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Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being

by George Lakoff and Rafael E. Núñez
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reviewed by David W. Henderson, Cornell University

This book is an attempt by cognitive scientists to launch a new discipline: *cognitive science of mathematics*. This discipline would include the subdiscipline of *mathematical idea analysis*.

What prompted me to read this book were the endorsements on the back cover by four well-known mathematicians (Reuben Hersh, Felix Browder, Bill Thurston, and Keith Devlin). I was deeply excited by the authors' purpose stated in their Preface and Introduction:

> Mathematical idea analysis, as we seek to develop it, asks what theorems *mean* and *why* they are true *on the basis of what they mean*. We believe it is important to reorient mathematics teaching toward understanding mathematical ideas and understanding *why* theorems are true. (page xv)

> We will be asking how normal human cognitive mechanisms are employed in the creation and understanding of mathematical ideas. (page 2)

It was with enthusiasm that I read the book together with the members of a mathematics department seminar at Cornell University. However, I became profoundly disappointed. In the end, I felt that the book does a disservice both to cognitive science and to mathematics because:

- There are numerous errors in mathematical fact. Only some of these are corrected on the book's web page: http://www.unifr.ch/perso/nunezr/welcome.html). There are so many errors that it seems inconceivable to me that the four mathematicians who have endorsements on the back cover could have read the book without noticing them. On the web page the authors blame the publisher for most of the errors. They report that the second printing has even more errors and has been recalled!
- The authors assert: "The cognitive science of mathematics asks questions **that mathematics does not, and cannot, ask about itself**. How do we understand such basic concepts as infinity, zero, lines, points, and sets using our everyday conceptual apparatus?" (page 7) [my emphasis]. I will show below that this statement is false. Most of the book after the Third Chapter provides a powerful argument that a mathematician that asks these questions is precisely what is needed.
- The authors seem to be working from a common misconception about the nature of what mathematicians do.

Let me add, however, that I see this book as an earnest attempt to understand the meaning of mathematics. I hope the book will encourage cognitive scientists and mathematicians to talk to one another. Perhaps *together* we can develop a clearer understanding of the meanings of mathematical concepts and of "what theorems *mean* and *why* they are true *on the basis of what they mean*." In addition, working together on mathematical idea analysis should help us toward a deeper understanding of mathematical intuition, such as expressed by David Hilbert

> If we now begin to construct mathematics, we shall first set our sights upon elementary number theory; we recognize that we can obtain and prove its truths through contentual intuitive considerations. ([5], page 469)

Cognitive Science – Cognitive Metaphor

The authors start in Chapter One by surveying discoveries by cognitive science of an innate arithmetic of the numbers 1 through 4 in most humans (and some animals). The problem is how to connect this innate arithmetic to the arithmetic of all numbers and to the rest of mathematics. According to the authors: "One of the principle results of cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts. This phenomenon has been studied scientifically for more than two decades and is in general as well established as any result in cognitive science." (page 39, 41)

For the authors, "metaphor" has a much more complex (and technical) meaning than it does for most of us. They describe a *cognitive metaphor* as "*inference-preserving cross-domain mapping*—a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic)." For some cognitive metaphors, cognitive scientists have detected actual neural connections in the brain.

To illustrate the authors' notion of cognitive metaphor, let us look at the "Arithmetic Is Object Collection" metaphor. This metaphor, as with all cognitive metaphors, consists of two domains and a mapping. In this case:

- a *source domain*: "collections of objects of the same size (based on our commonest experiences with grouping objects)"
- a *target domain*: natural numbers with addition and substraction (which the authors call "arithmetic".
- a *cross-domain mapping* as described in this table:

<u>Arithmetic Is Object Collection Metaphor</u>

Source Domain Object Collection		*Target Domain* Arithmetic
Collections of objects of the same size	→	Natural numbers
The size of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The smallest collection	→	The unit (One)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

It is basic to the author's arguments that the notions in the left-hand column have literal meaning, while the notions in the right-hand column do not. The notions in the right-hand column gain their meanings from the notions in the left-hand column via the metaphor. Each conceptual metaphor has *entailments*, which for this metaphor the authors describe as follows:

> Take the basic truths about collections of physical objects. Map them onto statements about numbers, using the metaphorical mapping. The result is a set of 'truths' about the natural numbers under the operations of addition and subtraction. (page 56)

They list 17 such entailments for arithmetic, that seem to me to be part of the what Hilbert called the "the truths" (involving only addition and subtraction) of elementary number theory "that we can obtain and prove through contentual intuitive considerations".

Mathematicians are Needed

As far as I can tell it is at this point (in Chapter 3 of 16) the authors leave results established by cognitive science research and move into the realm they describe as "hypothetical" and "plausible". The remainder of the book deals with plausible cognitive metaphors which the authors hypothesize account for our understanding of the meanings of real numbers, set theory, infinity (in varied forms), continuity, space-filling curves, infinitesimals, and the Euler equation $e^{\pi i} + 1 = 0$. It is also at this point that I think the authors' arguments and discussions need input from mathematicians and teachers of mathematics. I will illustrate by describing some of authors' metaphors and the improvements that I think mathematicians can make.

Actual Infinity: The authors "hypothesize" that the idea of infinity in mathematics is metaphorical. "Literally, there is no such thing as the result of an endless process: If a process has no end, there can be no 'ultimate result.' However, the mechanism of metaphor allows us to conceptualize the 'result' of an infinite process in terms of a process that does have an end." (page 158)

The authors "hypothesize that all cases of actual infinity are special cases of" a single cognitive metaphor which they call the *Basic Metaphor of Infinity* or BMI. BMI is a mapping from the domain, Completed Iterative Processes, to the domain, Iterative Process That Go On and On. A completed iterative process has 4 parts, all literal: the beginning state, the process that from an intermediate state produces the next state, an intermediate state, and the final resultant state that is unique and follows every non-final state. These are mapped onto 4 parts (with the same names and descriptions) of an Iterative Process That Goes On and On, where the first three parts have literal meaning but the last part (the 'final resultant state') has meaning only metaphorically from the cognitive mapping.

I illustrate with a special case from the book.

Parallel lines meet at infinity (using BMI): How do we conceptualize (or give meaning to) the notion in Projective Geometry that two parallel lines meet at infinity? If m and l are two parallel lines in the plane then let the line segment AB be a common perpendicular between them and consider the isosceles triangles on one side of AB. The authors call this the *frame*. They then construct the following special case of BMI.

Parallel Lines Meet At Infinity

Source Domain Completed Iterative Processes		*Target Domain* Isosceles Triangles with Base AB
The beginning state	→	Isosceles triangle ABC_0, where length of AC_0 (=BC_0) is D_0
The process that from $(n\text{-}1)^{\text{th}}$ state produces n^{th} state	→	Form ABC_n from $ABC_{n\text{-}1}$ by making "D_n arbitrarily larger than $D_{n\text{-}1}$"
Intermediate state	→	"$D_n > D_{n\text{-}1}$ and $(90^o - \alpha_n) < (90^o - \alpha_{n\text{-}1})$"
The final resultant state, unique and following every non-final state	→	**"$\alpha_\infty = 90^o$, D_∞ is infinitely long", and the sides meet at a unique C_∞, a point at ∞ (because $D_\infty > D_{n\text{-}1}$, for all finite n.**

They remind the reader that theirs is "not a mathematical analysis, is not meant to be one, and should not be confused with one." They state their "cognitive claim: The concept of 'point at infinity' in projective geometry is, form a cognitive perspective, a special case of the general notion of actual infinity." They admit that they "have at present no experimental evidence to back up this claim."

OK, let us look at this as mathematicians. There are several problems with this metaphor as presented—the three most important (from my perspective) are:

- As we teach in first-year calculus, not every monotone increasing sequence is unbounded. Thus we need more than "D_n arbitrarily larger than D_{n-1}" to insure that "D_∞ is infinitely long."
- This metaphor entails a unique point at infinity on *both* ends of a line, which does not agree with the usage in projective geometry nor with our intuitive finite experience with lines as I will now show.
- The metaphor only indirectly involves "lines" (the primary objects under consideration) and does not give meaning to the question: *Why does a line only have one point at infinity*?

A Mathematician's Metaphor (without BMI): I propose a different metaphor that I have used for years in my geometry classes. This metaphor more closely uses the main notions of projective geometry, lines and their intersections. For the frame of the metaphor we construct the *rotating line frame*: Take a line l in the Euclidean plane, a point A not on l, and a line m in the plane that is conceived of as free to rotate about A. As we rotate m about A, most positions of m result in a literal unique intersection with l and different positions result in different intersection points. There is no (literal) point of intersection when m is parallel to l. To be more specific: Imagine that m starts perpendicular to l and then rotates at a constant rate so that at time T it is parallel to l and then stops when it is again perpendicular to l. We now define a cognitive metaphor in the authors' sense with

- *Source Domain*: Continuous motion of a particle along a curve through a point P. (Let T be the time that the particle is at P.)
- *Target Domain*: The rotating line frame described above.
- *Cross Domain mapping*:

Projective Metaphor

Source Domain		*Target Domain*
Motion of the particle before T	→	Motion of the particle before T
Motion of the particle after T	→	Motion of the particle after T
At time T the particle is at a unique point P	→	**At time T the particle is at a unique point** (which we call the point at ∞ on l)

In a course presenting projective geometry, I show how a projective transformation can give a way of actually seeing (an image of) the point at ∞.

I see no need in this description for the authors' Basic Metaphor of Infinity. I propose this metaphor as a counterexample to the authors' hypothesis "that all cases of actual infinity are special cases of" the single cognitive metaphor BMI.

Not always metaphors: In addition, there are many cases (especially in geometry, which the authors considered only lightly) where our cognitive analysis does not produce cognitive metaphors. For example, look at the notion of "straightness". We say that "straight lines" in spherical geometry are the great circles on the sphere, but how do we understand what is the meaning of "straight" in this case? An answer sometimes given in textbooks is that, of course, great circles are not literally straight, but we will (metaphorically) call them straight. However, I have argued in [EG] and [DG] that great circles on a sphere are literally straight from an *intrinsic* proper point-of-view. *Extrinsically* (our ordinary view of an observer imaging the sphere from a position in three-space outside the sphere) the great circles are certainly not straight. They are *intrinsically* (the point of view of a 2-dimensional bug whose universe **is** the sphere) straight; that is, the 2-dimensional bug would experience the great circles as straight in its spherical universe. I would like to see a cognitive scientist analyze this situation which (at least on the surface) involves more centrally *imagination* and *point-of-view* rather than *metaphor*, per se.

Misconceptions about Mathematics

The above discussions of the projective metaphor and of straightness constitute a counterexample to authors' assertion that:

> The cognitive science of mathematics asks questions **that mathematics does not, and cannot, ask about itself**. How do we understand such basic concepts as infinity, zero, lines, points, and sets using our everyday conceptual apparatus? (page 7) [my emphasis].

On the book's web page [http://www.unifr.ch/perso/nunezr/warning.html] they explain further:

> [O]ur goal is to characterize mathematics in terms of cognitive mechanisms, not in terms of mathematics itself, e.g., formal definitions, axioms, and so on. Indeed, part of our job is to characterize how such formal definitions and axioms are themselves understood in cognitive terms.

These quotes contain two related misconceptions about mathematics:

- Misconception 1: Mathematics is formal, consisting of formal definitions, axioms, theorems, and proof.
- Misconception 2: Mathematics does not (and can not) ask what mathematical ideas *mean*, how they can be understood, and *why* they are true.

These misconceptions of mathematics are prevalent among non-mathematicians. The blame for this lies mostly with us, the mathematicians. Collectively, we have not done an effective job of communicating to the outside world the nature of our discipline. Nevertheless, many of us (including all four of the mathematicians whose endorsements are on the back cover) have in our writings attempted to dispel these misconceptions (See for example, [1], [2], [3], [4], [6], [7]). In particular, let me quote David Hilbert from the preface of *Geometry and the Imagination* [6]. This book is important because Hilbert is considered to be the "Father of Formalism" and yet he writes:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations. [Hilbert's emphasis]

On Hilbert's "one hand" is the tendency of formal mathematics that Lakoff and Núñez are looking at. On Hilbert's "other hand" is the tendency of mathematics to consider much of what Lakoff and Núñez says that it "does not, and cannot, ask about itself."

Philosophy of mathematics.

I find the authors' discussion of their *philosophy of embodied mathematics* to be profound and think any mathematician who studies it will find her/his own understandings of mathematics stimulated and challenged in constructive ways. But first the mathematicians must overcome their reactions to being told (incorrectly) that mathematicians do not and can not ask how they understand the meanings of mathematical ideas and results.

The authors summarize their view of the philosophy of mathematics with the statement:

> Mathematics as we know it is ... a product of the human mind.... It comes from us! We create it, but it is not arbitrary [because] it uses the basic conceptual mechanisms of the embodied human mind as it has evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history. (page 9)

In Part V of the book the authors expand on this summary and proceed to dismiss (or "disconfirm") other philosophies of mathematics. I recommend that the mathematical reader skip over all arguments dismissing various other philosophies of mathematics because for the most part these arguments are based on shallow summaries of what the various philosophies assert . Further, I do not think that the settling of these arguments is important or necessary for understanding the authors' main points. Regardless of one's philosophical beliefs, I think all mathematicians (and teachers of mathematics) would welcome a

> *conceptual foundations* [for mathematics that] would consist of a thorough mathematical ideas analysis that worked out in detail the conceptual structure of each mathematical domain, showing how those concepts are ultimately grounded in bodily experience and just what the network of ideas across mathematical disciplines looks like. (page 379)

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We need to work together

Cognitive scientists and mathematicians need to work together to develop *mathematical idea analysis*. I believe that most mathematicians and teachers of mathematics are concerned exactly with the things mentioned in these quotes from the authors:

> We believe that revealing the cognitive structure of mathematics makes mathematics more accessible and comprehensible. ... [M]athematical ideas ... can be understood for the most part in everyday terms. (page 7).

We, mathematicians and teachers, would certainly be thankful to cognitive scientists if they could help us in our grappling "not just with *what* is true but with what mathematical ideas *mean*, how they can be understood, and *why* they are true." (page 8)

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