i basis.

Dual basis $\{\tilde{b}^1,\tilde{b}^2,\tilde{b}^3\}$ for dual space such that $\tilde{b}^i(a_j)=\delta^i_j$. If $\tilde{b}^i = \sum_b Y^i_b \tilde{e}^b \qquad \tilde{e}^b(e_a)=\delta^b_a$

$$\delta^i_j = \tilde{b}^i(a_j) = Y^i_b\,\tilde{e}^b(X^a_j\vec{e}_a) = Y^i_b X^a_j \delta^a_b = (YX)^i_j \implies Y = X^{-1}.$$

Rotations can be specified by an axis $\hat{n}$ and angle ω

$$R(\hat{n},\omega)_{ij} = \underbrace{\hat{n}_i\hat{n}_j,\ \delta_{ij},\ \varepsilon_{ijk}\hat{n}_k}_{\text{linear combination of these things}}$$

parity of permutation (ijk).

$P_{ij} = \delta_{ij} - \hat{n}_i\hat{n}_j$ projection operator onto plane orthogonal to $\hat{n}$.

$$R(\hat{n},\omega)_{ij} = \hat{n}_i\hat{n}_j + (\delta_{ij}-\hat{n}_i\hat{n}_j)\cos\omega + \sin\omega\ \varepsilon_{ijk}\hat{n}_k$$

General form of rotation matrix.

$\mathrm{Tr}(R)\in\mathbb{Z}$ because it's integral in the crystal basis.

$\mathrm{tr}(R) = \pm(1+2\cos\omega)\in\mathbb{Z}$ ~~$\cos\omega$~~

Allowed rotations: $\cos\omega = \frac{n}{2}$ $n\in\mathbb{Z}$. If $\omega=\frac{2\pi}{n}$, then $n=1,2,3,4,6$.

Schönflies Notation (Again)

$$C_n \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$$

Groups are classified by number of rotatonal axes

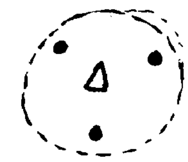
<u>Uniaxial</u>

C_n for $n = 1, 2, 3, 4, 6$

<u>Stereograms</u>

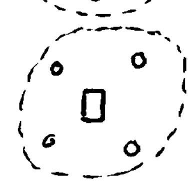
C_1

C_3

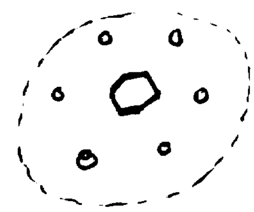


C_2

C_4



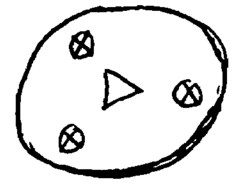
C_6



Mirror in the plane

C_{nh} h for horizontal reflection across plane

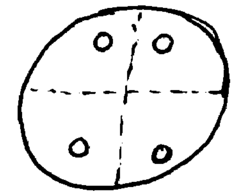
eg. C_{3h}



solid lines now (?)

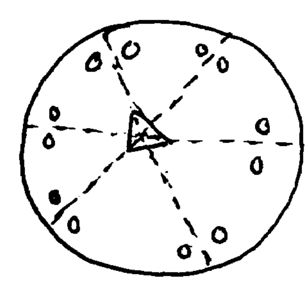
C_{nv} v for vertical mirror

eg. C_{2v}



need all mirrors that show symmetry

C_{3v}



Manifolds

What does it have to do with group theory?

Differentiating matrices in groups like $SO(3)$. Tangent bundles to a Lie group have more structure than a vector space.

Defn: Topological Space is a set X with a collection $\tau \subseteq 2^X$ such that

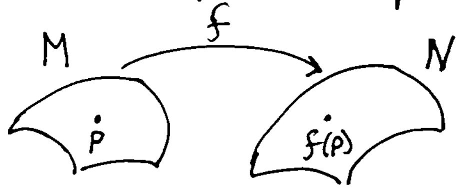
(1) $\phi, X \in \tau$

(2) τ closed under arbitrary union

(3) τ closed under finite intersection

(τ is open sets)

A Manifold is locally diffeomorphic to $\mathbb{R}^n$.



(1) How do we assign coordinates to M? A collection of open sets $\{U_\alpha\}$.

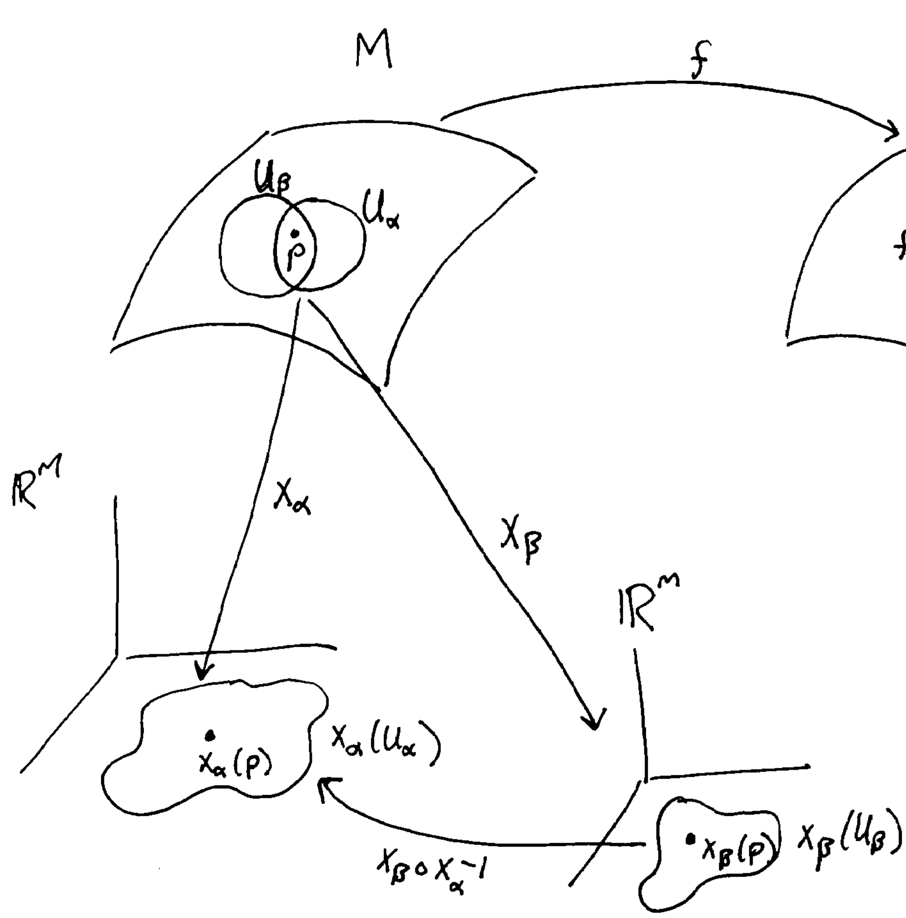
Manifold Data:

(1) Collection of Charts $\{U_\alpha\}$, $M = \bigcup_\alpha U_\alpha$

(2) Maps $\{X_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}$ which are continuous, with continuous inverse.

homeomorphisms

(3) $X_\beta \circ X_\alpha^{-1} : X_\alpha(U_\alpha \cap U_\beta) \longrightarrow X_\beta(U_\alpha \cap U_\beta)$ is C^∞ for all α, β



<u>Examples:</u>

$\mathbb{R}^n$ w/ identity

S^n by stereographic projection, with charts $\begin{cases} U_+ = S^N \setminus \{\text{north pole}\} \\ U_- = S^N \setminus \{\text{south pole}\}. \end{cases}$

- cannot have a single chart because its compact, would have image which is open and compact in $\mathbb{R}^n$.

<u>Defn:</u> $f: M \to N$ is <u>differentiable</u> map between manifolds (M, U_α, x_α) and (N, V_β, y_β) iff $f_{\beta\alpha} := y_\beta \circ f \circ x_\alpha^{-1}$ is differentiable at $x_\alpha(p)$ for all $p \in M$.

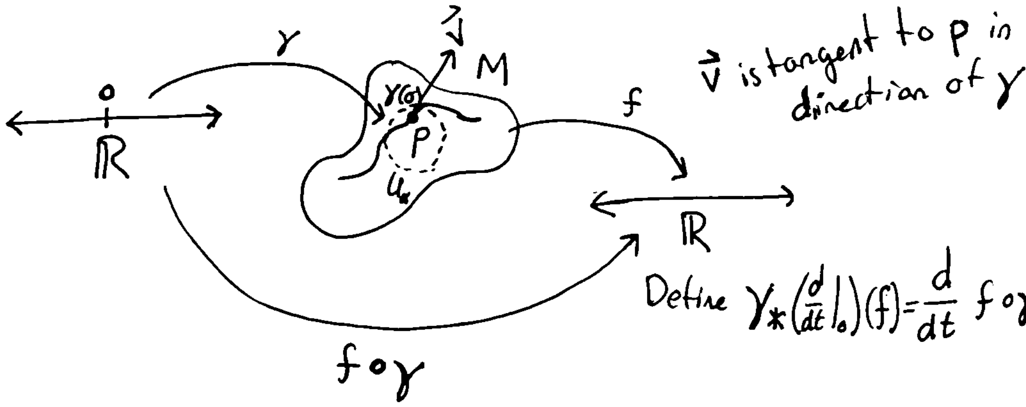
If $f_{\delta\gamma} = y_\delta \circ f \circ x_\gamma^{-1}$ is differentiable, then we know by C^∞-ness of the transition functions that

$$f_{\delta\gamma} = \underbrace{(y_\delta \circ y_\beta^{-1})}_{C^\infty} \circ \underbrace{y_\beta \circ f \circ x_\alpha^{-1}}_{f_{\beta\alpha} \text{ is } C^\infty} \circ \underbrace{(x_\alpha \circ x_\gamma^{-1})}_{C^\infty}.$$

Tangent Spaces:

A space "tangent" to a point on the manifold

$\gamma: \mathbb{R} \to M$ is a curve in M passing through p.



Define $\gamma_*\left(\frac{d}{dt}\Big|_0\right)(f) = \frac{d}{dt} f\circ\gamma(0)$

$$\gamma_*\left(\frac{d}{dt}\Big|_0\right)(f) = \frac{d}{dt}\Big|_0 f\circ\gamma = \frac{d}{dt}\Big|_0 (f\circ x_\alpha^{-1})\circ(x_\alpha\circ\gamma)$$

$f_\alpha: \mathbb{R}^m \to \mathbb{R}$ $\qquad$ $x_\alpha\circ\gamma: \mathbb{R}^m \to \mathbb{R}$

$(x_\alpha^1(\gamma(t)) \ldots x_\alpha^m(\gamma(t)))$

$$\frac{d}{dt} f_\alpha(x_\alpha^1(\gamma(t)), \ldots, x_\alpha^n(\gamma(t)))$$

$$= \sum_{i=1}^{n}\left(\frac{d x_\alpha^i(\gamma(t))}{dt}\right)\left(\frac{\partial}{\partial x^i} f_\alpha\right)$$ is a function $\mathbb{R}\to\mathbb{R}$

Defn: The tangent space to M at p, $T_p(M)$ is

$$T_p(M) := \left\{\gamma_*\left(\frac{d}{dt}\Big|_0\right) \text{ for all } C^\infty \text{ curves } \gamma: \mathbb{R}\to M \text{ with } \gamma(0)=p\right\}.$$

($\cdot$) $T_p(M)$ is a vector space. Given γ_1, γ_2, $f: M\to\mathbb{R}$, all C^∞.

$$\left[(\gamma_1)_* + (\gamma_2)_*\right]\left(\frac{d}{dt}\Big|_0\right)(f) := \frac{d}{dt}\Big|_0 (f\circ\gamma_1) + \frac{d}{dt}(f\circ\gamma_2) = \gamma_{1*}\left(\frac{d}{dt}\Big|_0\right)(f) + \gamma_{2*}\left(\frac{d}{dt}\Big|_0\right)(f).$$

Note that $\gamma_*\left(\frac{d}{dt}\Big|_0\right)(f) = \sum_i \frac{d x_\alpha^i(\gamma(0))}{dt} \frac{\partial}{\partial x^i} f = \left(\sum_i a^i \frac{\partial}{\partial x^i}\right) f$

Defn 2: $T_p(M)$ is the space of derivations on $F^\infty :\doteq \{f_i : M \to \mathbb{R}, C^\infty\}$

$X_p(af+g) = aX_p f + X_p g$

$X_p(fg) = (X_p f)g + f(X_p g)$ $\qquad$ They are the same!

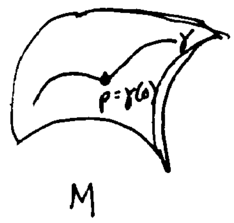
$(aX_p + Y_p)(f) = aX_p f + Y_p f$

<u>Mappings of manifolds and Lie groups</u>

Suppose we have a map $f: M \to N$ between manifolds M and N.
What does this map say about the corresponding tangent spaces at $p \in M$.

$$f_*: T_p(M) \longrightarrow T_{f(p)}(N)$$

Suppose that we have a path $\gamma(t)$ through M, $\gamma: \mathbb{R} \to M$, ~~$\gamma(0)=p$~~
$\gamma(0)=p$



If we have a function $h: M \to \mathbb{R}$, then we can pick out a particular tangent vector in $T_p(M)$, by having it act on h (evaluated on this path γ).

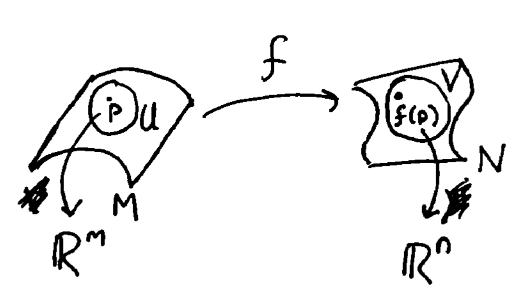
$$X_p(h) = \left[\frac{d}{dt}(h \circ \gamma(t))\right]_{t=0} = (h\circ\gamma)'(0)$$

Consider also $k: N \to \mathbb{R}$, another function on N. Can send the path γ to N by composing with f, $f \circ \gamma: N \to \mathbb{R}$. Defines a tangent vector in $T_{f(p)}(N)$:
$$Y_{f(p)}(k) = (f_*(X_p))(k) = (k \circ (f \circ \gamma))'(0) = X_p(k \circ f)$$

This idea defines the differential:

<u>Defn:</u> If $f: M \to N$ is a map between manifolds that is C^∞, then the <u>differential</u> of f at $p \in M$ is ~~defined by~~ given by the linear map
$$f_*: T_p(M) \longrightarrow T_{f(p)}(N)$$
defined by $\left(f_*(X_p)\right)(k) = X_p(k \circ f)$ for $X_p \in T_p(M)$ and $k: N \to \mathbb{R}$.

Why is it a "differential?"



$x: U \to \mathbb{R}^m$ $\quad$ $y: V \to \mathbb{R}^n$

Compute f_* in local coordinates x^i, y^j, which are coordinates of x in $\mathbb{R}^m$, y in $\mathbb{R}^n$, respectively. There is a coordinate basis

$$\frac{\partial}{\partial x^i} \in T_p(M), \quad i=1,\dots,m$$

$$\frac{\partial}{\partial y^j} \in T_{f(p)}(N), \quad j=1,\dots,n$$

In this basis, f_* is a matrix telling us where $\frac{\partial}{\partial x^i}$ is mapped:

$$f_*\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n} (f_*)_{ij} \frac{\partial}{\partial x^j}$$

Let k be the map which extracts the k^{th} coordinate, y^k ← not power/exponent, but index

$$\sum_{j=1}^{n} (f_*)_{ij} \frac{\partial}{\partial y^j} y^k = \sum_{j=1}^{n} (f_*)_{ij} \delta_{jk} = (f_*)_{ik}$$

$$f_*\left(\frac{\partial}{\partial x^i}\right) y^k = \frac{\partial}{\partial x^i}\left(\left(y^k \circ f \circ x^{-1}\right) \circ x\right)$$

where $x: U \to \mathbb{R}^m$ is coordinate map

$$x(p) = (x^1(p), \ldots, x^m(p))$$

$$\Longrightarrow \quad (f_*)_{ij} = \left[\frac{\partial}{\partial x^i}\left(y^j \circ f \circ x^{-1}\right)\right]_{x(p)} = \left(\partial_i \left(y^j \circ f \circ x^{-1}\right)\right)(x(p))$$

(f_*) is the Jacobian matrix of $y \circ f \circ x^{-1}: \mathbb{R}^m \longrightarrow \mathbb{R}^n$!

Suppose $f: M \longrightarrow \mathbb{R}$, what is f_*?

We let y be the global coordinates of $\mathbb{R}$. We can write:

$$\underset{\substack{\uparrow \\ j \in \{1\}}}{(f_*)_{ij}} = (f_*)_i = \partial_i (f \circ x^{-1}), \text{ so } f_*\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}(f \circ x^{-1}).$$

f_* maps the tangent space of M to the tangent space of $\mathbb{R}$, $\mathbb{R}^*$

$$f_*: T(M) \longrightarrow T(\mathbb{R}) = \mathbb{R}$$

f_* is an element of the dual space of $T(M)$, which is called $T^*(M)$, the <u>cotangent</u> <u>space</u>.

f_* is in this case given by df, a 1-form, e.g. with $\frac{\partial}{\partial x^i}$ a corresponding 1-form called dx^i.

Lie Groups:

Groups that are manifolds too!

Imagine the group of rotations in the complex plane.
To rotate by θ, multiply by $e^{i\theta}$.
Form a group labelled by a continuous parameter.
Identify with the circle in $\mathbb{C}$ to get manifold structure.

This is a simple example of a Lie Group.

Defn: A Lie Group G is a group that is also a C^∞ manifold, such that multiplication and inversion are C^∞ as well.

Once we have a Lie Group, we can define Lie Group homomorphisms.

Defn: If G_1, G_2 are Lie Groups, then
$\phi: G_1 \to G_2$ is a lie group homomorphism iff
(1) It is a group homomorphism
(2) It is a C^∞ map between G_1, G_2

Defn: if $\phi: G \to H$ is a Lie Group HM, invertible, and ϕ^{-1} is also a Lie Group HM, then ϕ is a Lie Group Isomorphism.

Representations:

The set of invertible matrices $GL(n,\mathbb{R})$ is also a Lie Group.
The elements are "vectors" in $\mathbb{R}^{n^2}$, and the group structure is given by matrix multiplication.

When we considered finite groups, a representation was a HM $\rho: G \to GL(n,\mathbb{R})$.
If ρ is a lie group HM, G is a Lie Group, then this defines a representation of G on $\mathbb{R}^n$.

For the finite groups we had particular matrices, but for Lie groups we get continuums of matrices.

Classical Lie Groups:

The classical lie groups are subsets of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$.

(1) Some have interpretations as symmetries of shapes

(2) All can be seen as symmetries of an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$.

Examples:

Special Orthogonal Group $SO(2n)$ or $SO(2n+1)$

Rotational Symmetries of S^{2n-1} or S^{2n}, respectively.

Orthogonal matrices with determinant $+1$.

Lie Groups

Suppose G is a ~~group~~ manifold and there is a group operation $\circ : G\times G \to G$.
If ~~$\circ$ is a~~ $\circ$ and -1 are C^∞ maps between manifolds, then G is a Lie Group.

Canonical Example:

C^∞ but not analytic

$$f(x) = \begin{cases} e^{-1/x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Examples:

$GL(n,\mathbb{R})$ or $Gl(n,\mathbb{C})$ with matrix multiplication

As topological structures, $Gl(n,\mathbb{C})$ is diffeomorphic to open subset of $\mathbb{C}^{n^2}$

Matrix multiplication is polynomial in each coordinate, so smooth

Open subset of a manifold is a manifold, so the compliment of $Gl(n,\mathbb{C})$ is $\{M : \det(M) = 0\} = \det^{-1}(\{0\})$ ← closed since det cts.

So $Gl(n,\mathbb{C})$ open subset of $\mathbb{C}^{n^2}$, hence a manifold.

By implicit function theorem, level sets of differentiable maps are also manifolds, so $\det^{-1}(\{1\}) = Sl(n,\mathbb{C})$ is also a manifold.

Orthogonal Group $O(n)$:

Define a quadratic form $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y}$.

Which transformations leave $\langle \cdot , \cdot \rangle$ invariant? Orthogonal Matrices $\mathcal{O}^T \mathcal{O} = I$.

Subgroup of $O(n)$ with determinant 1 is $SO(n)$.

$SO(n)$ is not simply connected, spin(n) is the simply connected covering space of $SO(n)$. $\dfrac{Spin(n)}{\mathbb{Z}/2\mathbb{Z}} \cong SO(n)$.

$Spin(3) = SU(2)$

$Spin(6) = SU(4)$

$\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$.

Unitary Group $U(n)$:

The matrices which preserve the quadratic form $\vec{x}^* \vec{y} = \langle x, y \rangle$.

$U^* U = I$, $|\det(U)| = 1 \Longrightarrow \det(U) = e^{i\theta}$

$SU(n)$ is subgroup with determinant 1.

$SU(n)$ is simply connected, with $\pi_1(SU(n)) = \langle e \rangle$

What about preserving other quadratic forms?

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \vec{y}$$

The operators preserving this quadratic form are $O(p,q)$. The Lorentz Group is $SO(3,1)$.

Symplectic Group:

$sp(2n)$ is the set of operators preserving $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T J \vec{y}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. (Hamiltonian Systems)

Cartan Notation

Classical Lie Groups
$$\begin{cases} A_n = SU(n+1) \\ B_n = SO(2n+1) \\ C_n = Sp(2n) \\ D_n = SO(2n) \end{cases}$$

Exceptional Lie Groups

G_2, F_4, E_6, E_7, E_8

"Kac-Moody Algebras"
Infinitely ~~Generated~~ Many Generators for Lie Group

There are no other finitely generated Lie Groups.

Lie Algebras

ex: $SO(3)$ has generators R_x, R_y, R_z (rotation about x, y, z respectively)

Construct the tangent space $T_I(SO(3))$

Curves $\alpha(t), \beta(t), \gamma(t)$

$$R_x(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(\alpha(t)) & -s(\alpha(t)) \\ 0 & s(\alpha(t)) & c(\alpha(t)) \end{pmatrix}$$ where $c(\theta) = \cos\theta$, $s(\theta) = \sin\theta$

The tangent space is ~~curves~~ derivatives of curves through 0:

$$\left[\frac{d}{dt} R_x(t)\right]_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = L_x$$

$$\left[\frac{d}{dt} R_y(t)\right]_{t=0} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = L_y$$

$$\left[\frac{d}{dt} R_z(t)\right]_{t=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = L_z$$

For any element $\theta \in SO(3)$, as a function of t,

$\theta(t)^T\theta(t) = I$, let $\theta(0) = 1$

So $\left[\frac{d}{dt}\theta(t)^T\theta(t)\right]_{t=0} = \cancel{\theta'(t)^T + \theta'(t)} \Longrightarrow \theta'(t)^T + \theta'(t)$

but also $\frac{d}{dt}\theta(t)^T\theta(t) = \frac{d}{dt} I = \frac{d}{dt} I = 0$

$\theta'(t)^T + \theta(t) = 0 \Longrightarrow \theta'$ is anti-symmetric.

So the dimension of the tangent space is $\frac{n(n-1)}{2}$. ← above diagonal

Since $\det\theta = 1$ in the Special orthogonal group, we must have

$$\left[\frac{d}{dt}\det\theta(t)\right]_{t=0} = 0 \Longrightarrow (\det\theta(t))\, Tr(\theta(t)^T\theta'(t)) = 0$$

$$\Longrightarrow Tr(\theta(t)^T\theta'(t)) = 0$$

Space of Antisymmetric $n \times n$ matrices.

Lie Algebra of a Lie Group

Defn: if G is a Lie group, its Lie Algebra is $T_e(G)$, the tangent space to G at the identity.

Measure the failure to commute of g_1, g_2 by $C(g_1, g_2) = g_1 g_2 g_1^{-1} g_2^{-1}$.
$g_1, g_2 \in G$

To move to the tangent space, take two curves $\gamma_1(t), \gamma_2(t)$ in G, and let $h(t) = \gamma_1(t)\gamma_2(t)\underbrace{\gamma_1^{-1}(t)}\underbrace{\gamma_2^{-1}(t)}$. Demand that $h(0) = e$.

inverse element of $\gamma_i(t)$!

~~[illegible] that~~, in local coordinates (X_α) we have a taylor expansion for γ_i:

$$X_\alpha \circ \gamma_i(t) = X_\alpha(e) + A_i t + \frac{1}{2}(X_\alpha \circ \gamma_i)''(0) t^2 + \dots$$

$$X_\alpha \circ \gamma_i(t)^{-1} = X_\alpha(e) - A_i t + -\tfrac{1}{2}\left((X_\alpha \circ \gamma_i)''(0) - 2A^2\right)t^2 + \dots$$

So then $X_\alpha \circ h(t)$ has the form

$$X_\alpha \circ h(t) = X_\alpha(e) + t^2(A_1A_2 - A_2A_1) + \dots$$

"$h(\tau) = e + \tau[A, B]$" where $[A,B] = AB - BA$ reparameterize

if $A \in T_e(G)$ then $[A, B] \in T_e(G)$.

Defn: A <u>Lie Algebra</u> is a vector space L with an operation $[\cdot,\cdot]$ satisfying:

(1) $[X, Y] = -[Y, X]$ for all $X, Y \in L$.

(2) $[\lambda x + y, z] = \lambda[x, z] + [y, z]$ and $[x, \lambda y + z] = \lambda[x, y] + [x, z]$.

(3) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

Defn: A <u>Lie sub-algebra</u> is a vector subspace K of a Lie Algebra L such that $\forall\, x, y \in K$, $[x, y] \in K$.

Defn: if L_1, L_2 are <u>Lie sub-algebras</u> of L, $L_1 \cap L_2 = \{0\}$ and every vector $v \in L$ is $v = v_1 + v_2$ where $v_1 \in L_1, v_2 \in L_2$ then $L = L_1 \oplus L_2$.

$[L_1, L_2] = \{0\}$

[redacted]

Defn: An <u>ideal</u> I of a Lie-Algebra L is a vector subspace of L such that

$[x, i] \in I$ for all $x \in L$.

If $L = L_1 \oplus L_2$, then L_1, L_2 are ideals of L.

Defn: A <u>simple</u> Lie Algebra has no non-trivial ideals.

Defn: A <u>semi-simple</u> Lie Algebra has no abelian ideals.

Theorem: Every lie Algebra can be decomposed as a direct sum of simple Lie Algebras plus some number of abelian Lie Algebras, of which there is one, $\mathfrak{u}(1)$.

$$L = \left(\bigoplus_{i=1}^{n} L_i\right) \oplus \left(\bigoplus_{i=1}^{k} \mathfrak{u}(1)\right)$$

To classify Lie Algebras, just classify simple ones.

The algebra of $SO(3)$ is simple, because $[L_i, L_j] = i\underbrace{\varepsilon_{ijk} L_k}_{\text{sum over } k} \longrightarrow [L_a, L_b] = i\,\varepsilon_{abc} L_c$

<u>Generators of a Lie Algebra</u>: A basis for the Lie Algebra as a vector space $\{e_i\}$ such that $[e_i, e_j] = i\underbrace{f_{ij}{}^{k} e_k}_{\text{sum over } k}$ where $f_{ij}{}^{k}$ are called the "structure constants"

$[e_a, e_b] = i \sum_c f_{ab}{}^{c} e_c$

"Natural" Generators for $SO(3)$ (in Quantum)

$$L_z = L_0 = L_3 \qquad [L_0, L_\pm] = \pm L_\pm$$

$$L_\pm = \frac{L_1 \pm i L_2}{2} \qquad [L_+, L_-] = 2L_0$$

Defn: A Lie Algebra <u>homomorphism</u> $\phi: L \to K$ is a linear map such that $\phi([\ell_1, \ell_2]_L) = [\phi(\ell_1), \phi(\ell_2)]_K$

if ϕ is bijective, <u>isomorphism</u>

Defn: a representation of a Lie Algebra L on a vector space V_R is a ___ homomorphism between L and $Aut(V_R)$.

Lie Algebra

Examples: SO(3)'s algebra has a 3d representation SO(3), and also a 2D representation $L_0, L_\pm \longmapsto \frac{1}{2}\sigma^1, \frac{1}{2}\sigma^2, \frac{1}{2}\sigma^3$ where $\sigma^{1,2,3}$ are the pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Aside
Standard Model: $SU(3) \oplus SU(2) \oplus U(1)$

03/25/14

Lie Algebra of $SU(2)$: (usually denoted $\mathfrak{su}(2)$)

Generated by J_1, J_2, J_3 which follow the relation

$[J_a, J_b] = i\varepsilon_{abc} J_c \qquad J_1 = \frac{1}{2}\sigma_1,\ J_2 = \frac{1}{2}\sigma_2,\ J_3 = \frac{1}{2}\sigma_3$

$\mathfrak{su}(2)$ is traceless, Hermitian 2×2 matrices over $\mathbb{C}$.

(1) How many mutually commuting generators are there?

Called the Rank of the Lie Algebra.

The Lie sub-algebra generated by the mutually commuting generators is the ~~[illegible]~~ Cartan sub-algebra.

In this algebra, $\mathfrak{su}(2)$, at most one. $\text{Rank}(\mathfrak{su}(2)) = 1$.

J_3 is generally picked to be the diagonal one.

Defn: $\text{Adj}_T(R) := [T, R]$ is a Lie Algebra homomorphism

$\text{Adj}_{J_3}: \mathfrak{su}(2) \longrightarrow \mathfrak{su}(2)$ is a linear transformation, so it has a matrix representation; plug in basis vectors.

$\mathrm{Adj}_{J_3}(J_1) = [J_3, J_1] = +iJ_2$

$\mathrm{Adj}_{J_3}(J_2) = [J_3, J_2] = -iJ_1$

$\mathrm{Adj}_{J_3}(J_3) = 0$

$$\mathrm{Adj}_{J_3} = \begin{array}{c} \begin{array}{ccc} J_1 & J_2 & J_3 \end{array} \\ \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

But it's not diagonal, and we don't like that.

Diagonalize!

The eigenvectors of this matrix are $J_\pm = \frac{J_1 \pm iJ_2}{\sqrt{2}}$

$[J_3, J_\pm] = \pm J_\pm$. Call $J_3 = J_0$.

Then $[J_+, J_-] = J_0$

Preview of Thursday

More Generally: The cartan sub-algebra is $C = \mathrm{span}\{H_i\}_{i=1}^{\mathrm{rank}\, g}$ where (g ← lie Algebra.)

$[H_i, H_j] = 0$. Assemble eigenvalues of H_i into rank(g) vectors $\vec{\alpha}$ (roots of g)

If $\{T^a\}$ is a basis for our Lie algebra, do the same to $\mathrm{Adj}_{H_i}(T^a)$, for all a.

"Raising and lowering operators" (?)

$|m; \alpha\rangle$ (m ↑ eigenvalue of J_0)

$J_0 |m;\alpha\rangle = m|m;\alpha\rangle$

$J_0(J_\pm|m;\alpha\rangle) = (J_\pm J_0 + [J_0, J_\pm])|m;\alpha\rangle = (m \pm 1) J_\pm |m;\alpha\rangle$

So therefore, $J_\pm |m;\alpha\rangle = N_m |m \pm 1; \alpha\rangle$ (N_m ↑ some normalization factor)

$\langle m;\alpha | m;\beta\rangle = \delta_{\alpha\beta}$

Since all irreps are finite dimensional, there is M_-, $J_-|M_-;\alpha\rangle = 0$.

M_+, $J_+|M_+;\alpha\rangle = 0$

Want to find N_m. ~~Say~~ $\cancel{\langle \ldots \rangle}$

Say $\langle M_+\alpha | M_+\alpha\rangle = 1$. Then we know

$$\langle M_+ - 1 | M_+ - 1\rangle |N_{M}|^2 = \langle M_+ | (J_-)^\dagger (J_-) | M_+\rangle = \langle M_+ | [J_+, J_-] | M_+\rangle = \langle M_+ | J_0 | M_+\rangle = M_+$$

$$\langle M_+-1 | M_+-1\rangle |N_{M_+}|^2 = M_+ \overbrace{\langle M_+ | M_+\rangle}^{=1} \implies |N_{M_+}|^2 = M_+.$$

$$\cancel{|N_{M_+}|}|^2\langle M_+-2 | M_+-2\rangle = \langle M_+-1 | J_+ J_- | M_+-1\rangle$$

N_{M_+-1}

$$= \langle M_+-1 | [J_+, J_-] | M_+-1\rangle$$

$$= \langle M_+-1 | M_+-1\rangle (M_+-1) + \langle M_+-1 | (J_+)^\dagger \underbrace{(J_+) | M_+-1\rangle}_{|N_{M_+}|^2}$$

$$|N_{M_+-1}|^2 - |N_{M_+}|^2 = M_+-1$$

$$\vdots$$

$$+ \quad |N_{M_+-k}|^2 - |N_{M_+-k+1}|^2 = M_+-k+1$$

$$|N_{M_+-k}|^2 = M_+(k+1) - \sum_{n=1}^{k-1} n \implies |N_{M_+-k}|^2 = M_+(k+1) + \frac{k(k-1)}{2}$$

$$= \frac{k+1}{2}(2M_+-k)$$

if gauss as a child was acting up, Holmon would have sent him to the headmaster for a caning.

Since $|N_{M_+-k}|^2 \geq 0$, $k_{max} = 2M_+. \implies M_+ =$ half-odd integer or $\mathbb{Z}$.
(Spin integer ~~or~~ off by $1/2$).

All representations spaces of $SU(2)$ are of the form V_M, $M = 2s+1$, $s = n, n/2$, $n \in \mathbb{Z}$.

If a Lie Algebra G has representations on V_1, V_2, can form $V_1 \oplus V_2$ and $V_1 \otimes V_2$. The tensor product is tricky to figure out, we can realize it by looking at curves $g(t)$ through the Lie Group. For representations of the Lie group D_1, D_2:

$$D(g(t)) = D_1 \otimes D_2(g(t)) \longrightarrow D(g(t))^{i_1 i_2}_{j_1 j_2} := D_1(g(t))^{i_1}_{j_1} D_2(g(t))^{i_2}_{j_2} \longrightarrow$$

So for the Algebra, look at tangents:

$$D(\vec{X}_1 \otimes \vec{X}_2)^{i_1 i_2}_{j_1 j_2} = (\vec{X}_1)^{i_1}_{j_1} \delta^{i_2}_{j_2} + \delta^{i_1}_{j_1} (\vec{X}_2)^{i_2}_{j_2} \qquad (7)$$

Clebsh-Gordon Coefficients

SU(2) is essentially the only rank 1 Lie Algebra.

Roots and Weights

3/27/14

We can generalize the construction of representations of $SU(2)$ to an arbitrary Lie Group. Recall the important details:

(1) The vector space on which a representation ρ_J acts, $\rho_J : su(2) \longrightarrow GL(D)$, where $D = 2J+1$, has a basis

$$|J,-J\rangle, |J,-J+1\rangle, \dots, |J,J-1\rangle, |J,J\rangle \leftarrow \text{vectors labelled by their eigenvalues w.r.t } J_3$$

(2) We have picked one of the generators, eg. J_3 and chosen our basis to diagonalize it

$$J_3 |J, \underset{\uparrow \text{eigenvalue}}{m}\rangle = m|J,m\rangle$$

(3) We can define two raising and lowering operators, J_+ and J_- so that

$$J_\pm |J,m\rangle \propto |J,m\pm 1\rangle$$

(4) By starting with the highest (or lowest) angular momentum state $|J,J\rangle$ (or $|J,-J\rangle$) we can build up the entire set of states by acting with with J^- (or J^+) multiple times.

The Cartan Sub-algebra

Why did we only diagonalize one matrix by our choice of basis? we can only simultaneously diagonalize commuting matrices.

In $su(2)$, $[J_i, J_j] = i\,\varepsilon_{ijk} J_k \neq 0$, so J_i and J_j don't commute. Can only do this for one of them.

For other Lie Algebras, there might be a larger set of commuting generators. The sub-algebra generated by these is called the Cartan sub-algebra. $\{H_1, \ldots, H_m\}$, where $[H_i, H_j] = 0$.

In a representation ρ of dimension D, $\rho: \mathfrak{g} \to GL(D)$, then the elements of the Cartan are mapped $\rho(H_i) = H_i$

sub-algebra

H_i is both the generator and matrix to which it is sent by ρ.

The generators $\{H_i \mid 1 \le i \le m\}$ satisfy

(1) They are Hermitian, $H_i^\dagger = H_i$

(2) They commute, $[H_i, H_j] = 0$ (and $\rho([H_i, H_j]) = [\rho(H_i), \rho(H_j)]$).

They are D by D matrices, and we may ortho-normalize them so that

(3) $\mathrm{tr}(H_i H_j) = k_D\, \delta_{ij} \quad 1 \le i, j \le m$ ($\mathrm{tr}(H_i H_j)$ is an inner product)

The number of Cartan generators, m, is called the <u>rank</u> of the algebra. We may choose a basis of vectors (eigenstates) that diagonalizes all of the generators H_i:

$|\vec{\mu}, D\rangle$ where $\vec{\mu}$ is a <u>weight vector</u> in $\mathbb{R}^m$.

The components of $\vec{\mu}$ are the eigenvalues of the H_i's, which are called <u>weights</u>.

$H_i |\vec{\mu}, D\rangle = \mu_i |\vec{\mu}, D\rangle$.

In treating the representations, there are two spaces to consider:

(1) $\mathbb{R}^D$, the vector space on which our representation acts.

(2) $\mathbb{R}^m$, the space of weight vectors.

Example: $su(2)$

Rank=1 The weight vectors are numbers (i.e. $\mathbb{R}^1$ vectors).

The Adjoint Representation (roots)

The basis of the Adjoint representation is the generators.
The action of the representation is given by the commutator.
Let us write X_a for generators of the algebra $\mathfrak{g}$.

$|X_a\rangle$ is the vector on which the representation acts.

Normalize them as $\langle X_a | X_b \rangle = \frac{1}{\lambda} tr\left(X_a^\dagger X_b\right)$

The natural action of the representation of X_a (i.e. really mean $Adj(X_a)$) is

$$X_a | X_b \rangle = |[X_a, X_b]\rangle$$

We call the weights of the adjoint representation roots. Because they commute,

$$H_i | H_j \rangle = |[H_i, H_j]\rangle = 0.$$

Let us choose them to be orthonormal, by Gram-Schmidt

$$\langle H_i | H_j \rangle = \frac{1}{\lambda} tr(H_i H_j) = \delta_{ij}$$

All the other generators we write as $E_{\vec{\alpha}}$

$$H_i | E_{\vec{\alpha}} \rangle = \alpha_i | E_{\vec{\alpha}} \rangle \qquad \alpha \in \mathbb{R}^m$$

↳ μ_i written as a weight in adjoint representation.

In the adjoint representation, $\vec{\alpha}$ is associated with a unique generator.
We saw that H_i corresponds to J_3 in $su(2)$. What does E_α correspond to?

Take the hermitian conjugate of $E_{\vec{\alpha}}$:

$$H_i \left| E_{\vec{\alpha}}^\dagger \right\rangle = \left| [H_i, E_{\vec{\alpha}}^\dagger] \right\rangle = \left| [H_i^\dagger, E_{\vec{\alpha}}^\dagger] \right\rangle = -\left| [H_i, E_{\vec{\alpha}}]^\dagger \right\rangle$$

$$= -\left| [H_i, E_{\vec{\alpha}}]^\dagger \right\rangle = \cancel{\ldots}$$

$$= -\left| (\alpha_i E_\alpha)^\dagger \right\rangle = -\overset{\text{assume real}}{\alpha_i} \left| E_\alpha^\dagger \right\rangle$$

So $\left| E_{\vec{\alpha}}^\dagger \right\rangle = \left| E_{-\vec{\alpha}} \right\rangle$ (Corresponds to J_+ and J_-).

Normalize these according to

$$\left\langle E_{\vec{\alpha}} \middle| E_{\vec{\beta}} \right\rangle = \frac{1}{\lambda} \operatorname{tr}\left(E_{\vec{\alpha}}^\dagger E_{\vec{\beta}} \right) = \delta_{\vec{\alpha}\vec{\beta}} := \prod_{i=1}^{m} \delta_{\alpha_i \beta_i}$$

<u>Raising and Lowering</u>

Let us return to a general D-dimensional representation. Suppose we have a vector with weight $\vec{\mu}$.

$$H_i \left| \vec{\mu}, D \right\rangle = \mu_i \left| \vec{\mu}, D \right\rangle$$

What is the weight vector for $E_{\pm\vec{\alpha}} \left| \vec{\mu}, D \right\rangle$? We can find it by writing the commutator:

$$H_i E_{\pm\vec{\alpha}} \left| \vec{\mu}, D \right\rangle = \left([H_i, E_{\pm\vec{\alpha}}] + E_{\pm\vec{\alpha}} H_i \right) \left| \vec{\mu}, D \right\rangle$$

$$= \left(\pm\alpha_i E_{\pm\vec{\alpha}} + \mu_i E_{\pm\vec{\alpha}} \right) \left| \vec{\mu}, D \right\rangle = \left(\pm\alpha_i + \mu_i \right) E_{\pm\vec{\alpha}} \left| \vec{\mu}, D \right\rangle$$

So $\boxed{E_{\pm\vec{\alpha}} \left| \vec{\mu}, D \right\rangle \propto \left| \vec{\mu} \pm \vec{\alpha}, D \right\rangle}$

This is like "raising" and "lowering" the eigenvalues in the weight vector. This is true of any representation. For the adjoint representation,

$E_{\vec{\alpha}} \left| E_{-\vec{\alpha}} \right\rangle$ will have root 0.

This must be an element of the Cartan Sub-algebra

$$E_{\vec{\alpha}} \left| E_{-\vec{\alpha}} \right\rangle = \left| [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] \right\rangle = \sum_{i=1}^{m} \beta_i \left| H_i \right\rangle = \vec{\beta} \cdot \left| \vec{H} \right\rangle$$

What is $\vec{\beta}$?

$\cancel{\beta_i = [H_i, E_{\vec{\alpha}}}$

What is $\vec{\beta}$?

$$\beta_i = \langle H_i | E_{\vec{\alpha}} | E_{-\vec{\alpha}} \rangle = \langle H_i | [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] \rangle = \frac{1}{\lambda} tr(H_i [E_{\vec{\alpha}}, E_{-\vec{\alpha}}])$$

$$= \frac{1}{\lambda} tr(H_i E_{\vec{\alpha}} E_{-\vec{\alpha}} - H_i E_{-\vec{\alpha}} E_{\vec{\alpha}})$$

$$= \frac{1}{\lambda} tr(E_{-\vec{\alpha}} H_i E_{\vec{\alpha}} - E_{-\vec{\alpha}} E_{\vec{\alpha}} H_i)$$

$$= \frac{1}{\lambda} tr(E_{-\vec{\alpha}} [H_i, E_{\vec{\alpha}}]) = \frac{1}{\lambda} tr(E_{-\vec{\alpha}} [H_i, E_{\vec{\alpha}}]) = \frac{1}{\lambda} tr(E_{-\vec{\alpha}} (\alpha_i E_{\vec{\alpha}}))$$

$$= \alpha_i \frac{1}{\lambda} tr(E_{-\vec{\alpha}} E_{\vec{\alpha}}) = \alpha_i \frac{1}{\lambda} tr(E_{\vec{\alpha}}^\dagger E_{\vec{\alpha}}) = \alpha_i.$$

So $\vec{\beta} = \vec{\alpha}$ and $[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha} \cdot \vec{H} = \sum_{i=1}^{m} \alpha_i H_i$.

This is the analogue of $[J_+, J_-] = J_3$ in $su(2)$.

Rows of $su(2)$'s

The way to write an $su(2)$ subalgebra for any root:

$$\boxed{\begin{aligned} E_\pm &= \frac{1}{|\vec{\alpha}|} E_{\pm\vec{\alpha}} \\ E_3 &= \frac{1}{|\vec{\alpha}|} \vec{\alpha} \cdot \vec{H} \end{aligned}}$$

$[E_3, E_\pm] = \pm E_\pm$

$[E_+, E_-] = E_3$

09/01/14

Simple Lie Algebras: no nontrivial ideals

Recall: if $\{H_1, \ldots, H_r\}$ are generators of the cartan subalgebra,

$[H_i, H_j] = 0$ and $\text{adj}_{H_i} E_{\vec{\alpha}} = \alpha_i E_{\vec{\alpha}}$ $\quad$ $\vec{\alpha}$ are called the roots of $\mathfrak{g}$.

Simultaneosly diagonalize generators of Cartan subalgebra.

Look at an irrep R, with states $|\vec{\mu}\rangle$. What is $H_i|\vec{\mu}\rangle$?

$H_i|\vec{\mu}\rangle = \mu_i|\vec{\mu}\rangle$. The $\vec{\mu}$s are the weights of the irrep R.

For the irrep adj, these kets are $E_{\vec{\alpha}} \mapsto |E_{\vec{\alpha}}\rangle$ and $H_i|E_{\vec{\alpha}}\rangle = [H_i, E_{\vec{\alpha}}]$
$= \alpha_i E_{\vec{\alpha}}$.

By this observation, we see that the roots of $\mathfrak{g}$ are the weights of adj.

$$[H_i, E_{\vec\alpha}^\dagger] = (-\alpha_i)\, E_{\vec\alpha}^\dagger . \quad \text{So,} \quad E_{\vec\alpha}^\dagger = E_{-\vec\alpha}$$

<u>Recall:</u> $[E_{\vec\alpha}, E_{-\vec\alpha}] = \sum_{i=1}^{r} \alpha_i H_i$

Another way to discover this fact:

$$[E_{\vec\alpha}, E_{-\vec\alpha}] = \mathrm{adj}_{E_{\vec\alpha}} |E_{-\vec\alpha}\rangle = E_{\vec\alpha} |E_{-\vec\alpha}\rangle$$

$$H_i\left(E_{\vec\alpha} | E_{-\vec\alpha}\rangle\right) = [H_i, E_{\vec\alpha}] |E_{-\vec\alpha}\rangle + E_{\vec\alpha} H_i | E_{-\vec\alpha}\rangle$$
$$= \alpha_i E_{\vec\alpha} | E_{-\vec\alpha}\rangle + -\alpha_i E_{\vec\alpha} | E_{-\vec\alpha}\rangle = 0.$$

Hence $[E_{\vec\alpha}, E_{-\vec\alpha}]$ commutes with H_i for all i. Thus, $[E_{\vec\alpha}, E_{-\vec\alpha}] = \sum_{i=1}^{r} \beta_i H_i$.

So take $\langle H_j | E_{\vec\alpha} | E_{-\vec\alpha}\rangle = \sum_i \beta_i \langle H_j | H_i\rangle$, defining $\langle X | Y\rangle = tr[X^\dagger, Y]$.

We may further normalize $\{H_i\}$ to be orthonormal by gram-schmidt, so $\langle H_i | H_j\rangle = \delta_{ij}$. This then means $\langle E_{\vec\alpha} | E_{\vec\beta}\rangle = \delta_{\vec\alpha, \vec\beta}$

So $\sum \beta_i \langle H_j | H_i\rangle = \sum_i \delta_{ij} \beta_i = \beta_j$ and

$$\langle H_j | E_{\vec\alpha} | E_{-\vec\alpha}\rangle = \langle H_j | [E_{\vec\alpha}, E_{-\vec\alpha}]\rangle = tr\,[H_j, [E_{\vec\alpha}, E_{-\vec\alpha}]]$$
$$= tr\,[E_{-\vec\alpha}, [H_j, E_{\vec\alpha}]] = \alpha_j\, tr\,[E_{\vec\alpha}^\dagger, E_{\vec\alpha}]$$
$$= \alpha_j \langle E_{\vec\alpha}, E_{\vec\alpha}\rangle = \alpha_j$$

Therefore, $\beta_j = \alpha_j$. ✓

Finally, we have to compute $[E_{\vec\alpha}, E_{\vec\beta}]$. To do this, observe

$$H_i\left(E_{\vec\alpha} | E_{\vec\beta}\rangle\right) = [H_i, [E_{\vec\alpha}, E_{\vec\beta}]] = -[E_{\vec\beta}, [H_i, E_{\vec\alpha}]] - [E_{\vec\alpha}, [E_{\vec\beta}, H_i]]$$
$$= -[E_{\vec\beta}, \alpha_i E_{\vec\alpha}] - [E_\alpha, -\beta_i E_{\vec\beta}]$$
$$= (\alpha_i + \beta_i)\, E_{\vec\alpha} | E_{\vec\beta}\rangle \propto E_{\vec\alpha+\vec\beta} \quad \underline{\text{if}}\ \vec\alpha+\vec\beta \text{ is a root}$$
$$= 0 \quad \underline{\text{else}}$$

Knowing this, in an irrep R, tag states by eigenvalues of H_i:

$$H_i|\vec{\mu},R\rangle = \mu_i|\vec{\mu},R\rangle.$$

Now consider

$$\langle\vec{\mu},R|\overbrace{[E_{\vec{\alpha}},E_{-\vec{\alpha}}]}^{\vec{\alpha}\cdot\vec{H}}|\vec{\mu},R\rangle = \vec{\alpha}\cdot\vec{\mu}$$

$$\langle\vec{\mu},R|[E_{\vec{\alpha}},E_{-\vec{\alpha}}]|\vec{\mu},R\rangle = \langle\vec{\mu},R|E_{\vec{\alpha}}E_{-\vec{\alpha}}|\vec{\mu},R\rangle - \langle\vec{\mu},R|\underbrace{E_{-\vec{\alpha}}}_{=E_{\vec{\alpha}}^{\dagger}}E_{\vec{\alpha}}|\vec{\mu},R\rangle$$

$$= \left|E_{-\vec{\alpha}}|\vec{\mu},R\rangle\right|^2 - \left|E_{\vec{\alpha}}|\vec{\mu},R\rangle\right|^2 \longleftarrow \text{hit w/ } H_i, \text{ \& } |\vec{\mu}+\vec{\alpha},R\rangle$$

$$= |N_{-\vec{\alpha},\vec{\mu}}|^2 - |N_{\vec{\alpha},\vec{\mu}}|^2 \longleftarrow \text{constants of proportionality}$$

So therefore $\vec{\alpha}\cdot\vec{\mu} = |N_{-\vec{\alpha},\vec{\mu}}|^2 - |N_{\vec{\alpha},\vec{\mu}}|^2$

~~Then~~ Note that $\langle\vec{\mu}-\vec{\alpha}|E_{-\vec{\alpha}}|\vec{\mu}\rangle = N_{-\vec{\alpha},\vec{\mu}}$ b/c $E_{-\alpha}|\vec{\mu}\rangle = N_{-\vec{\alpha},\vec{\mu}}|\vec{\mu}-\vec{\alpha}\rangle$

but also

$$\left.\langle\vec{\mu}-\vec{\alpha}|E_{\vec{\alpha}}^{\dagger}|\vec{\mu}\rangle = \langle\vec{\mu}|\underbrace{E_{\vec{\alpha}}|\vec{\mu}-\vec{\alpha}\rangle}_{N_{\vec{\alpha},\vec{\mu}-\vec{\alpha}}|\vec{\mu}\rangle}{}^{*}\right\} \Longrightarrow N_{-\vec{\alpha},\vec{\mu}} = N^{*}_{\vec{\alpha},\vec{\mu}-\vec{\alpha}}$$

Thus, $|N^{*}_{\vec{\alpha},\vec{\mu}-\vec{\alpha}}|^2 - |N_{\vec{\alpha},\vec{\mu}}|^2 = \vec{\alpha}\cdot\vec{\mu}$

$$\left.\begin{aligned} E_{\vec{\alpha}}|\vec{\mu}+p\vec{\alpha}\rangle &= 0 \\ E_{-\vec{\alpha}}|\vec{\mu}-q\vec{\alpha}\rangle &= 0 \end{aligned}\right\} \text{top + bottom states.}$$

raising topmost state? nope.

$$\left.\begin{aligned}
|N_{\vec{\alpha},\vec{\mu}+(p-1)\vec{\alpha}}|^2 - |\cancel{N_{\vec{\alpha},\vec{\mu}+p\vec{\alpha}}}^{\,0}|^2 &= \vec{\alpha}\cdot(\vec{\mu}+p\vec{\alpha}) \\
|N_{\vec{\alpha},\vec{\mu}+(p-2)\vec{\alpha}}|^2 - |N_{\vec{\alpha},\vec{\mu}+(p-1)\vec{\alpha}}|^2 &= \vec{\alpha}\cdot(\vec{\mu}+(p-1)\vec{\alpha}) \\
&\vdots \\
|N_{\vec{\alpha},\vec{\mu}-q\vec{\alpha}}|^2 - |N_{\vec{\alpha},\vec{\mu}-(q-1)\vec{\alpha}}|^2 &= \vec{\alpha}\cdot(\vec{\mu}-(q-1)\vec{\alpha}) \\
0 - |N_{\vec{\alpha},\vec{\mu}-q\vec{\alpha}}|^2 &= \vec{\alpha}\,(\vec{\mu}-q\vec{\alpha})
\end{aligned}\right\} \text{add all these equations, telescoping}$$

lowering lowermost state

Telescoping series gives

$$0 = (p+q+1)\,\vec{\alpha}\cdot\vec{\mu} + (\vec{\alpha}\cdot\vec{\alpha})\left(\frac{p(p+1)}{2} - \frac{q(q+1)}{2}\right)$$

$$= (p+q+1)\,\vec{\alpha}\cdot\vec{\mu} + (\vec{\alpha}\cdot\vec{\alpha})\,(p-q)(p+q+1) \Longrightarrow 2\frac{\vec{\alpha}\cdot\vec{\mu}}{\vec{\alpha}\cdot\vec{\alpha}} = -(p-q)$$

Take R to be the adjoint representation. Then for any two roots,

$$2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} = -(p-q) = m \in \mathbb{Z} \text{ and } 2\frac{\vec{\beta}\cdot\vec{\alpha}}{|\vec{\beta}|^2} = -(p'-q') = m' \in \mathbb{Z}$$

So $\cos^2\theta_{\vec{\alpha}\vec{\beta}} = \dfrac{(\vec{\alpha}\cdot\vec{\beta})^2}{|\vec{\alpha}|^2|\vec{\beta}|^2} = \dfrac{mm'}{4}$. So the allowed angles are:

θ		mm'
90	90	0
60	120	1
45	135	2
30	150	3
0	180	4

04/03/14

Lie Algebras:

$[H_i, H_j] = 0$

$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$

$[E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha}\cdot\vec{H}$

$$[E_{\vec{\alpha}}, E_{\vec{\beta}}] = \begin{cases} 0 & \vec{\alpha}+\vec{\beta} \text{ not a root} \\ N_{\vec{\alpha},\vec{\beta}}\, E_{\vec{\alpha}+\vec{\beta}} \end{cases}$$

Theorem: Given a weight $\vec{\alpha}$, the generator corresponding to it is unique.

Proof: $2\dfrac{(\vec{\alpha}\cdot\vec{\beta})}{|\alpha|^2} = -(p-q)$; $\vec{\beta}-q\vec{\alpha}$ is a weight ; $\vec{\beta}+p\vec{\alpha}$ is a weight

→

Proof Continued:

Suppose that $E_{\vec{\alpha}}, E'_{\vec{\alpha}}$ are two generators. Then:

$$E_{-\vec{\alpha}}|E'_{\vec{\alpha}}\rangle = \vec{\beta}\cdot\vec{H} \quad \text{where } \beta_i = \langle H_i | E^{\dagger}_{\vec{\alpha}} | E'_{\vec{\alpha}}\rangle = \langle E'_{\vec{\alpha}} | E_{\vec{\alpha}} | H_i\rangle^*$$

$$\operatorname{Tr}\left(H_i\, [\overset{E_{-\vec{\alpha}}}{E^{\dagger}_{\vec{\alpha}}}, E'_{\vec{\alpha}}]\right) = \operatorname{Tr}\left(E'_{\vec{\alpha}}\, [H_i, E_{-\alpha}]\right) = -\alpha \operatorname{tr}\left(E'_{\vec{\alpha}}, E_{-\vec{\alpha}}\right)$$

by gram-schmidt, we may assume $E'_{\vec{\alpha}}, E_{-\vec{\alpha}}$ are orthonormal, so $-\alpha\operatorname{tr}(E'_{\vec{\alpha}}, E_{-\vec{\alpha}}) = 0 \implies E_{-\vec{\alpha}}|E'_{\vec{\alpha}}\rangle = \vec{\beta}\cdot\vec{H} = 0$

According to master equation, we cannot lower E'_{α} even once w/ $E_{\vec{\alpha}}$, so $q=0$. Hence $2\dfrac{\vec{\alpha}\cdot\vec{\beta}}{|\alpha|^2} = -p$, but $\vec{\beta}=\vec{\alpha}$ and then $2=-p$, but $p\geq 0$ contradiction. ■

By the master equation, we have turned a problem about classifying Lie algebras into one about geometry.

Simple Roots

The "basis" for the possible roots. A root is <u>positive</u> if it's first nonzero component as a rank g vector is positive.

A positive root is ~~simple~~ <u>simple</u> if it cannot be decomposed as a sum of two positive roots.

If $\vec{\alpha}, \vec{\beta}$ are simple roots, <u>claim</u> $\vec{\beta}-\vec{\alpha}$ is not a roots. Else $\vec{\beta} = (\vec{\beta}-\vec{\alpha})+\vec{\alpha}$ would not be simple.

If $\vec{\alpha}, \vec{\beta}$ are simple, then $\vec{\beta}-\vec{\alpha}$ not a root, so in master equation, $q=0$ b/c we cannot lower even once. So

$$\left.\begin{aligned} 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\alpha|^2} &= -p \\ 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\beta|^2} &= -p' \end{aligned}\right\} \implies \cos^2\theta = -pp'/4 .$$

Do the simple roots form a basis?

(1) Span
(2) Linear Independence.

<u>Proof:</u> (2) Let $\{\vec{\Phi} \mid \vec{\Phi} \text{ simple}\} = B$. Finite set of B because the rank of the Lie Algebra is finite, and $\vec{\Phi} \in B$ is a rank R vector, so $|B| = \text{rank } \mathfrak{g}$.

$$\sum_{\phi \in B} k_\phi \vec{\phi} = 0 \qquad \text{Let } \vec{y} = \sum_{\substack{\vec{\phi} \in B \\ k_\phi > 0}} k_\phi \vec{\phi} \quad \text{and} \quad \vec{z} = \sum_{\substack{\vec{\phi} \in B \\ k_\phi < 0}} -k_\phi \vec{\phi}$$

Then $\vec{y} - \vec{z} = 0$, so $\vec{y} = \vec{z}$.

$$|\vec{y}|^2 = \vec{y} \cdot \vec{z} = \sum_{\substack{\vec{\alpha} \text{ simple}, k_\alpha > 0 \\ \vec{\beta} \text{ simple}, k_\beta < 0}} k_\alpha (-k_\beta) \underbrace{\vec{\alpha} \cdot \vec{\beta}}_{\leq 0 \text{ as before}} \implies |\vec{y}|^2 = 0 \implies \vec{y} = \vec{z} = 0.$$

∎

04/08/14

Simple Roots

- Last time we defined positive roots
- the simple roots are a set of linearly independent positive roots
  - form a basis for weight space
  - can reproduce the algebra from simple roots.
  - established that simple roots are linearly independent

<u>Claim:</u> To show simple roots span $\mathbb{R}^m$

<u>Proof:</u> suppose the contrary. Then there is a vector $\vec{v}$ that must be orthogonal to all the roots.

$$[\vec{v} \cdot \vec{H}, E_{\vec{\alpha}}] = v_i [H_i, E_{\vec{\alpha}}] = v_i \alpha_i E_{\vec{\alpha}} = \underbrace{\vec{v} \cdot \vec{\alpha}}_{=0} E_{\vec{\alpha}} = 0 \qquad \text{assume } \vec{v} \cdot \vec{\alpha} = 0 \text{ for all } \vec{\alpha}.$$

It also commutes with all the H_i's $\implies$ $\vec{v} \cdot \vec{H}$ commutes with all the generators of $\mathfrak{g}$, which means we can add another element to Cartan subalgebra, and hence $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$

$\implies$ $\mathfrak{g}$ is not a simple Lie Algebra.

So we have:

<u>Theorem</u>: The simple roots form a basis for the weight space of any simple Lie Algebra.

The entire Lie Algebra can be found from the simple roots.
We stated last time that any positive root could be written as a linear combination with nonnegative coefficients of the simple roots.

$$\vec{\gamma} = \sum_{i=1}^{m} k_i \vec{\alpha}_i \quad \text{with } k_i \geq 0$$

Not every choice of integers will work.
If we build them up iteratively:
→ take a set of positive roots
→ act on them with ~~rais~~ raising operators, and then check whether or not they are roots.

<u>Notation</u>: $\vec{\alpha}$ and $\vec{\beta}$ are simple roots.
More specifically, we can write $\{\vec{\alpha}_1, \ldots, \vec{\alpha}_m\}$
We can label a <u>class</u> of positive roots by an integer = # of simple roots counted w/ multiplicity
e.g. $\vec{\gamma} = \sum_{i=1}^{m} k_i \vec{\alpha}_i$, then $k = \sum_{i=1}^{m} k_i$ and $\vec{\gamma}_k = \{$all vectors that share the same $k\}$
e.g. $\vec{\gamma}_1 = \{\vec{\alpha}_1, \ldots, \vec{\alpha}_m\}$.

We now construct the positive roots inductively.

<u>Step 1</u>: We know that γ_1 are positive roots, b/c they are simple.
<u>Step k</u>: To take the next step, act on the previous step with all the simple roots. Several cases:
(i) We get a new root! $E_{\vec{\alpha}} |E_{\vec{\beta}}\rangle \propto |E_{\vec{\alpha}+\vec{\beta}}\rangle$

→

(ii) The state is annihilated $E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = 0$

The simple roots are always the lowest spin states of all the other $SU(2)$'s associated with the other roots.

From our general formula: $\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} = -(p-q) = -p$ at lowest possible b/c simple roots

if we know p, and $p \neq 0$ (which it doesn't, because we would then be in case (i)), then we have p more roots:

$$\vec{\beta}+p\vec{\alpha},\ \vec{\beta}+(p-1)\vec{\alpha},\ \ldots,\ \vec{\beta}+\vec{\alpha}.$$

What could go wrong?

There is a root in γ_{k+1} that is not something in γ_k ~~…~~ plus a simple root.

Note: ~~We cannot have~~ If such a state did exist, then it would be in the lowest spin state of all the ~~other~~ $su(2)$'s.

$$E_{-\vec{\alpha}}|E_{\vec{\gamma}_{k+1}}\rangle = 0$$

If nonzero, it would be in the class γ_k:

$$E_{-\vec{\alpha}}|E_{\vec{\gamma}^{p}_{k+1}}\rangle \propto |E_{\vec{\gamma}^{p}_{k}}\rangle$$

particular representative of the class γ_{k+1}, only one of them

So if we picked out a particular $\vec{\alpha}$ and evaluated its E_3 eigenvalue on γ_{k+1}, where $E_3 = \frac{1}{|\vec{\alpha}|^2}\vec{\alpha}\cdot\vec{H}$

$$E_3|\gamma_{k+1}\rangle = \frac{\vec{\alpha}\cdot\vec{\gamma}^{p}_{k+1}}{|\vec{\alpha}|^2}|\vec{\gamma}^{p}_{k+1}\rangle$$

here $\frac{\vec{\alpha}\cdot\vec{\gamma}_{k+1}}{|\vec{\alpha}|^2} \leq 0$

This leads to a contradiction: then $\vec{\gamma}^{p}_{k+1}$ would ~~mean~~ not have a positive norm, because

$$\|\vec{\gamma}^{p}_{k+1}\| = \vec{\gamma}^{p}_{k+1}\cdot\vec{\gamma}^{p}_{k+1} = \sum_{i=1}^{m} k_i\vec{\alpha}_i\cdot\vec{\gamma}^{p}_{k+1} = \sum_{i=1}^{M}\underbrace{k_i|\alpha|^2}_{\geq 0}\underbrace{\frac{\vec{\alpha}\cdot\vec{\gamma}^{p}_{k+1}}{|\alpha|^2}}_{\leq 0}$$

$$\leq 0 \quad \divideontimes.$$

So we are guaranteed to find the positive roots in this way.

We start with the simple roots $\{\vec{\gamma}_1\} = \{\vec{\alpha}_i\}$, act on them with the $\vec{\alpha}_i$ to get $\{\gamma_2\}$ and then act on these with all the $\vec{\alpha}_i$'s to get $\{\gamma_3\}$ and so on until we don't get a new root.

<u>Example</u>: $SU(3)$.

The simple roots are $\vec{\alpha}_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\vec{\alpha}_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, both unit vectors.

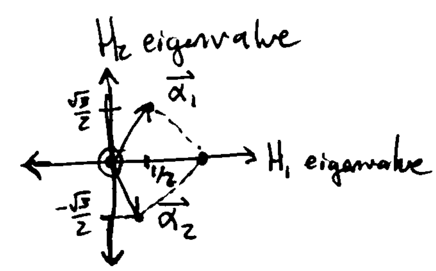
If we evaluate the dot product, $p = -2 \frac{\vec{\alpha}_1 \cdot \vec{\alpha}_2}{|\vec{\alpha}_1|^2} = -2\left(\frac{1}{4} - \frac{3}{4}\right) = 1$.

$$p' = -2 = \frac{\vec{\alpha}_1 \cdot \vec{\alpha}_2}{|\alpha_2|^2} = -2(1/4 - 3/4) = 1.$$

In either case, can raise each only once.

$$E_{\vec{\alpha}_1} | E_{\vec{\alpha}_2} \rangle \propto | E_{\vec{\alpha}_1 + \vec{\alpha}_2} \rangle$$

$$E_{\vec{\alpha}_2} | E_{\vec{\alpha}_1} \rangle \propto | E_{\vec{\alpha}_1 + \vec{\alpha}_2} \rangle$$



has reflections over y-axis too.

<u>Dynkin Diagrams</u>

A Dynkin diagram is a graphical way to represent angles between simple roots, only omitting the relative lengths of the roots.

Earlier, we showed that there are only four possibilities for angles between simple roots $\longrightarrow \cos^2\theta_{\alpha\beta} = 0, 1/4, 1/2, 3/4$.

For the simple roots, $\cos\theta_{\alpha\beta} \leq 0 \implies \cos\theta_{\alpha\beta} = 0, -1/2, -1/\sqrt{2}, -\sqrt{3}/2$

$$\downarrow 90^\circ \quad \downarrow 120^\circ \quad \downarrow 135^\circ \quad \downarrow 150^\circ$$

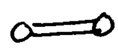
A Dynkin Diagram has a circle for each simple root, and n lines connecting them, where $\boxed{\cos\theta_{\alpha\beta} = -\frac{\sqrt{n}}{2}}$

Dynkin Diagram Examples:

rank 1: o $su(2)$

rank 2: o o $su(2) \times su(2)$

 $su(3)$

 $so(5)$ or $sp(4)$

 G_2 (exceptional)

The algebra of G_2:

So look at . Since $n=3$, the roots are $150°$ apart, two roots $\vec{\alpha}, \vec{\beta}$.

$\frac{2\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} = -3$ $\quad \frac{2\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|^2} = -1$ Take the product $\longrightarrow$ $\frac{(\vec{\alpha}\cdot\vec{\beta})^2}{(|\alpha||\beta|)^2} = \frac{3}{4} = \cos^2\theta_{\alpha\beta}$

$= -\frac{\sqrt{3}}{2}$

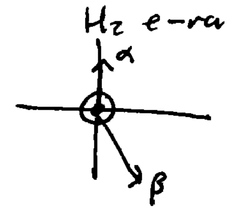
Take the ratio:

$$\frac{|\beta|^2}{|\alpha|^2} = 3.$$

So one is $\sqrt{3}$ longer than the other, let us choose a basis for $\mathbb{R}^2$.

$\vec{\alpha} = (0,1)$

$\vec{\beta} = (\frac{\sqrt{3}}{2}, -3/2)$



circle around dot is multiplicity, here means rank 2

$$\left.\begin{array}{l} 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} = -3 \Longrightarrow p = 3, \\ 2\frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|^2} = -1 \Longrightarrow p = 1. \end{array}\right\}$$

Hence $|\vec{\alpha}_2\rangle$ (β) is a $|\frac{3}{2}, -\frac{3}{2}\rangle$ state in the $\vec{\alpha}_1$ algebra.

$|\vec{\alpha}_1\rangle$ (α) is a $|1/2, -1/2\rangle$ state in the $\vec{\alpha}_2$ algebra

We then immediately know more roots:

$\vec{\alpha}_1 + \vec{\alpha}_2,\ 2\vec{\alpha}_1 + \vec{\alpha}_2,\ 3\vec{\alpha}_1 + \alpha_2$

$\{\vec{\gamma}_1\} = \{\vec{\alpha}_1, \vec{\alpha}_2\}$ $\quad \{\vec{\gamma}_3\} = \{2\vec{\alpha}_1 + \vec{\alpha}_2\}$ $\quad \{\vec{\gamma}_5\} = \{3\alpha_1 + 2\alpha_2\}$

$\{\vec{\gamma}_2\} = \{\vec{\alpha}_1 + \vec{\alpha}_2\}$ $\quad \{\vec{\gamma}_4\} = \{3\vec{\alpha}_1 + \vec{\alpha}_2\}$.

04/15/14

Representations of Lie Algebras in terms of weights / Fundamental weights

Weight vectors $\vec{\mu}$ satisfy, for any other weight $\vec{\alpha}$: $\dfrac{2\vec{\alpha}\cdot\vec{\mu}}{|\vec{\alpha}|^2} = -(p-q)$

how many times we can lower with $E_{-\vec{\alpha}}$.

how many times we can raise with $E_{+\vec{\alpha}}$.

For the highest weight state, $p=0$

$$2\frac{\vec{\alpha}\cdot\vec{\Lambda}}{|\vec{\alpha}|^2} = q \qquad \vec{\Lambda} \text{ is highest weight state}$$

Fundamental Weights $\Lambda^{(i)}$ are those that satisfy $\dfrac{2\vec{\alpha}^i\cdot\vec{\Lambda}^{(j)}}{|\vec{\alpha}^i|^2} = \delta^{ij}$

Also $\dfrac{2\vec{\alpha}^i\cdot\vec{\Lambda}}{|\vec{\alpha}^i|^2} = q^i$, so $\vec{\Lambda} = \sum_{i=1}^{\ell} q^i \Lambda^{(i)}$

Forms a representation R_Λ w/ $R_i = \Lambda^{(i)}$ the sub-representations.

Hence $R_\Lambda \subsetneq R_1 \otimes \ldots \otimes R_\ell$

Where is R_Λ in this tensor product?

Consider the SU(3) case:

Dynkin diagram o—o

Rank 2, with two roots $\vec{\alpha}_1$ and $\vec{\alpha}_2$ $\qquad \dfrac{2\vec{\alpha}_1\cdot\vec{\alpha}_2}{|\vec{\alpha}_1|^2} = \dfrac{2\vec{\alpha}_1\cdot\vec{\alpha}_2}{|\vec{\alpha}_2|^2} = -1$

And hence $|\vec{\alpha}_1|^2 = |\vec{\alpha}_2|^2$

Two fundamental representations (number is the inner product w/ fundamental weights)

$\overset{1}{\text{o}}$—o and o—$\overset{1}{\text{o}}$

$\vec{\Lambda} = a_1\vec{\alpha}_1 + a_2\vec{\alpha}_2 \Longrightarrow \vec{\Lambda} = 1/3(2\vec{\alpha}_1 + \vec{\alpha}_2)$

$\dfrac{2\vec{\Lambda}\cdot\vec{\alpha}_1}{|\vec{\alpha}_1|^2} = 1 \qquad \dfrac{2\vec{\Lambda}\cdot\vec{\alpha}_2}{|\vec{\alpha}_2|^2} = 0$

What are the other two states?

Lower by $\vec{\alpha}_1$ once; get $\frac{1}{3}(-\alpha_1 + \alpha_2)$

Find out how many times we can lower this new state by using master equation

$$\frac{2}{|\alpha_1|^2}\left(\tfrac{1}{3}(-\alpha_1+\alpha_2)\cdot\alpha_1\right) = -1$$

$$\frac{2}{|\alpha_2|^2}\left(\tfrac{1}{3}(-\alpha_1+\alpha_2)\cdot\alpha_2\right) = +1$$

can lower by α_2 again

$$\begin{array}{c} \Lambda = \tfrac{1}{3}(2\vec{\alpha}_1+\vec{\alpha}_2) \\ \downarrow \text{ lower by } \alpha_1 \\ \tfrac{1}{3}(-\vec{\alpha}_1+\vec{\alpha}_2) \\ \downarrow \text{ lower by } \alpha_2 \\ \tfrac{1}{3}(-\vec{\alpha}_1-2\vec{\alpha}_2) \end{array} \Bigg\} \text{ three fundamental weights of this representation}$$

This is called the $\underline{3}$ representation of SU(3). The other representation $\circ\!-\!\overset{1}{\circ}$ is the same, but interchanging α_1 and α_2. (The $\underline{\bar{3}}$ representation)

What about the representation $\overset{1}{\circ}\!-\!\overset{1}{\circ}$?

Should be contained in $\overset{1}{\circ}\!-\!\circ \otimes \circ\!-\!\overset{1}{\circ} = \underline{3}\otimes\underline{\bar{3}}$.

Denote the elements of the $\underline{3}$ representation as u_i, $|u\rangle = u_i|\Lambda^i\rangle$

Denote the elements of the $\underline{\bar{3}}$ representation as $\bar{v}^j$

Claim that $\underline{3}\otimes\underline{\bar{3}} = \underline{1}\otimes\underline{8}$, and that $\underline{8}$ is the adjoint representation.
Also, that $\underline{8} = \overset{1}{\circ}\!-\!\overset{1}{\circ}$.

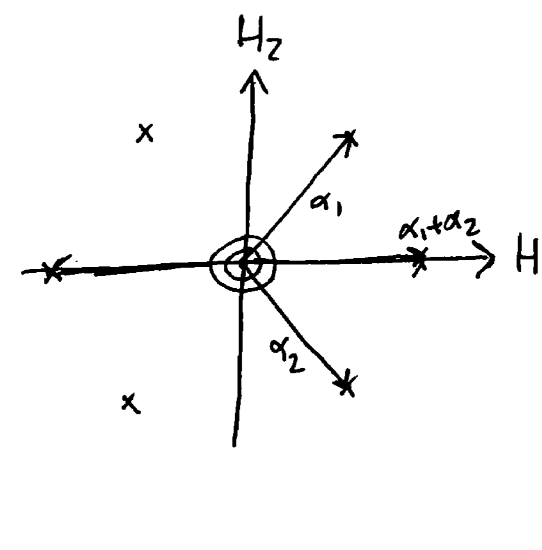
So now let Λ be the highest weight vector for $\overset{1}{\circ}\!-\!\overset{1}{\circ}$.

$$\frac{2\Lambda\cdot\alpha_1}{|\alpha_1|^2} = 1 \qquad \frac{2\Lambda\cdot\alpha_2}{|\alpha_2|^2} = 1 \qquad \frac{2\alpha_1\cdot\alpha_2}{|\alpha_{1,2}|^2} = -1$$

States are:

$$\begin{tikzcd}[column sep=small]
 & \alpha_1+\alpha_2 \arrow[dl, "-\vec{\alpha}_2"'] \arrow[dr, "-\vec{\alpha}_1"] & \\
\alpha_1 \arrow[d, "-\vec{\alpha}_1"'] & & \alpha_2 \arrow[d, "-\alpha_2"] \\
0 \arrow[d, "-\alpha_1"'] & & 0 \arrow[d, "-\alpha_2"] \\
-\alpha_1 \arrow[dr] & & -\alpha_2 \arrow[dl] \\
 & -(\alpha_1+\alpha_2) &
\end{tikzcd}$$

$\leftarrow$ since weight 2, should have 0 w/ multiplicity 2



(Pseudo) Real + Complex Irreps: D is a representation w/ generators T^a

$$[(-T^a)^*, (-T^b)^*] = [(T^a)^*, T^b]^* = (i f^{abc} T^c)^* = i f^{abc} (-T^c)^*.$$

can always choose the structure constants in $\mathbb{R}$.

$-(T^a)^*$ is the complex conjugate representation.

A representation R w/ generators T^a is <u>real</u> if R and $(-R^*)$ (w/ generators $(-T^a)^*$) are related by similarity transform.

If not, then complex.

$$R \otimes R \supseteq \underset{\text{symmetric}}{R_S} \oplus \underset{\text{antisymmetric.}}{R_A}$$

if $1 \subseteq R_S$, then real. if $1 \subseteq R_A$, then pseudoreal.

Depends on where the singlets are found.

How can we tell if its real using only the roots?

Given generators $H_1, \dots, H_\ell$ of the Cartan Algebra,

$$H_i |\vec{\mu}, R\rangle = \mu_i |\vec{\mu}, R\rangle$$

$$H_i |\vec{\mu}, \bar{R}\rangle = -\mu_i |\vec{\mu}, \bar{R}\rangle \longleftarrow \text{b/c } H_i \text{ hermition} \Rightarrow \mu_i \text{ real.}$$

The adjoint representation is always real.

<u>Tensor Products via Highest Weights</u>:

in SU(3): $\underline{3} \otimes \underline{3} = \underset{\text{symmetric}}{6} \oplus \underset{\text{antisymmetric}}{3}$

Highest weight in 3 is $\frac{1}{3}(2\alpha_1 + \alpha_2)$

In tensor product, they add: highest weight $= \frac{1}{3}(4\alpha_1 + 2\alpha_2) = \Lambda$.

$$\frac{2\Lambda \cdot \alpha_1}{|\alpha_1|^2} = 2 \qquad \frac{2\Lambda \cdot \alpha_2}{|\alpha_2|^2} = 0$$

$\longrightarrow$

04/15/14 page 4

Lowering gives

$$\tfrac{1}{3}(4\alpha_1+2\alpha_2)$$
$$\downarrow -\alpha_1$$
$$\tfrac{1}{3}(\alpha_1+2\alpha_2) \xrightarrow{-\alpha_2} \tfrac{1}{3}(\alpha_1-\alpha_2) \xrightarrow{-\alpha_1} \tfrac{1}{3}(-2\alpha_1-\alpha_2) \qquad \overline{3} \text{ representation}$$
$$\downarrow -\alpha_1$$
$$\tfrac{1}{3}(-2\alpha_1+2\alpha_2) \xrightarrow{-\alpha_2} \tfrac{1}{3}(-2\alpha_1-\alpha_2) \xrightarrow{-\alpha_2} \tfrac{1}{3}(-2\alpha_1-4\alpha_2)$$
$$\downarrow -\alpha_1 \qquad\qquad \downarrow -\alpha_1$$

Looking at this representation, we see that

$$\overset{1}{\circ}\!-\!\circ \otimes \circ\!-\!\overset{1}{\circ} = \overset{2}{\circ}\!-\!\circ \otimes \circ\!-\!\overset{1}{\circ} \qquad (?)$$

Physics of SU(3)

Generated by Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0\end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$
$$\lambda^7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

Generators:

$$T^a = \frac{\lambda^a}{2}$$

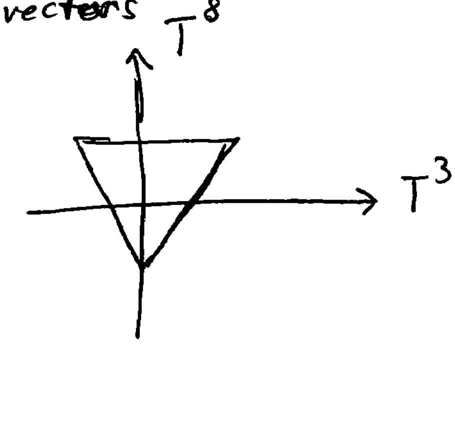
Normalized such that

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

Cartan Sub-algebra are $\frac{\lambda^8}{2}$ and $\frac{\lambda^3}{2}$

eigenvectors of λ^3 correspond to weight vectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \longmapsto \left|\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right\rangle$$
$$\begin{pmatrix}0\\-1\\0\end{pmatrix} \longmapsto \left|-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right\rangle$$
$$\begin{pmatrix}0\\0\\1\end{pmatrix} \longmapsto \left|0, -\tfrac{1}{\sqrt{3}}\right\rangle$$



Strong Interactions:

Heisenberg put proton, neutron in $SU(2)$ doublet: $\begin{pmatrix} P \\ N \end{pmatrix}$ "Strong Isospin"

04/17/14

Strangeness and Isospin

Heisenberg $SU(2)_I$ subscript I for strong isospin

Some interactions can't distinguish protons + neutrons. $\begin{pmatrix} P \\ N \end{pmatrix} \longmapsto U\begin{pmatrix} P \\ N \end{pmatrix}$ $SU(2)$ is exact symmetry of strong forces

In terms of quarks, $p = uud$ and $N = ddu$, symmetry under interchange of up and down quarks.

Baryon Numbers (conserved in any reaction)

$B=1$ P, N

$B=0$ $e^{\pm}, \mu^{\pm}, \nu_e, \cdots$

Also: charge conserved

Decay of proton (theorized in grand unification)

$p \longrightarrow e^+ \pi^0$ $t_p > 10^{42}$ years

time for decay

---

observe reaction $P + N \longrightarrow D + \pi^0$

$P + P \longrightarrow D + \pi^+$ $\longrightarrow$ postulate (π^+, π^0, π^-) form a $I=1$ triplet, $|1,-1\rangle$ $|1,0\rangle$ $|1,+1\rangle$

Deuterium

Each of π^+, π^0, π^- has Baryon number 0. π^- π^0 π^+

$$Q = I_3 + \frac{B}{2}$$

electric charge, isospin, $\frac{\text{Baryon \#}}{2}$

Scattering Reactions:

$\pi^+ p \longrightarrow \pi^+ p$
$\pi^+ N \longrightarrow \pi^+ N$
$\pi^+ N \longrightarrow \pi^0 P$
$\pi^- P \longrightarrow \pi^- P$
$\pi^- N \longrightarrow \pi^- N$

Rather than 5 reactions, predict based on symmetry and then check the experiments. Look at what $SU(2)_I$ predicts for scattering amplitudes.

Given an initial state, how likely is it to end up in a particular final state? (After time evolution)

$\langle \pi^0 P | S | \pi^+ N \rangle = B$ (**)

Final state: $\langle \pi^+ N, f|$

Initial state: $|\pi^+ N, i\rangle$ $\langle \pi^+ N, f | S | \pi^+ N, i \rangle = G$ (*)

Time Evolution Operator: S

$|\pi^+ N\rangle$

$\underbrace{I=1 \quad I=1/2}_{I = 3/2, 1/2, -1/2, -3/2}$ only permitted isospins

If I_a are generators of $SU(2)$, assume $[S, I_a] = 0$ (The isospin is unaffected by time evolution; invariant conserved)

Can we use the equations (*) and (**) to determine $\langle \pi^+ p | S | \pi^+ p \rangle$?

Well, $|\pi^+ P\rangle$ is the higest state in the representation; $|\pi^+ P\rangle = |3/2, 3/2\rangle$

$|\pi^+ P\rangle = |3/2, 3/2\rangle$

$\downarrow$ $\downarrow$

$|\pi^0 P\rangle + \sqrt{1/2}\,|\pi^+ N\rangle$ $\quad \frac{\sqrt{3}}{2}|3/2, 1/2\rangle$ $\Longrightarrow |3/2, 1/2\rangle = \sqrt{2/3}\,|\pi^0 P\rangle + \sqrt{1/3}\,|\pi^+ N\rangle$

$\longrightarrow$

$|1/2, 1/2\rangle = -\sqrt{1/3}\,|\pi^0 P\rangle + \sqrt{2/3}\,|\pi^+ N\rangle$

$|3/2, 1/2\rangle = \sqrt{2/3}\,|\pi^0 P\rangle + \sqrt{1/3}\,|\pi^+ N\rangle$

$\langle I', I_3', f | S | I, I_3, i\rangle = \delta_{II'}\delta_{I_3 I_3'}\, a_I$

$$\left.\begin{array}{l} |\pi^0 P\rangle = \sqrt{2/3}\,|3/2, 1/2\rangle + -\sqrt{1/3}\,|1/2, 1/2\rangle \\ |\pi^+ N\rangle = \sqrt{2/3}\,|1/2, 1/2\rangle + \sqrt{1/3}\,|3/2, 1/2\rangle \\ |\pi^+ P\rangle = |3/2, 3/2\rangle \end{array}\right\} \Longrightarrow G = \left(\sqrt{2/3}\langle 1/2, 1/2| + \sqrt{1/3}\langle 3/2, 1/2|\right)|S|\left(\sqrt{2/3}\,|1/2, 1/2\rangle + \sqrt{1/3}\,|3/2, 1/2\rangle\right)$$

So therefore $G = 2/3\, a_{1/2} + 1/3\, a_{3/2}$ and $B = \frac{\sqrt{2}}{3} a_{3/2} + -\frac{\sqrt{2}}{3} a_{1/2}$. G, B are measured quantities.

$\langle \pi^+ P, f | S | \pi^+ P, i\rangle = a_{3/2}$

So we can determine $a_{3/2}$ from measurement.

Other particles:

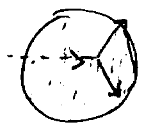
$\Delta^{++}, \Delta^0, \Delta^{-}$ spin $3/2$

Made through $\pi^+ P \to \Delta^{++}$

Can describe the ratio of Δ^{++} to Δ^0 through group theory, similar to what we did above.

Bubble chamber experiments

neutral



Occasionally two tracks from seemingly nowhere → new particles $\Lambda^0, \Sigma^+, \Xi^-$

Strong Interaction: $t \sim 10^{-23}$ s ← expected

Weak Interaction: $t \sim 10^{-10}$ s ← observed

Theres something else... new quantum number?

$$\boxed{\begin{array}{l} \Lambda^0 \to p + \pi^- \\ \Sigma^+ \to p + \pi^0 \\ \Xi^- \to \pi^- + \Lambda^0 \end{array}}$$

Pairs: Associated Production

$\pi^+ + N \to \Lambda^0 + K^+$

"Kaons"

Gell-Mann explained this by "strongeness" quantum number

$S(\Lambda^0) = S(K^-) = -1$ ← also $\underbrace{\Lambda^0, K^\pm, K^0, \overline{K^0}}_{\text{Baryon \# 0}}$

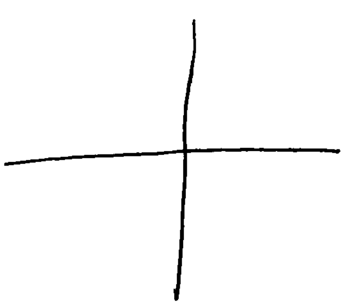
$S(K^+) = +1$ $S(\pi^+, \pi^-) = S(N, P) = 0$

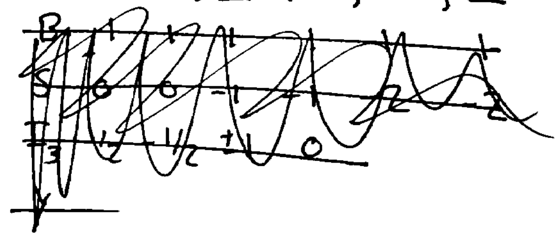
Earlier: $Q = I_3 + B/2$

Now $Q = I_3 + \frac{B+S}{2}$

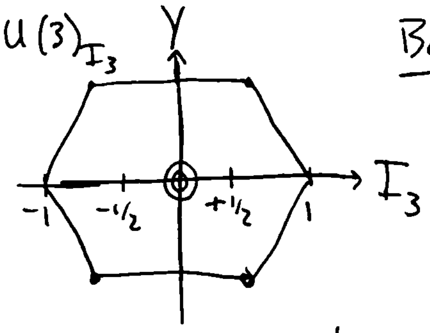
$Y = B + S$ is called "strong hypercharge"

Some clever guy plotted I_3 vs. $S+B$ for particles: $P, N, \Sigma^\pm, \Lambda^0, \Xi^-, \Xi^0$





$SU(3)$



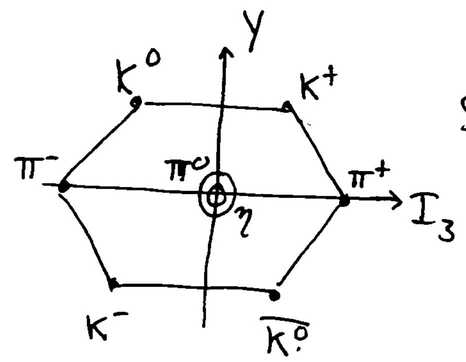
It's $SU(2)$!

Baryon Octet

<u>Baryons:</u>

	P	N	$\Sigma^{\pm}$	Σ^0	Λ^0	Ξ^-	Ξ^0
B	1	1	1	1	1	1	1
S	0	0	-1	-1	-1	-2	-2
I_3	$1/2$	$-1/2$	± 1	0	0	$-1/2$	$+1/2$
Y	1	1	0	0	0	-1	-1

Do the same for the Mesons:

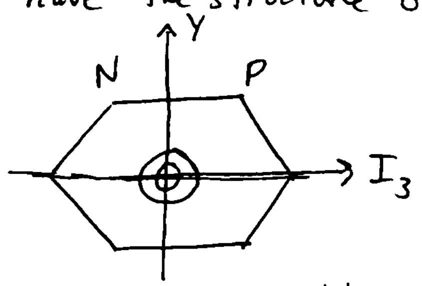


$SU(3)_F$

04/22/14

<u>Strong Interaction Physics</u> (of the 1960's)

<u>Recall:</u> Baryons have the structure of $SU(3)$ when you plot I_3 vs. Y.



$$Q = I_3 + \frac{B+S}{2}, \quad Y = B+S.$$

The particles are formed from the quarks $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$, where the u, d quarks form an $SU(2)$ subgroup. $\underline{3} = \underline{2} \oplus \underline{1}$

Horizontal lines form the breakdown of the adjoint representation $\underset{\substack{\uparrow \\ \text{adjoint} \\ \text{of } SU(3)}}{\underline{8}} = \underline{2} \oplus \underline{2} \oplus \underset{\substack{\uparrow \\ \text{adjoint} \\ \text{of } SU(2)}}{\underline{3}} \oplus \underline{1}$

Within $SU(2)$ irreps, Δmc^2 are small $\left(\frac{\text{MeV}}{\text{GeV}} \sim 10^{-3}\right)$

Think $\langle H \rangle$ the hamiltonian of Strong Interactions

$$\underset{\substack{\uparrow \\ \text{strong}}}{H_S} = \underset{\substack{\uparrow \\ \text{very strong}}}{H_{VS}} + \underset{\substack{\uparrow \\ \text{kinda strong}}}{H_{KS}}$$

$$[H_{VS}, T_a] = 0 \qquad [H_{KS}, T_a] = 0$$

$$a = 1, \dots, 8 \qquad a = 1, 2, 3$$

Tensor Operators

$$[T_a, \mathcal{O}^I] = (T_a^{(\theta)})^I{}_J \, \mathcal{O}^J$$

form the generators for adjoint rep.

Gell-Mann Matrices contain a subgroup isomorphic to SU(2)

$$T_1 = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \end{array}\right) \quad T_2 = \left(\begin{array}{cc|c} 0 & -i & \\ i & 0 & \\ \hline & & \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & \\ \hline & & \end{array}\right) = T_3$$

We want to find an operator H_{KS} which commutes with all of these matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = T_8$$

baryon octet matrix $\langle B | T_8 | B \rangle$

(1) Math part: what does $\langle B | T_8 | B\rangle$ mean in terms of 3×3 matrices?

$$\langle B | T_8 | B \rangle = \mathrm{Tr}\left(B^\dagger [T_8, B]\right).$$

$$= \mathrm{Tr}(B^\dagger T_8 B) - \mathrm{Tr}(B^\dagger B T_8)$$

Therefore, we see that the most general thing that commutes with the SU(2) generators is

$$H_{KS} = X_1 \,\mathrm{tr}(B^\dagger T_8 B) - \mathrm{tr}(B^\dagger B T_8)\, X_2$$

We also know (by experiment?)

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & P \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & \frac{-2\Lambda^0}{\sqrt{6}} \end{pmatrix}$$

$I = 1$ $\quad I = 1/2$ $\quad I = 1/2$

Mass of nucleons

$$M_N = M_0 - \tfrac{1}{6} X_2 + \tfrac{1}{12} X_1$$
$$M_\Lambda = M_0 - \tfrac{1}{2}(X_1 + X_2)$$
$$M_\Sigma = M_0 + \tfrac{1}{12}(X_1 + X_2)$$
$$M_\Xi = M_0 - \tfrac{1}{6} X_1 + \tfrac{1}{12} X_2$$

simplify

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma$$

"Gell-Mann-Okubo Relation"

But knowing that there are really six types of quarks what is the real symmetry going on?

$$\underbrace{u, d, s}_{0 \text{ to } 10\ \text{MeV}},\ \underset{\text{GeV}}{c},\ \underset{5\ \text{GeV}}{b},\ \underset{192\ \text{GeV}}{t}.$$

Given these masses, the symmetry of these quarks should not be SU(6), as you might naively guess.

The Standard Model:

$$SU_c(3) \times \underbrace{SU_L(2) \times U_Y(1)}_{\text{electroweak}}$$

QCD color symmetry, u^r, u^b, u^g

$\begin{pmatrix} u \\ d \end{pmatrix}_L \sim (\underset{C}{\underline{3}}, \underset{L}{\underline{2}}, \underset{Y}{+1})$ (color)

$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \sim (\underset{C}{\underline{1}}, \underset{L}{\underline{2}}, \underset{Y}{-1})$

$u_R \sim (\underline{3}, \underline{1}, 4/3)$

$d_R \sim (\underline{3}, \underline{1}, -2/3)$

$e_R \sim (\underline{1}, \underline{1}, -2)$

$\nu_R \sim (1, 1, 0)$

gluons $\sim (\underline{8}, \underline{1}, 0)$ $\qquad B \sim (\underline{1}, \underline{1}, 0)$

$W^{\pm}, W^0 \sim (\underline{1}, \underline{3}, 0)$

$SU(3) \times SU(2) \times U(1)$ has rank 4: What is the smallest group of rank 4 that has this as a subgroup? $SU(5)$ ← working in $SU(5)$ instead is "grand unification". (Maybe...)

04/24/14

$su(N)$ is the algebra for $SU(N)$, dimension N^2-1

$N \times N$ matrices
- traceless (comes from group has determinant 1)
- hermition (group elements are unitary)

$(\tilde{E}_{ij})_{kl} = \delta_{ik}\delta_{jl}$ ← basis "vectors" $\qquad Tr_F(T^a, T^b) = \frac{1}{2}\delta^{ab} \qquad E_{ij} = \frac{1}{\sqrt{2}}\tilde{E}_{ij}$ ← normalize basis vectors

Cartan Subalgebra: N-1 diagonal generators

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \qquad H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \qquad H_3 = \frac{1}{2\sqrt{6}}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -3 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

normalization factor is from demanding $Tr(T^a, T^b) = \frac{1}{2}\delta^{ab}$

$$H_{N-1} = \frac{1}{\sqrt{2N(N-1)}}\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 0 & & & & -(N-1) \end{pmatrix}$$

Weights of the fundamental representation

$$H_i|\vec{\mu}; N\rangle = \mu_i|\vec{\mu}; N\rangle$$

$\vec{\lambda}_1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \ldots, \frac{1}{\sqrt{2N(N-1)}}\right)$ ← first diagonal component of these matrices

$\vec{\lambda}_2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}, \ldots, \frac{1}{\sqrt{2N(N-1)}}\right)$ ← second diagonal component of these matrices.

$$\vec{\lambda}_{N-1} = \left(0, 0, \ldots 0, \frac{-(N-2)}{\sqrt{2(N-1)(N-2)}}, \frac{1}{\sqrt{2N(N-1)}}\right)$$

N λ's in an $N-1$ dimensional space $\Rightarrow$ nontrivial linear relation.

$$\vec{\lambda}_N = \left(0, \ldots, 0, \frac{-(N-1)}{\sqrt{2N(N-1)}}\right)$$

How do we find the roots?

How many should we have? $N^2-1 = \underbrace{(N-1)}_{\substack{\text{0 roots} \\ \text{for the} \\ \text{Cartan} \\ \text{subalgebra}}} + \underbrace{N(N-1)}_{\text{other roots}}$

Conjecture $\vec{\lambda}_i - \vec{\lambda}_j$ are the roots.

There are $N(N-1)$ of them, and they obey the correct relation

$$E_{-(\lambda_1-\lambda_2)}\lambda_1 \propto E_{\lambda_2}$$

$\alpha_1 = \vec{\lambda}_1 - \vec{\lambda}_2 \leftarrow$ most positive root

$\alpha_2 = \vec{\lambda}_2 - \vec{\lambda}_3$

$\vdots$ and sums of these!

$\alpha_{N-1} = \vec{\lambda}_{N-1} - \vec{\lambda}_N$

What generators do the roots correspond to?

guess $\vec{\lambda}_i - \vec{\lambda}_j \longleftrightarrow E_{ij}$ no sum over i, j, m!

to verify: $[H_m, E_{ij}] \stackrel{?}{=} (\vec{\lambda}_i - \vec{\lambda}_j)_m E_{ij}$

m^{th} component of $\vec{\lambda}_i - \vec{\lambda}_j$

First, simplify the H_m matrices.

$$(H_m)_{ij} = \sum_{k=1}^{m} \frac{(\delta_{ik}\delta_{jk} - m\,\delta_{i,m+1}\delta_{j,m+1})}{\sqrt{2m(m+1)}}$$

$$(H_m)_{kl}(E_{ij})_{lp} - (E_{ij})_{kl}(H_m)_{lp} \stackrel{?}{=} (\vec{\lambda}_i - \vec{\lambda}_j)_m (E_{ij})_{kp}$$

Yeah, it works. Check later.

To get the Dynkin Diagram

simple roots are $\alpha_i = \vec{\lambda}_i - \vec{\lambda}_{i+1}$

We want $\dfrac{2\vec{\alpha}_i \cdot \vec{\alpha}_j}{|\vec{\alpha}_i|^2}$. Really need to know $\vec{\lambda}_i \cdot \vec{\lambda}_j = \begin{cases} \frac{N-1}{2N} & i=j \\ -\frac{1}{2N} & i \neq j \end{cases}$

Some magic handwaving shows

$$\frac{2\vec{\alpha}_i \cdot \vec{\alpha}_j}{|\vec{\alpha}_i|^2} = \begin{cases} 0 & \text{if } j \neq i+1 \\ -1 & \text{if } j = i+1 \end{cases}$$

So there are only relations between α_i and α_{i+1}, so the diagram is:

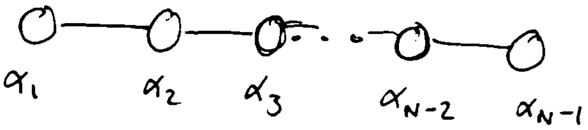


Diagram shows: $SU(N) \supseteq SU(M) \times SU(N-M) \times U(1)$

For any representation with highest weight $\vec{\mu}$, $\dfrac{2\vec{\alpha}_i \cdot \vec{\mu}}{|\vec{\alpha}_i|^2} = \mu_i$

Fundamental representation is the one with $\vec{\lambda}_i$, $i = 1 \ldots N$ as the weights. Breaking it up as $SU(N) \supseteq SU(M) \times SU(N-M)$

$$\underline{N} = (\underline{M}, \underline{1}) \oplus (\underline{1}, \underline{N-M})$$

the $\vec{\lambda}_i$ — first M λ_i's, M-dimensional rep — ~~first~~ last N-M λ_i's, N-M dimensional rep

04/29/14

The Classification Theorem

(1) Started w/ the adjoint representation

$$|E_{\vec{\alpha}}\rangle \qquad E_{\vec{\alpha}}|E_{\vec{\beta}}\rangle = |[E_{\vec{\alpha}}, E_{\vec{\beta}}]\rangle \qquad H_i|E_{\vec{\alpha}}\rangle = \alpha_i |E_{\vec{\alpha}}\rangle$$

(2) Labelled the generators by their roots $\vec{\alpha}$ (m=rank of algebra)

(3) positive and negative roots

(4) Simple roots form a basis for the root space
 - generate all roots from simple ones

(5) The angles and relative lengths between simple roots are important, encoded in the master formula:

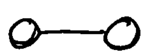
$$\frac{2\,\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2} \in \mathbb{Z} \implies \cos^2\theta_{\alpha\beta} = \frac{n}{4} \qquad n = 0,1,2,3$$

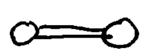
(6) Dynkin diagrams:

Each circle corresponds to a simple root.
Number of circles is the rank.
Number of lines connecting circles is the same n in $\cos\theta_{\alpha\beta} = -\frac{\sqrt{n}}{2}$

For example:

 $\rightarrow n=0$ 90°

 $\rightarrow n=1$ 120°

 $\rightarrow n=2$ 135°

 $\rightarrow n=3$ 150°

If one root is shorter in length than another, shade it in

$1/\sqrt{2}$ shorter than other

Π-systems

Definition: A π-system is a set of vectors that satisfies
(a) The vectors are linearly independent
(b) If $\vec{\alpha}, \vec{\beta}$ are distinct elements of the π-system, then $\frac{2\,\vec{\alpha}\cdot\vec{\beta}}{|\vec{\alpha}|^2}$ is a nonpositive integer
(c) The set is indecomposable: it cannot be written as the direct sum of mutually orthogonal subspaces.

The last requirement restricts us to the <u>simple</u> Lie Algebras, those that cannot be written as the direct product of other Lie Algebras (called semisimple).

$$\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \leftarrow \text{semisimple}$$

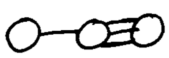
<u>Example of semi-simple:</u>

The problem of classifying simple Lie Algebras can be restated as one of classifying Dynkin diagrams, which correspond to π-systems.

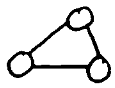
<u>We shall do this with four lemmas:</u>

<u>Lemma 1:</u> The only possible π-systems with three vectors are

<u>Proof:</u> The sum of angles between 3 linearly independent vectors must be $\lneq$ 2π. Hence we can eliminate:

 $\sum$ angles $= 360°$

 $\sum$ angles $= 360°$

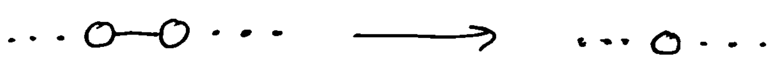
 $\sum$ angles $= 360°$

Can also eliminate the semisimple ones: (decomposable)

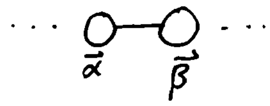
This lemma is useful because any connected subdiagram of a π-system is also a π-system. Hence any more complicated connected Dynkin diagram must <u>only</u> have these as subdiagrams.

<u>Corollary:</u> The only π-system that contains a triple line is , which is G_2.

<u>Lemma 2:</u> The next step is to reduce big π-systems to smaller π-systems. If a π-system has two vectors connected with a single line, the diagram that we get by merging them together is also a π-system.



Proof: Let us denote the two vectors as $\vec{\alpha}$ and $\vec{\beta}$.

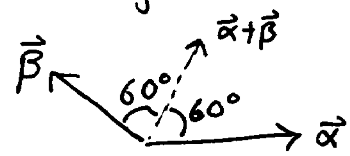


Because of Lemma 1, no other vector can be connected to both $\vec{\alpha}$ and $\vec{\beta}$ in the Dynkin diagram. If $\vec{\gamma}$ connects to $\vec{\alpha}$, then

$\vec{\alpha}\cdot\vec{\gamma} \neq 0$ and $\vec{\alpha}\cdot\vec{\beta} = 0$, or if $\vec{\gamma}$ connects to $\vec{\beta}$, $\vec{\gamma}\cdot\vec{\beta}\neq 0$, $\vec{\gamma}\cdot\vec{\alpha} = 0$.

Because $|\vec{\alpha}| = |\vec{\beta}|$, and the angle between them is $120°$,

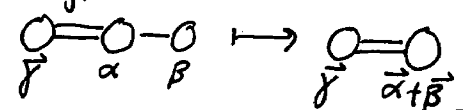
$|\vec{\alpha}+\vec{\beta}| = |\vec{\alpha}| = |\vec{\beta}|$



Let Γ be the rest of the diagram. WLOG $\vec{\gamma}$ is connected to $\vec{\alpha}$.

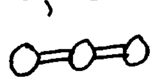
$$\vec{\gamma}\cdot(\vec{\alpha}+\vec{\beta}) = \vec{\gamma}\cdot\vec{\alpha} \implies \frac{2\vec{\gamma}\cdot(\vec{\alpha}+\vec{\beta})}{|\vec{\alpha}+\vec{\beta}|^2} = \frac{2\vec{\gamma}\cdot\vec{\alpha}}{|\vec{\alpha}|^2}$$

The subset $\Gamma \cup \{\vec{\alpha}+\vec{\beta}\}$ is also a π-system: e.g.

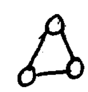


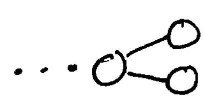
■

Corollary 2.1: No π-system has more than one double line.

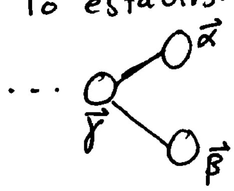
Proof: Otherwise, can shrink a π-system along single lines until we get , which cannot be a π-system, $\divideontimes$.

Corollary 2.2: No π-system has a closed loop.

Proof: Otherwise, shrink w/ lemma until we get , not possible. $\divideontimes$

Lemma 3: If the configuration  is a π-system, then  is also a π-system.

Proof: To establish this, we need to take dot products.



$$\frac{2\vec{\alpha}\cdot\vec{\gamma}}{|\vec{\alpha}|^2} = -1, \quad \frac{2\vec{\alpha}\cdot\vec{\gamma}}{|\vec{\gamma}|^2} = -1 \qquad \vec{\alpha}\cdot\vec{\beta} = 0$$

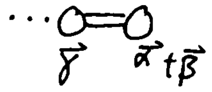
$$\frac{2\vec{\beta}\cdot\vec{\gamma}}{|\vec{\beta}|^2} = -1, \quad \frac{2\vec{\beta}\cdot\vec{\gamma}}{|\vec{\gamma}|^2} = -1$$

What are the dot products with $\vec{\alpha}+\vec{\beta}$?

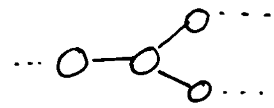
$$\frac{2\vec{\gamma}\cdot(\vec{\alpha}+\vec{\beta})}{|\vec{\gamma}|^2} = \frac{2\vec{\gamma}\cdot\vec{\alpha}}{|\vec{\gamma}|^2} + \frac{2\vec{\gamma}\cdot\vec{\beta}}{|\vec{\gamma}|^2} = -2 \qquad \underline{\text{and}}$$

$$\frac{2\vec{\gamma}\cdot(\vec{\alpha}+\vec{\beta})}{|\vec{\alpha}+\vec{\beta}|^2} = \frac{2\vec{\gamma}\cdot\vec{\alpha}}{|\vec{\alpha}|^2}\frac{|\vec{\alpha}|^2}{|\vec{\alpha}+\vec{\beta}|^2} + \frac{2\vec{\gamma}\cdot\vec{\beta}}{|\vec{\beta}|^2}\frac{|\vec{\beta}|^2}{|\vec{\alpha}+\vec{\beta}|^2} = -1\left(\frac{|\vec{\alpha}|^2+|\vec{\beta}|^2}{|\vec{\alpha}|^2+\underbrace{2\vec{\alpha}\cdot\vec{\beta}}_{=0 \text{ b/c } \vec{\alpha},\vec{\beta} \text{ orthogonal.}}+|\vec{\beta}|^2}\right) = -1$$

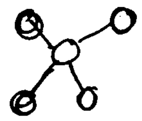
This corresponds to the diagram for the π-system



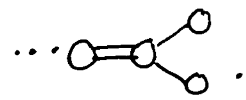
<u>Corollary 3.1:</u> The only branch in a π-system is



In particular, we cannot have



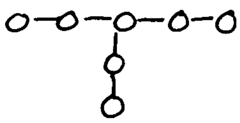
or



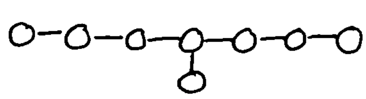
<u>Corollary 3.2:</u> No π-system can have more than **one** branch.

<u>Lemma 4:</u> No π-system contains either of the following as subdiagrams:

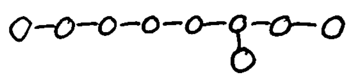
(a)



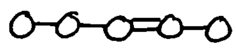
(b)



(c)



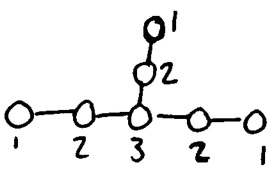
(d)



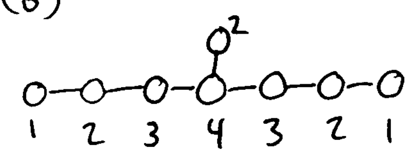
<u>Proof:</u> Each of these fails linear independence, that is, $\sum_{i=1}^{n} \mu_i \vec{\alpha_i} = 0$ for μ_i not all zero. Instead we show $\left(\sum_i \mu_i \vec{\alpha_i}\right)^2 = 0$.

In the first three cases: (value of μ_i on the dots).

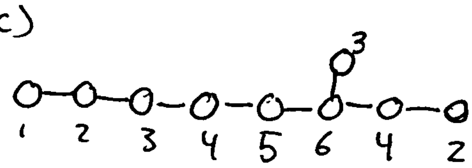
(a)



(b)



(c)



05/01/14

For the last case, we must assume that one set of roots is larger or smaller than the others

e.g.  longer by $\sqrt{2}$ or  longer by $\sqrt{2}$

Explicitly check the first **case**: $(\vec{\alpha}_1 + 2\vec{\alpha}_2 + 3\vec{\alpha}_3 + 2\vec{\alpha}_4 + \vec{\alpha}_5)^2$

$$= 1+4+9+4\cdot 2+2 + 4\vec{\alpha}_1\cdot\vec{\alpha}_2 + 12\vec{\alpha}_2\cdot\vec{\alpha}_3 + 12\vec{\alpha}_3\cdot\vec{\alpha}_4 + 4\vec{\alpha}_4\cdot\vec{\alpha}_5$$

$$= 24 + 4(-1/2) + 12(-1/2) + 12\left(\tfrac{-1}{\sqrt{2}}\sqrt{2}\right) - 4(1/2)(\sqrt{2})^2$$

$$= 24 - 2 - 6 - 12 - 4 = 0$$

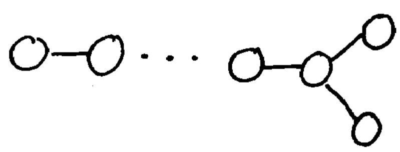
Concludes proof of Lemma 4. ∎

With these 4 lemmas, we can classify all the π-systems, and therefore all Dynkin Diagrams and therefore all simple Lie Algebras.

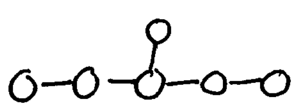
Theorem: The possible Lie Algebras are:

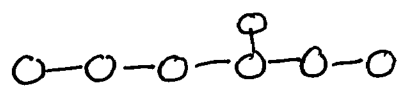
Class (1) Algebras with only single lines

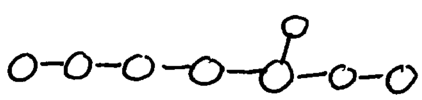
A chain:  $A_n = su(n+1)$

A tree with a single branch:  $D_n = so(2n)$

Some exceptional cases:

 E_6

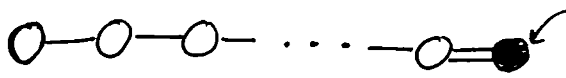
 E_7

 E_8

continuing this pattern is forbidden by lemma 4.

Class (2) Algebras with a double line.

We showed that diagrams with a double line cannot branch and can have no more than one double line.

 shorter by $\sqrt{2}$ $\quad B_n = so(2n+1)$

Shorter by $\sqrt{2}$ $\quad C_n = sp(2n)$

More exceptional cases:

 F_4

Class (3) Algebras with a triple line:

Only one possibility  G_2

Equivalences:

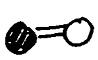
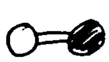
The assignment of particular Dynkin diagrams to an Algebra is not unique.

e.g.

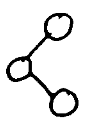
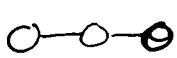
 $= A_1, B_1, C_1 \qquad su(2) = so(3) = sp(2) \leftarrow$ groups are not the same, algebras are.

The case of D_2 is a little peculiar (diagram isn't well defined)

 $= D_2 = so(4) = so(3) \times so(3)$ is not simple!

Trivially,  $=$  $\Longrightarrow B_2 = C_2 \Longrightarrow so(5) = sp(4)$

One last case:

$D_3 =$  $= so(6) =$  $= A_3 = su(4)$.

<u>Regular Subalgebrass</u>

The <u>regular</u> <u>subalgebras</u> of a Lie Algebra g is a subalgebra that satisfies two requirements.

(a) the roots of the subalgebra are a subset of the roots of g, <u>and</u>

(b) the generators of the Cartan subalgebra are a linear combination of the generators of the Cartan subalgebra of g.

There are a few techniques involving Dynkin diagrams to find these.

(1) Remove a circle from the diagram (remove simple root)

$A_{n+m-1} = su(n+m)$

$su(m) \times su(n) \subseteq su(n+m)$.

Always produces a lower rank algebra.

(2) Merge Dynkin diagrams

Use one of the old merging lemmas

(a)

merge along single line

(b)

merge along the branches

(a) Shows $su(n) \subseteq su(n+1)$

(b) shows that $so(2n-1) \subseteq so(2n)$

} produces a lower rank algebra.

Extended Dynkin Diagrams

A maximal subalgebra has the same rank as the original algebra. How do we find these? Add ~~the~~ a new lowest root $\vec{\alpha}_0$ to our set of roots. For all other i, $\vec{\alpha}_0 - \vec{\alpha}_i$ is not a root, i.e., cannot lower by $\vec{\alpha}_i$ for any other i. So

$$\frac{2\,\vec{\alpha}_0 \cdot \vec{\alpha}_i}{|\vec{\alpha}_i|^2} \quad \text{is a non-negative integer.}$$

But there is a linear relation $-\vec{\alpha}_0 = \sum_{j=1}^{n} k_j \vec{\alpha}_j$ because the lowest root is the opposite of the highest root. Diagrammatically:

Dynkin

A_n

B_n

C_n

D_n

F_4

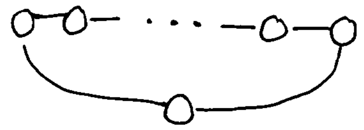
G_2

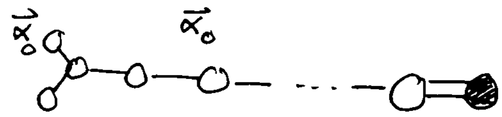
E_6

E_7

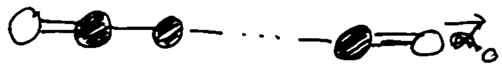
E_8

Extended Dynkin

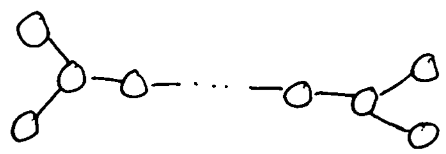

A_n'

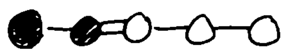


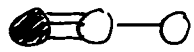
B_n'

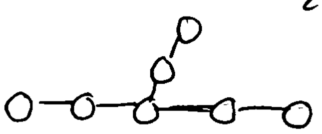


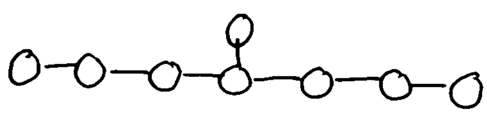
C_n'

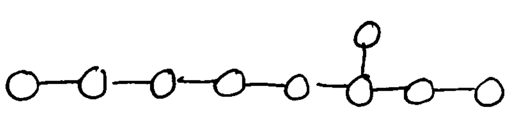

D_n'



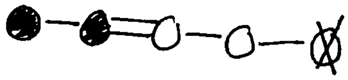

G_2'

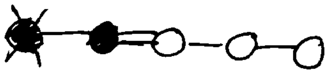

E_6'

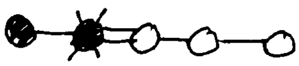

E_7'

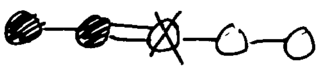

E_8'

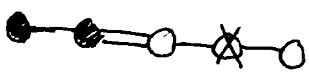
Example: $F_4 \longrightarrow F_4'$

 $\longrightarrow F_4$

 $\longrightarrow B_4 = so(9)$

 $\longrightarrow A_1 \times A_3 = su(2) \times su(3)$

 $\longrightarrow A_2 \times A_2 = su(3) \times su(3)$

 $\longrightarrow C_3 \times A_1 = sp(6) \times su(2)$

These are the maximal sub-algebras of F_4:

$so(9) \subsetneq F_4$, $su(2) \times su(3) \subseteq F_4$, $su(3) \times su(3) \subseteq F_4$, $sp(6) \times su(2) \subseteq F_4$.