Math 3040 HW 10 - Due Nov. 25, 2019

1. Let m be a nonnegative integer, and n a positive integer. Using the division algorithm we can write $m = qn + r$, with $0 \leq r \leq n-1$. As in class define $S_{(m,n)} = \{mx + ny : x, y \in \mathbb{Z}\}$ and $S_{(n,r)} = \{nu + rv : u, v \in \mathbb{Z}\}$. Prove that $S_{(m,n)} = S_{(n,r)}$.

   (Remark: If we add to the definition of gcd that $\gcd(n, 0) = \gcd(0, n) = n$, then this proves that $\gcd(m, n) = \gcd(n, r)$. This result leads to a fast algorithm for computing $\gcd(m, n)$ and $[a]^{-1}$ if $[a]$ has a multiplicative inverse in $\mathbb{Z}/n\mathbb{Z}$.)

2. (Dust off your linear algebra! You can use results from linear algebra, especially orthogonal complements, in this problem. Page 350 and thereabouts of the Lay textbook for math 2210 and math 2940 may be of particular interest ...) Let V be a subspace of $\mathbb{R}^n$ and let W be the orthogonal complement of V, $W = V^{\perp}$. Define $\sim$ on $\mathbb{R}^n$ by $x \sim y$ if and only if $x - y \in V$.

   (a) Prove that $\sim$ is an equivalence relation on V.

   (b) Prove that if w and w' are in W and $w \neq w'$, then $w \not\sim w'$.

   (c) Prove that if $x \in \mathbb{R}^n$, then there exists a unique $w \in W$ such that $x \sim w$.

   (d) Prove that if $r \in \mathbb{R}$, $x, y, z, w \in \mathbb{R}^n$, and $x \sim y, z \sim w$, then $(x + z) \sim (y + w)$ and $rx \sim ry$.

3. Let $(Z, +, \cdot)$ be a commutative ring. Define $U(Z)$ to be all of the elements of Z with a multiplicative inverse. Prove that $U(Z)$ with $\cdot$ is a group.