Math 4420 HW 2 - Due Thursday, Feb. 13, 2020

1. True or False. Explain. In a $2-(v,3,\lambda)$ design either v is odd or λ is even.
2. Prove there does not exist a $2-(46,10,2)$ design with $b=v$.
3. The 2-designs in discussion question #2 are called *finite projective planes*. Prove that if a linear space is a finite projective plane, then there exists n such that every point is in $n+1$ lines and the number of points is n^2+n+1. Hence the Fano plane is the smallest finite projective plane.
4. Throughout this problem A is a $3\times n$ matrix. For each pair $1\le i<j\le n$, define

$$B_{i,j}=\{k : v_k \in \text{ span }\{v_i,v_j\}.\}.$$

Example:

$$A=\begin{bmatrix}1&0&0&1&0\\0&1&0&1&1\\0&0&1&0&1\end{bmatrix}.$$

Then

$$B_{1,2}=B_{1,4}=B_{2,4}=\{1,2,4\},$$
$$B_{2,3}=B_{2,5}=B_{3,5}=\{2,3,5\},$$
$$B_{1,3}=\{1,3\}, B_{1,5}=\{1,5\}, B_{3,4}=\{3,4\}, B_{4,5}=\{4,5\}.$$

Finally, define an incidence structure $\mathcal{L}_A$ with $\mathbf{P}=[n]$ and $\mathcal{B}$ the *distinct* sets of the $B_{i,j}$.

Example continued: Using A above, the blocks of $\mathcal{L}_A$ are

$$\{1,2,4\},\{2,3,5\},\{1,3\},\{1,5\},\{3,4\},\{3,5\}.$$

   (a) Prove that if no two columns of A are scalar multiples of each other, then $\mathcal{L}_A$ is a linear space.

   (b) Prove that if B is obtained from A by elementary row operations, then $\mathcal{L}_{\mathcal{B}}=\mathcal{L}_A$.

   (c) Prove that if B is obtained from A by multiplying a column by a nonzero scalar, then $\mathcal{L}_{\mathcal{B}}=\mathcal{L}_A$.

5. Prove that if there exists a 3×7 matrix A such that $\mathcal{L}_A$ is the Fano plane (add one to the labels of figure 19.2), then there exists another 3×7 matrix B such that $\mathcal{L}_B=\mathcal{L}_A$ and is of the form

$$\begin{bmatrix}1&0&0&1&0&1&1\\0&1&0&a&1&d&0\\0&0&1&0&c&e&b\end{bmatrix},$$

where a,b,c,d,e are nonzero scalars.

   (a) Prove that $a=d, e=b$ and $e=cd=ca$.

   (b) Prove that $2ca=0$.

   (c) So we conclude that the Fano plane is not equal to $\mathcal{L}_A$ for any A. If only $2=0\ldots$