g-ELEMENTS OF MATROID COMPLEXES

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Abstract. A g-element for a graded R-module is a one-form with properties similar to a Lefschetz class in the cohomology ring of a compact complex projective manifold, except that the induced multiplication maps are injections instead of bijections. We show that if $k(\Delta)$ is the face ring of the independence complex of a matroid and the characteristic of k is zero, then there is a non-empty Zariski open subset of pairs (Θ, ω) such that Θ is a linear set of paramenters for $k(\Delta)$ and ω is a g-element for $k(\Delta)/<\Theta>$. This leads to an inequality on the first half of the h-vector of the complex similar to the g-theorem for simplicial polytopes.

1. Introduction

The combinatorics of the independence complex of a matroid can be approached from several different directions. The f-vector directly encodes the number of independent sets of every cardinality, while the h-vector contains the same information encoded in a way which is more appropriate for reliability problems [4]. In either case the fundamental question is the same. What vectors are possible?

Let $(h_0, h_1, \dots h_r)$ be the h-vector of the independence complex of a rank r matroid without coloops. Using a PS-ear decomposition of the complex Chari [6] proved that for all $i \le r/2$, $h_i \le h_{r-i}$ and $h_{i-1} \le h_i$. By showing that the h-vector was the Hilbert function of $k(\Delta)/<\Theta>$, where $k(\Delta)$ is the face ring of the complex and Θ is a linear system of parameters for $k(\Delta)$, Stanley [9] proved that $h_{i+1} \le h_i^{<i>}$ (see section 3 for a definition of the $<i>$-operator). By combining these two methods we show in Theorem 4.3 that if we define $g_i = h_i - h_{i-1}$, then $g_{i+1} \le g_i^{<i>}$ for all $i < r/2$. All of these inequalities are immediate consequences of the existence of pairs (Θ, ω) such that Θ is a linear set of parameters for $k(\Delta)$ and ω is a g-element for $k(\Delta)/<\Theta>$. Using a different approach, toric hyperkähler varieties, Hausel and Sturmfels proved the existience of g-elements for $k(\Delta)/<\Theta>$ when the matroid is representable over the rationals [7]. A g-element is a one-form which acts like a Lefschetz class of a compact complex projective manifold except that it induces injections instead of bijections (Definition 4.1).

The broken circuit complex of a matroid is a subcomplex of the independence complex and directly encodes the coefficients of the characteristic polynomial of the matroid. Every broken circuit complex is a cone, and if we remove the cone point we obtain a reduced broken circuit complex. Any independence complex is also a reduced broken circuit complex. Since the h-vector is unchanged by the removal of a cone point, the set of h-vectors of independence complexes is a (strict) subset of the set of h-vectors of broken circuit complexes. A natural question is whether or not Theorem 4.2 holds for broken circuit complexes. In Section 5 we

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show that even if broken circuit complexes satisfy the corresponding combinatorial inequalities, there may be no set of linear parameters for the face ring such that there exist g-elements for the quotient ring.

Matroid terminology and notation closely follows [8]. The main exception to this is that we use $M - A$ for the deletion of a subset instead of $M \setminus A$. The ground set of the matroid M is always E.

2. Complexes

Let Δ be a finite abstract simplicial complex with vertices $V = \{v_1, \dots, v_n\}$. The *f-vector* of Δ is the sequence $(f_0(\Delta), \dots, f_s(\Delta))$, where $f_i(\Delta)$ is the number of simplices of cardinality i and $s-1$ is the dimension of Δ. The *h-vector* of Δ is the sequence $(h_0(\Delta), \dots, h_s(\Delta))$ defined by,

$$h_i(\Delta) = \sum_{k=0}^{i} (-1)^{i+k} f_k(\Delta) \binom{s-k}{i-k}.$$

Equivalently, if we let $f_\Delta(t) = f_0 t^s + f_1 t^{s-1} + \dots f_{s-1} t + f_s$, then $h_\Delta(t) = h_0 t^s + h_1 t^{s-1} + \dots h_{s-1} t + h_s$ satisfies $h_\Delta(1+t) = f_\Delta(t)$.

The *independence complex* of M is the simplicial complex whose vertices are the non-loop elements of E and whose simplices are the independent subsets of E. We let $\Delta(M)$ represent the independence complex of M.

In order to define the broken circuit complex for M, we first choose a linear order $\mathbf{n}$ on the elements of the matroid. Given such an order, a *broken circuit* is a circuit with its least element removed. The *broken circuit complex* is the simplicial complex whose simplices are the subsets of E which do not contain a broken circuit. We denote the broken circuit complex of M with linear order $\mathbf{n}$ by $\Delta^{BC}(M)$ or, if necessary, $\Delta^{BC}(M, \mathbf{n})$. Different orderings may lead to different complexes, see [1, Example 7.4.4]. Conversely, distinct matroids can have the same broken circuit complex. For instance, let $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and let $\mathbf{n}$ be the obvious order. Let M_1 be the matroid on E whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$ and let M_2 be the matroid on E whose bases are all triples except $\{e_1, e_2, e_3\}$ and $\{e_1, e_5, e_6\}$. Then M_1 and M_2 are non-isomorphic matroids but their broken circuit complexes are identical. Both $h_{\Delta(M)}(t)$ and $h_{\Delta^{BC}(M)}(t)$ satisfy similar contraction-deletion formulas.

Proposition 2.1. [1], [3]

1. *If e is a loop of M, then $h_{\Delta(M)}(t) = h_{\Delta(M-e)}(t)$, and $h_{\Delta^{BC}(M)}(t) = 1$.*
2. *If e is a coloop of M, then $h_{\Delta(M)}(t) = h_{\Delta(M-e)}(t)$, and $h_{\Delta^{BC}(M)}(t) = h_{\Delta^{BC}(M-e)}(t)$.*
3. *If e is neither a loop nor a coloop of M, then $h_{\Delta(M)}(t) = h_{\Delta(M-e)}(t) + h_{\Delta(M/e)}(t)$ and $h_{\Delta^{BC}(M)}(t) = h_{\Delta^{BC}(M-e)}(t) + h_{\Delta^{BC}(M/e)}(t)$.*
4. *If $M = M_1 \oplus M_2$, then $h_{\Delta(M)}(t) = h_{\Delta(M_1)}(t) \cdot h_{\Delta(M_2)}(t)$.*
5. *If S is an independent series class of M, then $h_{\Delta(M)}(t) = h_{\Delta(M/S)}(t) + h_{\Delta(M-S)}(t)(1 + t + \cdots + t^{|S|-1})$.*

3. Face rings

Let k be a field and let $R = k[x_1, \dots, x_n]$.

Definition 3.1. *The face ring of Δ is the graded k-algebra*

$$k[\Delta] = R/I_\Delta,$$

where I_Δ is the ideal generated by all monomials $x_{i_1} \cdot \cdots \cdot x_{i_l}$ such that $\{v_{i_1}, \dots, v_{i_l}\}$ is not a simplex of Δ.

Let $s-1$ be the dimension of Δ. Let $\Theta = \{\theta_1, \dots, \theta_s\}$ be a set of one-forms in R. Write each $\theta_i = k_{i1}x_1 + \dots k_{is}x_s$ and let $K = (k_{ij})$. To each simplex in Δ there is a corresponding set of column vectors in K. If for every simplex of Δ the corresponding set of column vectors is independent, then Θ is a *linear set of parameters* (l.s.o.p.) for $k(\Delta)$. If k is infinite, then it is always possible to choose Θ such that every set of s columns of K is independent.

Given a l.s.o.p. Θ for $k(\Delta)$ let $R(\Delta, \Theta) = k(\Delta)/ < \Theta >$. If Θ is unambiguous, then we just use $R(\Delta)$. Since Θ is homogeneous, $R(\Delta)$ is a graded k-algebra.

Theorem 3.2. [9] *Let Θ be a l.s.o.p. for $\Delta(M)$ and let $R(\Delta(M))_i$ be the i^{th} graded component of $R(\Delta(M))$. Then $h_i(\Delta(M)) = \dim_k R(\Delta(M))_i$. Similarly, if Θ is a l.s.o.p. for $\Delta^{BC}(M)$, then $h_i(\Delta^{BC}(M)) = \dim_k R(\Delta^{BC}(M))_i$.*

Given any two integers $i, j > 0$ there is a unique way to write

$$j = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_l}{l}, a_i > a_{i-1} > \cdots > a_l \geq l \geq 1.$$

Given this expansion define,

$$j^{<i>} = \binom{a_i+1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_l+1}{l+1}, a_i > a_{i-1} > \cdots > a_l \geq l \geq 1.$$

Theorem 3.3. [11, Theorem 2.2] *Let $Q = R/I$, where I is a homogeneous ideal. Let Q_i be the forms of degree i in Q and let $h_i = \dim_k Q_i$. Then $h_{i+1} \leq h_i^{<i>}$.*

Corollary 3.4. [9] *For any independence or broken circuit complex $h_{i+1} \leq h_i^{<i>}$.*

4. The ring $R(\Delta(M))$

In order to study the properties of $h_i(\Delta(M))$ we will look for elements with properties slightly weaker than those provided by Lefschetz elements of the cohomology ring of a compact complex projective manifold.

Definition 4.1. *Let N be a (non-negatively) graded R-module whose dimension over k is finite. Let r be the last non-zero grade of N and let ω be a one-form in R. Then ω is a g-element for N if for all $i, 0 \leq i \leq r/2$, multiplication by ω^{r-2i} is an injection from N_i to N_{r-i}.*

If we replace injection with bijection in the above definition, then we obtain the strong Stanley property in [12].

Let M be a rank r matroid without coloops and k a field of characteristic zero. Let $n = |E|$. Write the elements of $k^{n\times(r+1)}$ in the form (Θ, ω), where Θ consists of r elements in k^n and ω is also in k^n. Identify elements of k^n with the one-forms in R in the canonical way. Let U be the set of all pairs $(\Theta, \omega) \in k^{n\times(r+1)}$ such that Θ is a l.s.o.p. for $k(\Delta(M))$ and ω is a g-element for $R(\Delta(M), \Theta)$.

Theorem 4.2. *Let M, U and k be as above. Then, U is a non-empty Zariski open subset of $k^{n\times(r+1)}$.*

Proof. We first note that Θ is a l.s.o.p. for $k(\Delta)$ if and only if the determinants of the appropriate $r \times r$ minors of K are non-zero. Secondly, the multiplication maps ω^{r-2i} can be encoded as matrices which are polynomial in the coefficients of K and ω. Thus, U is the intersection of two Zariski open subsets of $k^{n\times(r+1)}$.

To show that U is not empty we proceed by induction on n. However, we use a slightly different (but equivalent) induction hypothesis. Let $C(j)$ be the circuit with j elements. Let P be a direct sum of circuits, so we can write $P = C(j_1) \oplus \cdots \oplus C(j_m)$. The rank of $M \oplus P$ is $r' = r + j_1 + \cdots + j_m - m$ and its cardinality is $n' = n + j_1 + \cdots + j_m$. The induction hypothesis is that given any such P, then $U = \{(\Theta, \omega) \in k^{n' \times (r'+1)} : \Theta$ is a l.s.o.p. for $k(\Delta(M \oplus P))$ and ω is a g-element for $R(\Delta(M \oplus P), \Theta)\}$ is not empty. If M consists of a single loop, then $k(\Delta(M \oplus P)) \simeq k(\Delta(P))$. As a simplicial complex $\Delta(P)$ is $\partial(\Delta^{j_1-1}) * \cdots * \partial(\Delta^{j_m-1})$, where Δ^j is the j-simplex. Since this is the boundary of a convex rational polytope we can apply the Hard Lefschetz theorem as in [10] to see that U is not empty when $k = \mathbb{Q}$, and hence is not empty for any field of characteristic zero.

For the induction step, let S be a series class of M. Reordering M if necessary, we assume that $S = \{e_1, \ldots, e_s\}$ consists of the first s elements of M. If S is a circuit, then $M = (M - S) \oplus C(s)$. Hence, $M \oplus P = (M - S) \oplus (C(s) \oplus P)$ and the induction hypothesis applies to $M - S$. So assume that S is independent . Let $x^S = x_1 \cdot \cdots \cdot x_s$. For Θ a l.s.o.p. for $k(\Delta(M \oplus P))$ consider the following short exact sequence.

(1)
$$0 \to \frac{< x^S >}{< x^S > \cap < \Theta + I_{\Delta(M \oplus P)} >} \to R(\Delta(M \oplus P, \Theta)) \to \frac{R(\Delta(M \oplus P), \Theta)}{< x^S >} \to 0.$$

Since S is a series class, a subset of $M - S$ is independent if and only if its union with any proper subset of S is independent. Hence, the right-hand side is just $R(\Delta((M - S) \oplus (C(s) \oplus P), \Theta)$. Therefore, we can apply the induction hypothesis to $M - S$ to obtain a non-empty Zariski open subset U' of $k^{n\times(r+1)}$ consisting of pairs (Θ', ω') such that ω' is a g-element for $R(\Delta((M - S) \oplus C(s) \oplus P), \Theta')$.

In order to analyze the left-hand side of (1) choose generators $\{\theta_1, \ldots, \theta_s, \ldots, \theta_{r'}\}$ for $< \Theta >$ so that in the corresponding matrix K, $k_{ij} = \delta_{ij}$ for $1 \le i \le s$. Now define an R-module structure on $R' = k[x_{s+1}, \ldots, x_{n'}]$ by defining $(x_i) \cdot f = (x_i - \theta_i) \cdot f$ for $1 \le i \le s$ and $f \in R'$. Let $\phi : R' \to < x^S > / < \Theta + I_{\Delta(M \oplus P)} >$ be multiplication by x^S. Since S is independent, every polynomial in $< x^S > / < \Theta + I_{\Delta(M \oplus P)} >$ is equivalent to a polynomial in $< x^S > R'$. So, ϕ is surjective. The kernel of ϕ contains $\Theta' = \{\theta \in \Theta : \theta = k_{s+1} x_{s+1} + \cdots + k_{n'} x_{n'}.\}$ In addition, $\ker \phi$ contains all monomials in $I_{\Delta((M/S) \oplus P)}$. Since Θ' is a l.s.o.p. for $k(\Delta(M/S))$, we see that ϕ is a degree s graded surjective R-module homomorphism from $R' / < I_{\Delta((M/S) \oplus P)} + \Theta' >$ to the left-hand side of (1). Proposition 2.1 and $h_{\Delta(C(s))}(t) = 1 + t + \cdots + t^{s-1}$ show that the k-dimension of $R'(\Delta(M/S), \Theta')$ and the l.h.s. of (1) are the same. Hence ϕ is an isomorphism. Therefore, by the induction hypothesis applied to M/S, there is a non-empty Zariski open subset U'' of $k^{n\times(r+1)}$ consisting of pairs (Θ'', ω'') such that if ψ is multiplication by $\omega''^{(r'-2i-s)}$, then

$$\psi : \left(\frac{< x^S >}{< x^S > \cap < \Theta'' + I_{\Delta(M \oplus P)} >}\right)_{i+s} \to \left(\frac{< x^S >}{< x^S > \cap < \Theta'' + I_{\Delta(M \oplus P)} >}\right)_{r'-i}$$

is an injection for $1 \leq i \leq (r' - s)/2$. Now, $U' \cap U'' \subseteq U$. Since the intersection of two non-empty Zariski open subsets of $k^{n\times(r+1)}$ is not empty, U is also not empty. □

Theorem 4.3. *Let M be a rank r matroid without coloops. Let $h_i = h_i(\Delta(M))$. Then,*

1. $h_0 \leq \cdots \leq h_{\lfloor r/2 \rfloor}$.
2. $h_i \leq h_{r-i}$ *for all* $i \leq r/2$.
3. *Let* $g_i = h_i - h_{i-1}$. *Then, for all* $i < r/2$, $g_{i+1} \leq g_i^{<i>}$.

Proof. The first two inequalities follow from the injectivity properties of any g-element ω for $R(\Delta(M), \Theta)$. Since $g_i = (R(\Delta(M), \Theta)/ < \omega >)_i$ when $i < r/2$, the last inequality follows from Theorem 3.3. □

The first two inequalities were obtained by Chari using a PS-ear decomposition of $\Delta(M)$. See [5] for details on PS-ear decompositions. Hausel and Sturmfels used toric hyperkähler varities to prove the last inequality for matroids representable over the rationals [7]. The proof of Theorem 4.2 is essentially an algebraic version of a PS-ear decomposition [6, Theorem 2]. Indeed, the proof works with a much simpler induction hypothesis for any simplicial complex with a PS-ear decomposition.

5. The ring $R(\Delta^{BC}(M))$

As shown in [2] the cone on any independence complex is a broken circuit complex (for some other matroid). Since the h-vector of the cone of a simplicial complex is the same as the h-vector of the original complex, the h-vectors of independence complexes form a (strict) subset of the the h-vectors of broken-circuit complexes. It is natural to ask whether or not Theorem 4.2 holds for $\Delta^{BC}(M)$. The last non-zero element of the h-vector of $\Delta^{BC}(M)$ is $r-m$, where m is the number of components of M. It is not difficult to modify the proof of Theorem 4.2 to produce injections from $R(\Delta^{BC}(M))_0$ to $R(\Delta^{BC}(M))_{r-m}$ and from $R(\Delta^{BC}(M))_1$ to $R(\Delta^{BC}(M))_{r-m-1}$. Since the first possible problem is in degree 2, the smallest possible rank of M for which Theorem 4.2 does not hold for $\Delta^{BC}(M)$ is six.

Let $G(s)$ be the graph obtained by subdividing each edge of the graph consisting of s parallel edges into two edges. Let $M(s)$ be the cycle matroid of $G(s)$. The rank of $M(s)$ is $s+1$ and $M(s)$ has $2s$ elements.

Proposition 5.1. *Let Θ be a l.s.o.p. for $M(s)$, $\mathbf{n}$ a linear ordering of the elements of $M(s)$ and ω a linear form in $k[x_1, \ldots, x_{2s}]$. Then, multiplication by ω has a non-trivial kernel in $R(\Delta^{BC}(M))_2$.*

Proof. Let E_l consist of the greatest l elements of $M(s)$ with respect to $\mathbf{n}$. Let $\{e_i, e_j\}$ be the first pair of edges to appear in E_l as l goes from 1 to $s+1$ such that they come from the subdivision of one of the parallel edges used to construct $G(s)$. Consider the ideal $< x_i x_j > \subseteq R(\Delta^{BC}(M, \mathbf{n}))$. Using the same reasoning as in the proof of Theorem 4.2, the choice of $\{e_i, e_j\}$ implies that $< x_i x_j >$ is isomorphic as an R-module to $R'(\Delta^{BC}(\Delta(M(s)/\{e_i, e_j\}, \mathbf{n}'), \Theta')$, where R' and Θ' are defined as in the proof of Theorem 4.2, and $\mathbf{n}'$ is the order on $M(s)/\{e_i, e_j\}$ induced from $\mathbf{n}$. Now, $M(s)/\{e_i, e_j\}$ is the cycle matroid of the $G(s)$ with the two edges $\{e_i, e_j\}$ contracted. For any such pair and any linear order $\Delta^{BC}(M(s)/\{e_i, e_j\}$ is an $s-2$ dimensional simplex. Hence $< x_i x_j > \simeq k$ and will vanish under any multiplication map. □

Repeated application of Proposition 2.1 shows that $h_i(\Delta^{BC}(M(s))) = \binom{s}{i}$ when $i \neq 1$ and $h_1(\Delta^{BC}(M(s))) = s - 1$. When $s \geq 5$, the h-vector of the broken circuit complex of $M(s)$ satisfies the combinatorial conditions of Theorem 4.3 but there is no l.s.o.p. for the face ring such that the quotient ring has g-elements. Thus, the comments following Theorem 4.3 show that $\Delta^{BC}(M(s))$ does not have a PS-ear decomposition when $s \geq 5$. As far as we know, whether or not broken circuit complexes satisfy the combinatorial inequalities of Theorem 4.3 remains an open question.

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