Topological finiteness for edge-vertex enumeration

Ed Swartz *

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Abstract

The number of PL-homeomorphism types of combinatorial manifolds in a fixed dimension with an upper bound on g_2 is finite.

At the intersection of topological and enumerative combinatorics is the relationship between the f-vector of simplicial complex and its topology. The Euler-Poincaré formula is perhaps the oldest such result. More recently, the complete characterization of all possible f-vectors of Cohen-Macaulay complexes [13] is another example.

Combinatorial manifolds (also called combinatorial triangulations), simplicial complexes whose vertex links have a common subdivision with the boundary of the simplex, are a natural class of spaces in which to study this type of question. Every smooth compact manifold has such a triangulation which is unique up to PL-equivalence [17]. However, the sheer variety of topological phenomena which occur in manifolds makes these types of questions much more difficult. For instance, there is no Turing machine which given as input an arbitrary five-dimensional combinatorial manifold Δ and any sequence $(f_0, \dots, f_5)$ of positive integers, outputs whether or not there exists a triangulation Δ' which is PL-homeomorphic to Δ and has exactly f_i faces of dimension i. Since the only combinatorial 5-manifold with the f-vector of $\partial\Delta^6$, the boundary of the 6-simplex, is $\partial\Delta^6$, the existence of such a Turing machine would allow the construction of an algorithm which can determine whether or not a given combinatorial manifold is PL-homeomorphic to $\partial\Delta^6$. As shown by Novikov, this is impossible [4, Section 10].

Our focus is on affine invariants of f_0 and f_1. For α, β, γ real numbers let $L_{\alpha,\beta,\gamma}(\Delta) = \alpha f_1 + \beta f_0 + \gamma$ be an affine invariant of the number of edges and vertices in a simplicial complex. What *qualitative* information does a lower or upper bound on $L_{\alpha,\beta,\gamma}$ give? For instance, suppose $\alpha = 0$. Then knowledge of $L_{0,\beta,\gamma}$ is equivalent to being given the number of vertices. A lower bound for f_0 reveals **no** topological information since repeatedly subdividing a facet increases the number of vertices without changing the PL-homeomorphism type of the complex. An upper bound for the number of vertices evidently restricts one to a finite number of complexes. At that point attention changes to *quantitative* results which delineate exactly the possible spaces and/or restrictions on topological invariants.

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A good example of this is due to Brehm and Kühnel. They proved that if Δ is $(d-1)$-dimensional with $d \geq 4$ and $f_0 < 2d+1$, then the fundamental group of Δ is trivial [3].

When $\alpha \neq 0$ we can, by appropriate scaling and translation, assume that $\alpha = 1$ and $\gamma = 0$, so our invariant is of the form $L_\beta = f_1 + \beta f_0$. From a qualitative point of view there are three cases to consider: 1) $\beta < -d$, 2) $\beta > -d$, 3) $\beta = -d$, where $d = \dim \Delta + 1$.

1) $\beta < -d$. For sufficiently large N, any PL-manifold has combinatorial triangulations with N vertices and $\binom{N}{2}$ edges [15, Corollary 5.15]. Hence a lower bound for L_β carries no topological information. Similarly, since $\beta < -d$, repeatedly subdividing a facet with one new vertex produces triangulations with L_β tending toward $-\infty$. Thus an upper bound for L_β also says nothing about the topology of Δ.

2) $\beta > -d$. As in the first case, large two-neighborly triangulations imply that a lower bound for L_β does not impart any topological information. Write $\beta = -d + \epsilon$, $\epsilon > 0$. Then a lower bound for L_β is a lower bound for $(f_1 - df_0) + \epsilon f_0$. By Theorem 1.1 below $f_1 - df_0 \geq -\binom{d+1}{2}$. Therefore an upper bound limits the number of vertices and there are only a finite number of possible complexes.

3) $\beta = -d$. Our main result, Theorem 2.1, says that for a given upper bound there are only finitely many PL-homeomorphism types. As repeatedly subdividing a facet with one new vertex does not change L_β, there are infinitely many possible complexes. So, up to translation and scaling, $L_{1,-d,0}$ is the unique affine invariant involving f_0 and f_1 which for a given upper bound admits infinitely many combinatorial manifolds, but only finitely many PL-homeomorphism types.

For historical reasons we will study $L_{1,-d,\binom{d+1}{2}}$. This particular invariant is usually called g_2. It has algebraic interpretations (Theorem 1.5 below) and connections to framework rigidity [6].

1 Background

Many of the results in this section hold in much more generality than we state. Throughout this section Δ is a combinatorial manifold whose vertex set is $V = \{x_1, \ldots, x_n\}$ and whose dimension is $d-1$. So, maximal faces, or *facets* all have d vertices. The geometric realization of Δ is $|\Delta|$ and we say Δ is PL-homeomorphic to a space X if $|\Delta|$ is. The *link* of a face $F \in \Delta$ is

$$\operatorname{lk} F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset.\}$$

The f-vector of Δ is $(f_{-1}, f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces in Δ. In particular, $f_{-1} = 1$ (corresponding to the empty set) and $f_0 = n$. The h-vector, $(h_0, \ldots, h_1)$ is defined by the functional equation,

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{d-1}(t-1)^{d-i}.$$

The g-vector is $(g_0, \ldots, g_d)$ and is given by $g_i = h_i - h_{i-1}$. Of particular interest is

$$g_2 = f_1 - df_0 + \binom{d+1}{2}.$$

The first serious study of g_2 in this setting was by Walkup [16]. He proved the following theorem in dimension three and classified all three-dimensional combinatorial manifolds with $g_2 \leq 17$. In addition to Theorem 1.1 here, the nonnegativity of g_2 was also shown independently by Gromov [5].

Theorem 1.1 *[6, Theorem 1.1] Let Δ be a combinatorial manifold of dimension at least three. Then $g_2 \geq 0$. Furthermore, if $g_2 = 0$, then Δ is a stacked sphere.*

A *stacked sphere* is any complex that can be obtained from the boundary of a simplex by repeatedly subdividing a facet with one new vertex. Any stacked sphere is PL-homeomorphic to the boundary of a simplex. Except for the boundary of the simplex a stacked sphere always has at least one missing facet. A *missing facet* is a subset σ of V with $|\sigma| = d$, $\sigma \notin \Delta$, but every proper subset of σ is a face of Δ. Two obvious ways for Δ to have a missing facet is if it was formed via handle addition, or as the connected sum along a facet of two other combinatorial manifolds. Starting with a combinatorial manifold Δ', we say Δ is formed by *handle addition* from Δ' if it is the quotient space derived by identifying the vertices of two disjoint facets of Δ' and their associated lower dimensional faces and then removing the (open) identified facet. As long as the distance in the graph theoretical sense between each pair of identified vertices is at least three, the resulting complex is a combinatorial manifold. In this case we write $\Delta = \Delta'_H$. If $\Delta = \Delta'_H$, then the PL-homeomorphism type of Δ is determined up to at most two possibilities by the PL-homeomormphsim type of Δ'.

Connected sum along a facet is a similar construction. Let Δ_1 and Δ_2 be two $(d-1)$-dimensional combinatorial manifolds with disjoint sets of vertices. Identify the vertices and their corresponding faces for two facets, one from each complex. Remove the (open) identified facet and denote the resulting complex by $\Delta_1 \# \Delta_2$. Then $\Delta_1 \# \Delta_2$ is a combinatorial manifold and its PL-homeomorphism type is determined up to at most two possibilities depending on the PL-homeomorphism types of $|\Delta_1|$ and $|\Delta_2|$.

While it is clear that both of the above constructions leave a missing facet, the following well-known theorem says that the converse also holds. For a detailed proof see [1].

Theorem 1.2 *Suppose Δ has a missing facet and is a $(d-1)$-dimensional combinatorial manifold with $d \geq 4$. Then Δ was obtained via handle addition or connected sum along a facet.*

One advantage of studying g_2 (as opposed to other scalings or translations of $L_{1,-d,0}$) is that it behaves very well with handle addition and connected sum along a facet. Direct computation shows that $g_2(\Delta_H) = g_2(\Delta) + \binom{d+1}{2}$ and $g_2(\Delta_1 \# \Delta_2) = g_2(\Delta_1) + g_2(\Delta_2)$. Another advantage of g_2 is its connection to the face ring of Δ.

Definition 1.3 *The* **face ring** *of* Δ *(also known as the* **Stanley-Reisner ring***) is*

$$\mathbb{C}[\Delta] \equiv \mathbb{C}[x_1, \ldots, x_n]/I_\Delta,$$

where $I_\Delta \equiv < x_{i_1} \cdots x_{i_k} : \{v_{i_1}, \ldots, v_{i_k}\} \notin \Delta >$.

Since I_Δ is a homogeneous ideal $\mathbb{C}[\Delta]$ is graded. We denote the degree i piece of $\mathbb{C}[\Delta]$ by $\mathbb{C}[\Delta]_i$. A set $\Theta = \{\theta_1, \ldots, \theta_d\}$ of linear forms in $\mathbb{C}[x_1, \ldots, x_n]$ is a *linear system of parameters* (l.s.o.p.) for $\mathbb{C}[\Delta]$ if $\mathbb{C}[\Delta]/(\Theta)$ is finite dimensional as a vector space over $\mathbb{C}$. If we write each $\theta_i = \sum \theta_{ij} x_j$, then Θ is a l.s.o.p. whenever every $d \times d$ minor of $(\theta)_{ij}$ is nonsingular. The connections between h-vectors, g_2 and $\mathbb{C}[\Delta]$ are given by the following two formulas.

Theorem 1.4 *(Schenzel's formula) [12] If* Δ *is a connected combinatorial manifold, then for any l.s.o.p.* Θ*,*

$$\dim_{\mathbb{C}}(\mathbb{C}[\Delta]/(\Theta))_i = h_i + \binom{d}{i} \sum_{j=1}^{j=i-1} (-1)^{i-j-1} \beta_{j-1}(\Delta),$$

where the β_{j-1} *are the Betti numbers of* $|\Delta|$.

Theorem 1.5 *[9] If* Δ *is a connected combinatorial manifold,* $d \geq 3$ *and* ω *is a generic linear form, then*

$$\dim_{\mathbb{C}}(\mathbb{C}[\Delta]/(\Theta, \omega))_2 = g_2.$$

The last preliminary result we need is Macaulay's characterization of Hilbert functions of homogeneous quotients of polynomial rings. The following weaker statement will suffice.

Theorem 1.6 *[10] Let* $R = \mathbb{C}[x_1, \ldots, x_n]/I$ *be a homogeneous quotient of a polynomial ring. Set* $F(i) = \dim_{\mathbb{C}} R_i$*. If* $F(i) \leq \binom{a}{i}$*, then* $F(i+1) \leq \binom{a+1}{i+1}$.

2 Finiteness

The goal of this section is to prove our main result.

Theorem 2.1 *Fix* $d \geq 3$ *and* $g \geq 0$*. Then there are only finitely many PL-homeomorphism classes of connected* $(d-1)$*-dimensional combinatorial manifolds without boundary and* $g_2 \leq g$.

When $d = 3$, $g_2 = -6\chi(|\Delta|)$ and hence determines the Euler characteristic of Δ, so the theorem holds.

Definition 2.2 *A triangulation* Δ *is* **irreducible** *if it does not contain a missing facet and has minimal* g_2 *among all combinatorial manifolds PL-homeomorphic to* Δ.

For a fixed homemorphism type there may be no irreducible triangulations. For instance, when $d \geq 4$ all triangulations of $S^{d-2} \times S^1$ which minimize g_2 contain at least one missing facet [15, Theorem 4.30].

Proposition 2.3 *For fixed g and $d \geq 4$ there are only finitely many connected irreducible $(d-1)$-dimensional combinatorial manifolds Δ with $g_2(\Delta) \leq g$.*

Proof:
First we prove this for $d \geq 5$.

Definition 2.4 *Let V be the vertex set of Δ. Define*

$$\tilde{h}_i(\Delta) = \sum_{v \in V} h_i(lk\ v). \tag{1}$$

Let Δ be a pure simplicial complex. Then,[14, Proposition 2.3]

$$\tilde{h}_{i-1}(\Delta) = i\ h_i(\Delta) + (d-i+1)h_{i-1}(\Delta). \tag{2}$$

The above equation implies

$$\begin{aligned} \tilde{h}_2 - \tilde{h}_1 &= 3h_3 + (d-2)h_2 - 2h_2 - (d-1)h_1 \\ &= 3(h_1 + g_2 + g_3) + (d-2)(h_1 + g_2) - 2(h_1 + g_2) - (d-1)h_1 \\ &= 3g_3 + (d-1)g_2. \end{aligned} \tag{3}$$

Now we show that g_3 can be bounded from above by g_2. Let Θ be a linear system of parameters for $\mathbb{C}[\Delta]$ and let ω be a one-form such that the conclusion of Theorem 1.5 holds. Also, let $F(i) = \dim_{\mathbb{C}}(\mathbb{C}[\Delta]/(\Theta, \omega))_i$. Then $F(2) = g_2$ and $F(3) \geq (\mathbb{C}[\Delta]/(\Theta\rangle)_3 - (\mathbb{C}[\Delta]/(\Theta)_2 = g_3 + \binom{d}{3}\beta_1 \geq g_3$. The equality is Schenzel's formula. Theorem 1.6 implies that $F(3)$ is bounded above by $M(g_2)$, where $M(x)$ is a well-defined function. Hence $g_3 \leq M(g_2)$.

Suppose that Δ has more than $3M(g) + (d-1)g$ vertices. Since $h_2 \geq h_1$ in each link, equation (3) implies that for the link of some vertex $v, h_2 = h_1$. By Theorem 1.1 this link is a stacked sphere. Let σ be a missing $d-2$-dimensional face in the link of v. There are two possibilities:

- $\sigma \in \Delta$. In this case $v * \sigma$ is a missing facet in Δ. So Δ is not irreducible.
- $\sigma \notin \Delta$. We retriangulate Δ. First remove v. Now introduce σ. This divides the sphere which was the link of v into two spheres. Specifically, $\sigma \cup lk\ v$ is the union of two spheres, S_1 and S_2 whose intersection is σ. Cone off these two spheres with new vertices v_1 and v_2. The new complex is a combinatorial manifold PL-homeomorphic to Δ, has one extra vertex and $d-1$ extra edges. As g_2 has been reduced by one, Δ is not irreducible.

Therefore every $(d-1)$-dimensional irreducible triangulation of a homology manifold with $g_2 \leq g$ has at most $3M(g) + (d-1)g$ vertices and thus there are only finitely many of them.

To prove the proposition for $d = 4$ we rephrase [16, Lemma 11.12].

Theorem 2.5 *[16] Let Δ be an irreducible triangulation of a 3-manifold other than the boundary of the 4-simplex. Then $f_1 > 4.5 f_0$.*

Since $f_1 = 4f_0 + g_2 - 10$, no irreducible 3-manifold can have more than $2g_2 - 20$ vertices. □

Proof: (of Theorem 2.1) The proof is by induction on g and n. When $g = 0$ Theorem 1.1 implies that Δ is a stacked sphere. If Δ is irreducible then it comes from a finite list of complexes. So assume Δ is not irreducible. If Δ does not minimize g_2 among PL-homeomorphic complexes, then apply the induction hypothesis. If it does, then it must contain a missing facet. Hence it can be written as either Δ'_H or $\Delta_1 \# \Delta_2$. In the former case $g_2(\Delta) = g_2(\Delta') + \binom{d+1}{2}$. By the induction hypothesis there are only finitely many possible PL-homeomorphism types for Δ' and for each such Δ' there are at most two possible homeomorphism types for handle addition. In the second case, if $g_2(\Delta_1) > 0$ and $g_2(\Delta_2) > 0$, then the induction hypothesis applies to both since $g_2(\Delta) = g_2(\Delta_1) + g_2(\Delta_2)$. So there are only finitely many PL-homeomorphism types for Δ_1 and Δ_2 and again, up to PL-homeomorphism, there are only two possible connected sums for each pair (Δ_1, Δ_2). Lastly, if $g_2(\Delta_1) > 0$, but $g_2(\Delta_2) = 0$ (and by symmetry, if $g_2(\Delta_2) > 0$ and $g_2(\Delta_1) = 0$), then Δ_2 is a stacked sphere and Δ is PL-homeomorphic to Δ_1, so we can induct on the number of vertices. □

3 Quantitative aspects and higher dimensional faces

In view of Theorem 2.1 it is natural to ask for specific topological types and/or invariants associated to varying values of g_2. By Theorem 1.1 $g_2 = 0$ implies that Δ is a stacked sphere. The proof of Theorem 2.1 shows that irreducible complexes with $g_2 = 1$ have at most $d + 2$ vertices, and hence are PL-homeomorphic to spheres [2]. In dimension three Walkup proved the following classification:

upper bound for g_2	space
9	sphere
16	sphere or S^2-bundle over S^1
17	sphere or S^2-bundle over S^1 or $\mathbb{R}P^3$.

Problem 3.1 *For $d \geq 5$ determine the smallest $g(d)$ such that there exists a combinatorial manifold with $g_2 = g(d)$ and is not PL-homeomorphic to a sphere.*

An infinite series of examples due to Kühnel show that $g(d) \leq \binom{d+1}{2}$ [8]. However, there are some reasons to believe that $g(d)$ is closer to $d^2/4$ than $d^2/2$.

One topological invariant where the role of g_2 is well-understood is the first Betti number of $|\Delta|$. The following theorem was originally conjectured by Kalai.

Theorem 3.2 *[11] Let Δ be a $(d-1)$-dimensional combinatorial manifold with $d \geq 4$. Then $g_2 \geq \binom{d+1}{2}\beta_1$, where β_1 is the first Betti number with respect to any field.*

The following extension of this to the fundamental group has been suggested by Kalai.

Conjecture 3.3 *If Δ is a $(d-1)$-dimensional combinatorial manifold with $d \geq 4$, then $g_2 \geq \binom{d+1}{2}m$, where m is the minimum number of generators of the fundamental group of $|\Delta|$.*

Another obvious question raised by finiteness for g_2 is whether or not similar results hold for $g_i,\ i \geq 3$. At first sight, the answer is no. In dimension four, $g_3 = 10\chi(|\Delta|)$ [7] and there can be infinitely many PL-homeomorphism types for a fixed Euler characteristic. Even if one takes the view that dimension four is too small and that the question should not be asked for g_3 until dimension five, there are combinatorial manifolds homeomorphic to $S^1 \times S^4$ with $g_3 < 0$ [8]. Repeatedly taking connected sum along a facet produces g_3 which tends to minus infinity, so an upper bound does not produce topological finiteness. However, the proper generalization of topological finiteness for g_2 to higher g_i should involve bounding Betti numbers. Indeed, if instead of insisting on connected complexes, but instead an upper bound on β_0, then the analog of Theorem 2.1 still holds. This leaves as one possible generalization of Theorem 2.1 this suggestion of Kalai.

Problem 3.4 *Fix $d, g, b_0, \dots, b_{i-2}$, and $i < d/2$. Is the number of PL-homeomorphism types of combinatorial manifolds Δ with $\dim \Delta = d, g_i \leq g$, and $\beta_j(|\Delta|) \leq b_j$ for $0 \leq j \leq i-2$ finite?*

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