§8.2: Trigonometric Integrals
§8.3: Trigonometric Substitution
§8.5: Partial Fractions

Name: Solutions

Math 1910

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Rapid Review

(1) **Power-reducing identitites**

$$\cos^2(x) = \frac{1+\cos(2x)}{2}, \qquad \sin^2(x) = \frac{1-\sin(2x)}{2}$$

(2) **Completing the square.** If you have an integral with a $1/\sqrt{ax^2+bx+c}$ in it, you need to complete the square. Rewrite

$$ax^2+bx+c = a(x-h)^2+k$$

where

$$h = \boxed{-\frac{b}{2a}}^{(1)}, \qquad k = \boxed{c-\frac{b^2}{4a}}^{(2)}$$

(3) **Partial Fractions:** if you have an expression that looks like

$$\frac{f(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)}$$

where there are no repeats in the a_i's, then you can write

$$\frac{f(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}$$

If there are repeats in the a_i's, then $(x-a)^n$ contributes

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_n}{(x-a)^n}.$$

And $(x^2+b)^n$ contributes

$$\frac{A_1x+B_1}{x^2+b}+\frac{A_2x+B_2}{(x^2+b)^2}+\ldots+\frac{A_nx+B_n}{(x^2+b)^n}.$$

PROBLEMS

(1) For each of the following integrals, should you use substitution, integration by parts, trig substitution, partial fractions, or something else?

(a) $\int \ln(x)\,dx$

SOLUTION: Integration by parts, with $u = \ln(x)$ and $dv = dx$.

(b) $\int \sqrt{4x^2 - 1}\,dx$

SOLUTION: Trig substitution, with $x = \frac{1}{2}\sec\theta$.

(c) $\int \frac{x}{\sqrt{12 - 6x - x^2}}\,dx$

SOLUTION: Complete the square under the radical, $12 - 6x - x^2 = 21 - (x+3)^2$, and then substitute $u = x + 3$.

(d) $\int \sin^3(x)\cos^3(x)\,dx$

SOLUTION: Rewrite $\sin^3(x) = (1 - \cos^2(x))\sin(x)$, and let $u = \cos(x)$.

(e) $\int x\sec^2(x)\,dx$

SOLUTION: Use integration by parts, with $u = x$ and $dv = \sec^2(x)\,dx$.

(f) $\int \frac{1}{\sqrt{9 - x^2}}\,dx.$

SOLUTION: Either substitute $u = 3x$ and use the formula for the derivative of $\sin^{-1}(u)$, or substitute $x = 3\sin\theta$.

(g) $\int x^2\sqrt{x+1}\,dx$

SOLUTION: Make the substitution $u = x + 1$. Then $du = dx$ and $x^2 = (u-1)^2 = u^2 - 2u + 1$.

(h) $\int \frac{1}{(x+1)(x+2)^3}\,dx$

SOLUTION: Use partial fractions to decompose

$$\frac{1}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}.$$

(i) $\int \frac{1}{(x+12)^4}\,dx$

SOLUTION: Substitute $u = x + 12$.

(2) Evaluate the integral.

(a) $\int \frac{1}{\sqrt{x^2 + 9}}\,dx$

SOLUTION: Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$, and $x^2 - 9 = 9\sec^2\theta - 9 = 9(\sec^2\theta - 1) = 9\tan^2\theta$, so we have

$$\int \frac{1}{\sqrt{x^2+9}}\,dx = \int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \boxed{\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C}$$

(b) $\int x\sqrt{x^2-5}\,dx.$

SOLUTION: Substitute $u = x^2 - 5$, so then

$$\int x\sqrt{x^2-5}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}u^{3/2} + C = \boxed{\frac{1}{3}(x^2-5)^{3/2} + C}$$

(c) $\int \frac{3x+5}{x^2-4x-5}\,dx$

SOLUTION: Factor the denominator as $x^2 - 4x - 5 = (x+1)(x-5)$. So we're trying to do partial fractions with

$$\frac{3x+5}{x^2-4x-5} = \frac{3x+5}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x-1}.$$

Clearing denominators, we have

$$3x + 5 = A(x+1) + B(x-5).$$

Set $x = 5$ to get $A = \frac{10}{3}$. Set $x = -1$ to get $B = -\frac{1}{3}$. Then we have

$$\frac{3x+5}{x^2-4x-5} = \frac{\frac{10}{3}}{x-5} + \frac{-\frac{1}{3}}{x+1}.$$

Therefore,

$$\int \frac{3x+5}{x^2-4x-5}\,dx = \frac{10}{3}\int \frac{1}{(x-5)}\,dx - \frac{1}{3}\int \frac{1}{x+1}\,dx = \boxed{\frac{10}{3}\ln|x-5| - \frac{1}{3}\ln|x+1| + C}.$$

(d) $\int e^{2x}\cos(x)\,dx$

SOLUTION: Use integration by parts with $u = e^{2x}$ and $dv = \cos(x)\,dx$. Then

$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\,dx.$$

Do integration by parts again, this time with $u = e^{2x}$ and $dv = \sin(x)\,dx$. So we have

$$\begin{aligned}\int e^{2x}\cos x\,dx &= e^{2x}\sin(x) - \int 2e^{2x}\sin(x)\,dx \\ &= e^{2x}\sin(x) - 2\left(-e^{2x}\cos(x) - \int (-\cos(x))2e^{2x}\,dx\right) \\ &= e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,dx\end{aligned}$$

Now add $4\int e^{2x}\cos(x)\,dx$ to both sides, so we have

$$5\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) + 2e^{2x}\cos(x) + C$$

Divide both sides by 5 to get the answer,

$$\boxed{\frac{1}{5}e^{2x}\sin(x) + \frac{2}{5}e^{2x}\cos(x) + C}$$

(e) $\int \cos^2\theta \sin^2\theta \, d\theta$

SOLUTION: First use the identity $\cos^2\theta = 1 - \sin^2\theta$ to write

$$\int \cos^2\theta \sin^2\theta \, d\theta = \int (1-\sin^2\theta)\sin^2\theta \, d\theta = \int \sin^2\theta \, d\theta - \int \sin^4\theta \, d\theta.$$

Using the reduction formula for $\sin^m(x)$,

$$\begin{aligned}
\int \cos^2\theta \sin^2\theta \, d\theta &= \int \sin^2\theta \, d\theta - \left(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta \, d\theta\right) \\
&= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\int \sin^2\theta \, d\theta \\
&= \frac{1}{4}\sin^3\theta\cos\theta + \frac{1}{4}\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\int d\theta\right) \\
&= \boxed{\frac{1}{4}\sin^3\theta\cos\theta - \frac{1}{8}\sin\theta\cos\theta + \frac{1}{8}\theta + C}
\end{aligned}$$

(f) $\int \cos(x)\sin^5(x)\,dx$

SOLUTION: Substitute $u = \sin x$, $du = \cos(x)\,dx$.

$$\int \cos(x)\sin^5(x)\,dx = \int u^5\,du = \frac{u^6}{6} + C = \boxed{\frac{\sin^6(x)}{6} + C.}$$

(g) $\int \frac{1}{x(x-1)^2}\,dx$

SOLUTION: Use partial fractions to write

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing denominators gives

$$1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Setting $x = 0$ gives $A = 1$; setting $x = 1$ gives $C = 1$ and setting $x = 2$ gives $B = -1$. The result is

$$\frac{1}{x(x-1)^2} = \frac{1}{x} + \frac{-1}{x-1} + \frac{1}{(x-1)^2}.$$

Now we can integrate.

$$\int \frac{1}{x(x-1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx = \boxed{\ln|x| - \ln|x-1| - \frac{1}{x-1} + C.}$$

(h) $\int \cos^2(4x)\,dx$

SOLUTION: Use the substitution $u = 4x$ and $du = 4\,dx$. Then

$$\begin{aligned}
\int \cos^2(4x)\,dx &= \frac{1}{4}\int \cos^2(u)\,du \\
&= \frac{1}{4}\left(\frac{1}{2}u + \frac{1}{2}\sin(u)\cos(u)\right) + C \\
&= \boxed{\frac{1}{2}x + \frac{1}{8}\sin(4x)\cos(4x) + C}
\end{aligned}$$

(i) $\displaystyle\int \frac{3}{(x+1)(x^2+x)}\,dx$

SOLUTION: Do partial fractions

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{(x+1)(x)(x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Clearing denominators gives $3 = A(x+1)^2 + Bx(x+1) + Cx$, and setting $x = 0$ give $A = 3$; setting $x = -1$ give $C = -3$. Now plug in $A = 3$ and $C = -3$ to get

$$3 = 3(x+1)^2 + Bx(x+1) - 3x$$

Then set $x = 1$ to get $B = -3$. Therefore,

$$\begin{aligned}\int \frac{3}{(x+1)(x^2+x)}\,dx &= 3\int \frac{1}{x}\,dx - 3\int \frac{1}{x+1}\,dx - 3\int \frac{1}{(x+1)^2}\,dx \\ &= \boxed{3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + C}\end{aligned}$$

(j) $\displaystyle\int (\ln x + 1)\sqrt{(x\ln x)^2 + 1}\,dx$

SOLUTION: Let $u = x\ln x$. Then $du = (1 + \ln x)\,dx$, and

$$\int (\ln x + 1)\sqrt{(x\ln x)^2 + 1}\,dx = \int \sqrt{u^2+1}\,du.$$

Then substitute $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $u^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$. Therefore,

$$\int \sqrt{u^2+1}\,du = \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

Substitute back $\tan\theta = u$ and $\sec\theta = \sqrt{u^2+1}$, so

$$\int \sqrt{u^2+1}\,du = \frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln|u + \sqrt{u^2+1}| + C.$$

Finally substitute back $u = x\ln x$.

$$\boxed{\frac{1}{2}x\ln x\sqrt{(x\ln x)^2+1} + \frac{1}{2}\ln\left|x\ln x + \sqrt{(x\ln x)^2+1}\right|}$$

Fun fact: Mathematica wouldn't do this integral for me, but I could do it by hand!