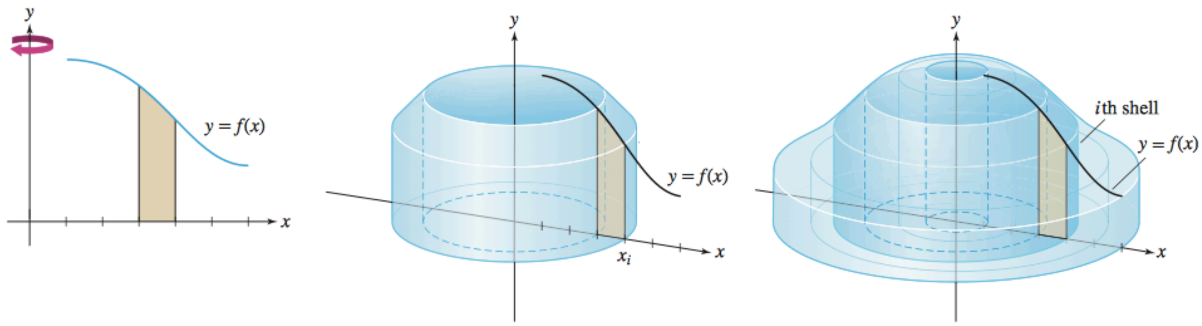
§6.4: Shell Method
§6.5: Work and Energy
Math 1910

Name: Solutions
September 26, 2017

One-page Review

(1) **Shell Method**: When you rotate the region between two graphs around an axis, the segments **parallel** to the axis generate cylindrical shells. The volume V of the solid of revolution is the integral of the surface areas of the shells.

$$V = \int \boxed{2\pi(\text{radius})(\text{height of shell})\, dr} \ (1).$$



(2) What is the volume of

(a) the region between $f(x)$, $f(x) \geq 0$, and the x-axis for $x \in [a, b]$, rotated around the y-axis?

$$V = \boxed{2\pi \int_a^b x f(x)\, dx.} \ (2)$$

(b) the region between $f(x)$ and $g(x)$, $f(x) \geq g(x) \geq 0$, for $x \in [a, b]$ rotated around the y-axis?

$$V = \boxed{2\pi \int_a^b (f(x) - g(x))\, dx.} \ (3)$$

(c) the region between $f(x)$, $f(x) \geq 0$ and the x-axis for $x \in [a, b]$, rotated around the line c?

If $c \leq a$, $V = \boxed{2\pi \int_a^b (x - c) f(x)\, dx.}$ (4) If $c \geq a$, $V = \boxed{2\pi \int_a^b (c - x) f(x)\, dx.}$ (5)

(3) The work W performed to move an object from a to b along the x-axis by applying a force of magnitude $F(x)$ is $W = \int_a^b F(x)\, dx$.

(4) To compute work against gravity, first decompose an object into N layers of equal thickness Δy, and then express the work performed on a thin layer as $L(y)\Delta y$, where

$$L(y) = g \times \text{density} \times \text{area of } y \times \text{distance lifted}.$$

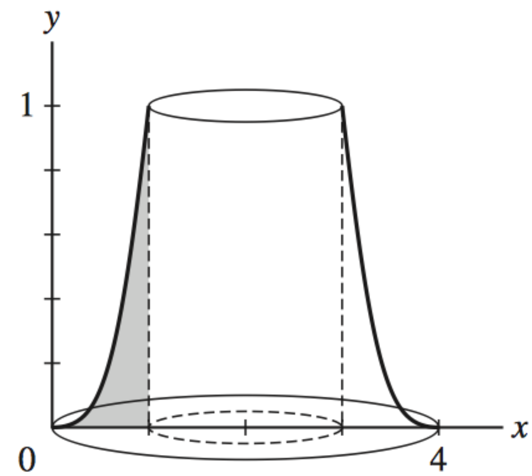
The total work performed to lift the object from height a to height b is $W = \boxed{\int_a^b L(y)\, dy}$ (6).

PROBLEMS

(1) Sketch the solid obtained by rotating the region underneath the graph of f over the interval about the given axis, and calculate its volume using the shell method.

(a) $f(x) = x^3$, $x \in [0, 1]$, about $x = 2$.

SOLUTION:

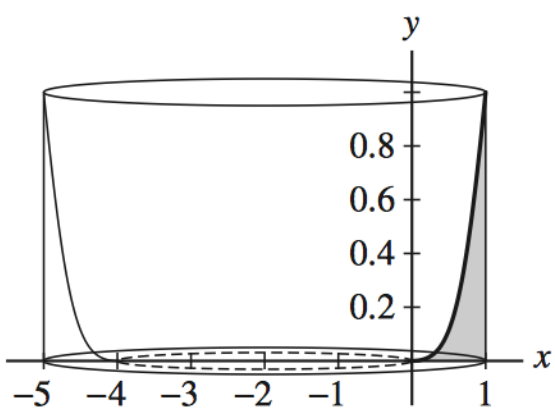


Each shell has radius $2 - x$ and height x^3, so the volume of this solid is

$$2\pi \int_0^1 (2-x)(x^3)\,dx = 2\pi \int_0^1 (2x^3 - x^4)\,dx = 2\pi \left(\frac{x^4}{2} - \frac{x^5}{2} \right) \Big|_0^1 = \frac{3\pi}{5}.$$

(b) $f(x) = x^3$, $x \in [0, 1]$, about $x = -2$.

SOLUTION:

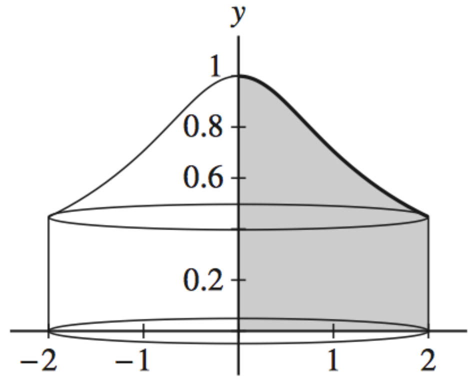


Each shell has radius $x - (-2) = x + 2$ and height x^3, so the volume of this solid is

$$2\pi \int_0^1 (2+x)(x^3)\,dx = 2\pi \int_0^1 (2x^3 + x^4)\,dx = 2\pi \left(\frac{x^4}{2} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{7\pi}{5}.$$

(c) $f(x) = \frac{1}{\sqrt{x^2+1}}$, $x \in [0, 2]$, about $x = 0$.

SOLUTION:



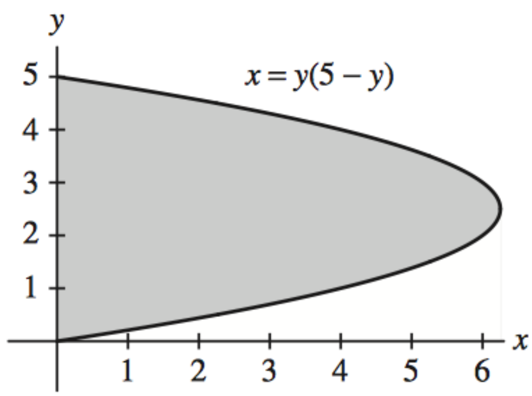
Each shell has radius x, and height $\frac{1}{\sqrt{x^2+1}}$, so the volume of the solid is

$$2\pi \int_0^2 x \left(\frac{1}{\sqrt{x^2+1}} \right) dx = 2\pi \sqrt{x^2+1} \Big|_0^2 = 2\pi \left(\sqrt{5} - 1 \right).$$

(2) Use the most convenient method (disk/washer or shell) to find the given volume of rotation.

(a) Region between $x = y(5-y)$ and $x = 0$, rotated about the y-axis.

SOLUTION: Examine the picture below, which shows the region in question. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizonally, then the right endpoint of a slice is always along the parabola and the left endpoint is on the y-axis. So it's easier to do horizontal slices.

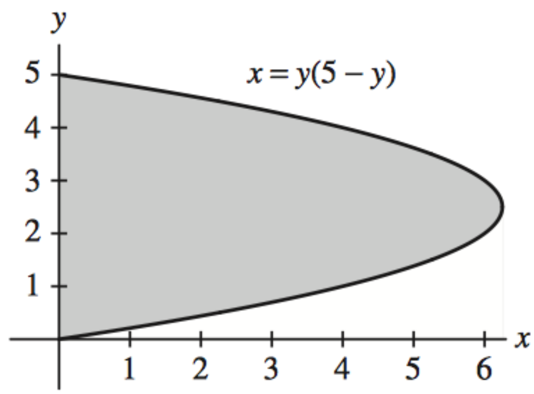


Now suppose the region is rotated about the y-axis. Because a horizontal slice is perpendicular to the y-axis, we will calculate the volume of the resulting solid using the disk method. Each cross section is a disk of radius $R = y(5-y)$, so the volume is

$$\pi \int_0^5 y^2(5-y)^2\,dy = \pi \int_0^5 (25y^2 - 10y^3 + y^4)\,dy = \pi \left(\frac{25}{3}y^3 - \frac{5}{2}y^4 + \frac{1}{5}y^5 \right) \Big|_0^5 = \frac{625\pi}{6}.$$

(b) Region between $x = y(5-y)$ and $x = 0$, rotated around the x-axis.

SOLUTION: Examine the ?gure below, which shows the region bounded by $x = y(5?y)$ and $x = 0$. If the indicated region is sliced vertically, then the top of the slice lies along one branch of the parabola $x = y(5-y)$ and the bottom lies along the other branch. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along the parabola and left endpoint always lies along the y-axis. Clearly, it will be easier to slice the region horizontally.

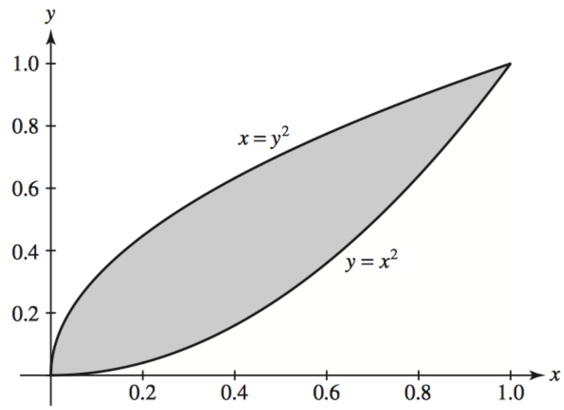


Now, suppose the region is rotated about the x-axis. Because a horizontal slice is parallel to the x-axis, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of y and a height of $y(5-y)$, so the volume is

$$2\pi \int_0^5 y^2(5-y)\,dy = 2\pi \int_0^5 (5y^2 - y^3)\,dy = 2\pi \left(\frac{5}{3}y^3 - \frac{1}{4}y^4 \right) \Big|_0^5 = \frac{625\pi}{6}.$$

(c) Region between $y = x^2$ and $x = y^2$, rotated about $x = 3$.

SOLUTION: Examine the figure below, which shows the region bounded by the $y = x^2$ and $x = y^2$. If the indicated region is sliced vertically, then the top of the slice lies along $x = y^2$ and the bottom lies along $y = x^2$. On the other hand, if the region is sliced horizontally, then the right endpoint of the slice always lies along $y = x^2$ and left endpoint always lies along $x = y^2$. Thus, for this region, either choice of slice will be convenient. To proceed, let's choose a vertical slice.

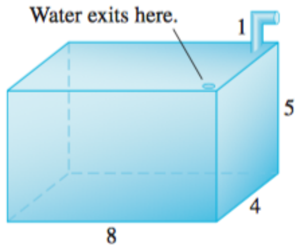


Now rotate the region about $x = 3$. Because a vertical slice is parallel to $x = 3$, we will calculate the volume of the resulting solid using the shell method. Each shell has a radius of $3 - x$ and a height of $\sqrt{x} - x^2$, so the volume is

$$\begin{aligned} 2\pi \int_0^1 (3 - x)(\sqrt{x} - x^2)\,dx &= 2\pi \int_0^1 (3x^{1/2} - x^{3/2} - 3x^2 + x^3)\,dx \\ &= 2\pi \left(2x^{3/2} - \frac{2}{5}x^{5/2} - x^3 + \frac{1}{4}x^4 \right) \Bigg|_0^1 = \frac{17\pi}{10}. \end{aligned}$$

(3) Calculate the work (in Joules) required to pump all of the water out of a full tank with the shape described. Distances are in meters, and the density of water is 1000 kg/m^3.

(a) A rectangular tank, with water exiting from a small hole in the top.



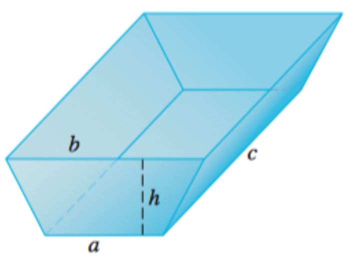
SOLUTION: Place the origin at the top of the box, and let the positive y-axis point downward. The volume of one layer of water is $32\Delta y$ cubic meters, so the force needed to lift it is

$$(9.8)(1000)(32)\Delta y = 313600\Delta y \text{ Newtons.}$$

Each layer must be lifted y meters, so the total work needed to empty the tank is

$$\int_0^5 313600y\,dy = 156800y^2\Big|_0^5 = 3.92 \times 10^6 \text{ Joules.}$$

(b) A trough as in the picture, where the water exits by pouring over the sides.



SOLUTION: Place the origin along the bottom edge of the trough, and let the positive y-axis point upward. From similar triangles, the width of a layer of water at height y meters is

$$w = a + \frac{y(b-a)}{h} \text{ meters,}$$

so the volume of each layer is

$$wc\Delta y = c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ meters}^3.$$

Thus, the force needed to lift a layer is

$$9800c\left(a + \frac{y(b-a)}{h}\right)\Delta y \text{ Newtons.}$$

Each layer must be lifted $h - y$ meters, so the total work needed to empty the tank is

$$\int_0^h 9800(h-y)c\left(a + \frac{y(b-a)}{h}\right) dy = 9800c\left(\frac{ah^3}{3} + \frac{bh^2}{6}\right) \text{ Joules.}$$