§7.1: Exponential Functions
§7.2: Inverse Functions
§7.3 Logarithms

Math 1910

Name: Solutions

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One-page Review

(1) $f(x) = b^x$ is increasing if $b > 1$ (1) and decreasing if $b < 1$ (2).

(2) The derivative of $f(x) = b^x$ is $\frac{d}{dx} b^x =$ $b^x \ln(b)$ (3).

(3) $\frac{x^x}{e} =$ e^x (4) and $\frac{x^{f(x)}}{e} =$ $f'(x)e^{f(x)}$ (5) and $\frac{x^{kx+b}}{e} =$ ke^{kx+b} (6).

(4) $\int e^x \, dx =$ $e^x + C$ (7) and $\int e^{kx+b} =$ $\frac{1}{k}e^{kx+b} + C$ (8) for constants k, b.

(5) A function f with domain D is **one to one** if $f(x) = c$ has at most one solution with $x \in D$. (9)

(6) Let f have domain D and range R. The **inverse** f^{-1} is the unique function with domain R and range D such that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ (10).

(7) The inverse of f exists if and only if f is one-to-one (11) on its domain.

(8) **Horizontal Line Test:** f is one-to-one if and only if every horizontal line intersects the graph of f only once. (12)

(9) To find the inverse function, solve $y = f(x)$ for x (13) in terms of y (14).

(10) The graph of f^{-1} is obtained by reflecting (15) the graph of f through the line $y = x$ (16).

(11) If f is differentiable and one-to-one with inverse g, then for x such that $f'(g(x)) \neq 0$,

$$g'(x) = \frac{1}{f'(g(x))}.$$

(12) The inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b(x)$ (17).

(13) Logarithm Rules

(a) $\log_b(1) =$ 0 (18) and $\log_b(b) =$ 1 (19).

(b) $\log_b(xy) =$ $\log_b(x) + \log_b(y)$ (20) and $\log_b\left(\frac{x}{y}\right) =$ $\log_b(x) - \log_b(y)$ (21)

(c) **Change of Base:** $\frac{\log_a(x)}{\log_a(b)} =$ $\log_b(x)$ (22).

(d) $\log_b(x^n) =$ $n \log_b(x)$ (23).

(14) $\frac{x}{\ln}(x) =$ $\frac{1}{x}$ (24) and $\frac{x}{\log_b}(x) =$ $\frac{1}{\ln(b)x}$ (25)

(15) $\int \frac{1}{x} \, dx =$ $\ln|x| + C$ (26).

PROBLEMS

(1) Calculate the derivative.

(a) $f(x) = 7e^{2x} + 3e^{4x}$

SOLUTION: $f'(x) = 14e^{2x} + 12e^{4x}$.

(b) $f(x) = e^{e^x}$

SOLUTION: $f'(x) = e^x e^{e^x}$

(c) $f(x) = 3^x$

SOLUTION: $f'(x) = 3^x \ln(3)$

(d) $f(t) = \dfrac{1}{1 - e^{-3t}}$

SOLUTION: $f'(t) = -3(1 - e^{-3t})^{-2} e^{-3t}$

(e) $f(t) = \cos(te^{-2t})$

SOLUTION: $f'(t) = -\sin(te^{-2t})(e^{-2t} + -2te^{-2t})$

(f) $\displaystyle\int_4^{e^x} \sin t \, dt$

SOLUTION: Recall that $\frac{x}{\int}{}^{f(x)}_{a} g(t)\,dt = g(f(x))f'(x)\,dx$. So

$$\frac{d}{dx} \int_4^{e^x} \sin t \, dt = \sin(e^x) e^x.$$

(g) $f(x) = x \ln x$

SOLUTION: $f'(x) = \ln x + 1$

(h) $f(x) = \ln(x^5)$

SOLUTION: $f'(x) = \dfrac{5}{x}$

(i) $f(x) = \ln(\sin(x) + 1)$

SOLUTION: $f'(x) = \dfrac{\cos(x)}{\sin(x) + 1}$

(j) $f(x) = e^{\ln(x)^2}$

SOLUTION: $f'(x) = e^{(\ln x)^2} 2 \dfrac{\ln(x)}{x}$

(k) $f(x) = \log_a(\log_b(x))$

SOLUTION: $f'(x) = \dfrac{1}{\ln(a)\, x \ln(x)}$

(l) $f(x) = 16^{\sin x}$

SOLUTION: $f'(x) = \ln(16)\cos(x) 16^{\sin x}$

(2) Calculate the integral.

(a) $\displaystyle\int \frac{7}{x}\,dx$

SOLUTION: $7 \ln|x| + C$

(b) $\displaystyle\int e^{4x}\,dx$

SOLUTION: $\dfrac{1}{4} e^{4x} + C$

(c) $\displaystyle\int \frac{\ln x}{x}\,dx$

SOLUTION: Set $u = \ln x$, so $du = \frac{1}{x} dx$. Therefore,

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\ln(x)^2 + C.$$

(d) $\displaystyle\int \frac{1}{9x - 3}\,dx$

SOLUTION: Let $u = 9x - 3$. Then $du = 9dx$ and substituting gives

$$\int \frac{1}{9u}\,du = \frac{1}{9}\ln|u| + C = \frac{1}{9}\ln|9x - 3| + C.$$

(e) $\displaystyle\int_2^3 (e^{4t-3})\,dt$

SOLUTION: $\displaystyle\int_2^3 (e^{4t-3})\,dt = e^{-3}\int_2^3 e^{4t}\,dt = e^{-3}\left(\frac{1}{4}e^{4t}\right)\Big|_2^3 = \frac{e^{-3}}{4}\left(e^{12} - e^8\right) = \frac{1}{4}(e^9 - e^5)$

(f) $\displaystyle\int e^t \sqrt{e^t + 1}\,dt$

SOLUTION: Let $u = e^t + 1$. Then $du = e^t\,dt$, so the integral becomes

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(e^t + 1)^{3/2} + C$$

(g) $\displaystyle\int e^x \cos e^x\,dx$

SOLUTION: Let $u = e^x$. Then $du = e^x\,dx$, so

$$\int e^x \cos e^x\,dx = \int \cos u = \sin u + C = \sin e^x + C.$$

(h) $\displaystyle\int \tan(4x + 1)\,dx$

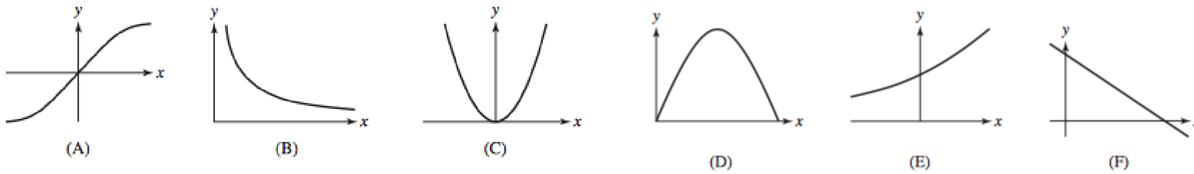
SOLUTION: First, rewrite the integral as

$$\int \tan(4x + 1)\,dx = \int \frac{\sin(4x + 1)}{\cos(4x + 1)}\,dx$$

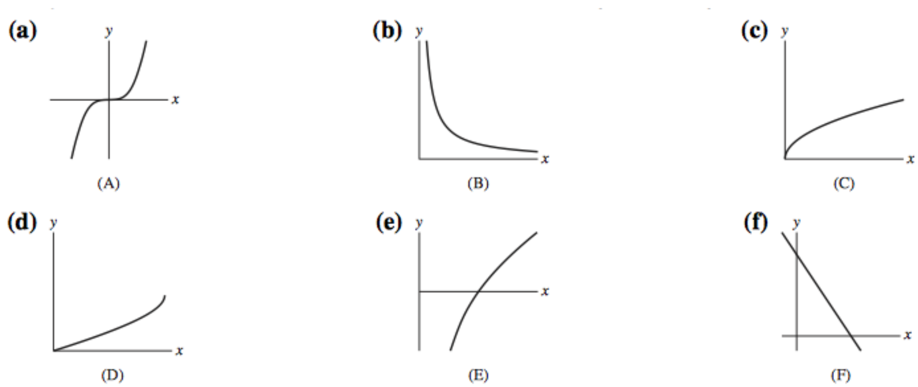
then let $u = \cos(4x + 1)$, so $du = -4\sin(4x + 1)\,dx$. Hence,

$$\int \frac{\sin(4x + 1)}{\cos(4x + 1)}\,dx = -\frac{1}{4}\int \frac{1}{u}\,du = -\frac{1}{4}\ln|\cos(4x + 1)| + C$$

(3) For each function shown below, sketch the graph of the inverse.



SOLUTION:



(4) Calculate $g(b)$ and $g'(b)$, where g is the inverse of f.

(a) $f(x) = x + \cos x$, $b = 1$.
SOLUTION: $g(1) = 0$, $g'(1) = 1$.

(b) $f(x) = 4x^3 - 2x$, $b = -2$.
SOLUTION: $g(-2) = -1$, $g'(-2) = \frac{1}{10}$.

(c) $f(x) = \sqrt{x^2 + 6x}$ for $x \geq 0$, $b = 4$.
SOLUTION: $g(4) = 2$, $g'(4) = \frac{4}{5}$.

(d) $f(x) = \frac{1}{x+1}$, $b = \frac{1}{4}$.
SOLUTION: $g(1/4) = 3$, $g'(1/4) = -16$.

(5) Which of the following statements are true and which are false? If false, modify the statement to make it correct.

(a) If f is increasing, then f^{-1} is increasing.
SOLUTION: True.

(b) If f is concave up, then f^{-1} is concave up.
SOLUTION: False. Reflecting the graph of f across the line $y = x$ to get the graph of f^{-1} means that if the graph of f is concave up, then the graph of f^{-1} is concave down.

(c) If f is odd then f^{-1} is odd.
SOLUTION: Think of what the graph of an odd function looks like. Reflecting the graph across the line $y = x$ preserves this property.

(d) Linear functions $f(x) = ax + b$ are always one-to-one.
SOLUTION: True. The inverse is $f^{-1}(x) = \frac{1}{a}(x - b)$.

(e) $f(x) = \sin(x)$ is one-to-one.

SOLUTION: False. The graph of $f(x) = \sin(x)$ fails the horizontal line test. But if we restrict the domain to $(-\pi, \pi)$, then this is true and $\arcsin(x)$ is the inverse of $\sin(x)$ on this domain.