Placeholder Title

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ABSTRACT.

1. Introduction

2. Formulation of the Problem

Let f be a continuous function on the circle $\mathbb{R}/2\pi\mathbb{Z}$. The partial sums of the Fourier series of f may be written as convolutions of f with the Dirichlet Kernel

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(t) f(x-t)\ dt \tag{2.1}$$

for

$$D_n(t) = \frac{1}{2\pi}(1 + 2\sum_{k=1}^{n} cos(kt)) = \frac{1}{2\pi}\frac{sin(n+\frac{1}{2})t}{sin(\frac{1}{2}t)} \tag{2.2}$$

We have

$$\int_{-\pi}^{\pi} D_n(t)dt = 1 \tag{2.3}$$

but

$$\int_{-\pi}^{\pi} |D_n(t)|dt = O(logn) \tag{2.4}$$

so the Dirichlet Kernel fails to be an approximate identity, and in general the partial sums do not converge to f. Fejèr observed that the averages

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$$\sigma_n(f)(x) = \frac{1}{N+1}\sum_{k=0}^{N} s_k(f)(x) \tag{2.5}$$

are also given by convolution with Kernels K_n given by

$$K_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \frac{1}{2\pi(N+1)}\left(\frac{sin(\frac{N+1}{2}t)}{sin(\frac{1}{2}t)}\right)^2 \tag{2.6}$$

But now $K_n(t)$ is nonnegative, so

$$1 = \int_{-\pi}^{\pi} K_n(t)dt = \int_{-\pi}^{\pi} |K_n(t)|dt \tag{2.7}$$

and in fact K_n is an approximate identity. Specifically, we have the estimate

$$\int_{|t|\geqslant\epsilon} |K_n(t)|dt \leqslant \frac{c}{N\epsilon} \tag{2.8}$$

for some c and all $\epsilon > 0$.

So if we define the modulus of continuity of f by

$$m_\epsilon(f) = \sup_x \sup_{|t|\leqslant\epsilon} |f(x-t) - f(x)| \tag{2.9}$$

then

$$\begin{aligned} ||K_n * f - f||_\infty &\leqslant \int_{|t|\leqslant\epsilon} |f(x-t)-f(x)|K_n(t)dt + \int_{|t|\geqslant\epsilon} 2||f||_\infty K_n(t)dt \\ &\leqslant m_\epsilon(f) + 2||f||_\infty \frac{1}{N+1}\left(\frac{1}{sin(\frac{1}{2}\epsilon)}\right)^2 \end{aligned} \tag{2.10}$$

Thus we obtain Fejèr's theorem that $K_n * f$ converges uniformly to f as $N \to \infty$ in a quantitative form.

Now suppose we are given a sequence

$$0 = n_0 < n_1 < n_2 < n_3 < ... \tag{2.11}$$

and we consider the sparse averages

$$\widetilde{\sigma_n}(f) = \frac{1}{N+1}\sum_{k=0}^{N} s_{n_k}(f) \tag{2.12}$$

analogous to (2.5). Then, analogous to (2.6), we have

$$\widetilde{\sigma_n}(f) = Q_n * f \text{ for} \tag{2.13}$$

$$Q_N(t) = \frac{1}{N+1}\sum_{k=0}^{N} D_{n_k}(t) = \frac{1}{N+1}\frac{1}{2\pi}\sum_{k=0}^{N}\frac{sin(n_k+\frac{1}{2})t}{sin(\frac{1}{2}t)} \tag{2.14}$$

In order to show that $\widetilde{\sigma_n}(f) \to f$ uniformly for continous f we need to verify that Q_n is an approximate identity:

$$\int_{-\pi}^{\pi} Q_n(t)dt = 1 \tag{2.15}$$

$$\int_{-\pi}^{\pi} |Q_n(t)|dt \leqslant M \text{ for all n} \tag{2.16}$$

$$\int_{|t|\geqslant\epsilon} |Q_n(t)|dt \leqslant \varphi_\epsilon(N) \tag{2.17}$$

with $\lim\limits_{N\to\infty} \varphi_\epsilon(N) = 0$ for all $\epsilon > 0$ Indeed, just like (2.10) we obtain

$$||Q_n * f - f||_\infty \leqslant Mm_\epsilon(f) + 2||f||_\infty \varphi_\epsilon(N) \tag{2.18}$$

and hence $Q_n * f \to f$ uniformly.

Of course, (2.15) is an immediate consequence of (2.3).

Main Question:
Under what conditions on the sequence n_j do we have (2.16) and (2.17)?

3. A Counterexample

In this section, we show how to modify a construction of the Fejer of a continuous function whose Fourier series diverges at a point to exhibit a sequence n_j and a continuous function such that $\widetilde{\sigma_n}(f)(0)$ is unbounded. The basic building block is the function

$$\begin{aligned} F_{n,m}(x) = {} & \frac{cos(m)x}{n} + \frac{cos(m+1)x}{n-1} + ... + \frac{cos(m+n-1)x}{1} \\ & - \frac{cos(m+n+1)x}{1} - \frac{cos(m+n+2)x}{2} - ... - \frac{cos(m+2n)x}{n} \end{aligned} \tag{3.1}$$

Note that

$$S_n F_{n,m}(x) = \begin{cases} 0 & \text{if } N < m \\ F_{n,m}(x) & \text{if } N \geqslant m+2n \end{cases} \tag{3.2}$$

$$F_{n,m}(0) = 0 \tag{3.3}$$

$$S_{n+m}F_{n,m}(0) = O(logn) \tag{3.4}$$

We also have the uniform boundedness of all $F_{n,m}$ as a consequence of the uniform boundedness of $\sum_{k=1}^{n} \frac{sin(kx)}{k}$.

Now we choose n_k, m_k so there is no overlap between the exponentials in F_{n_k,m_k}. For example, this will hold if $m_k \geqslant 1 + m_{k-1} + 2n_{k-1}$. Then we choose positive coefficients a_k such that

$$\sum_{k=1}^{\infty} a_k < \infty \text{ and set} \tag{3.5}$$

$$f = \sum_{k=1}^{\infty} a_k F_{n_k,m_k} \text{ , the series converging uniformly} \tag{3.6}$$

Note that

$$S_{n_k+m_k} f(0) = a_k S_{n_k+m_k} F_{n_k,m_k}(0) \tag{3.7}$$

since all the other terms vanish. Thus

$$S_{n_k+m_k} f(0) = O(a_k log(n_k)) \tag{3.8}$$

and we can make this diverge by the appropriate choice of n_k and a_k. For example, $a_k = k^2$ and $n_k = m_k = 2^{(k^3)}$. Thus the sequence $2 * 2^{(k^3)}$ gives a negative answer to the Main Question in the previous section.

4. The Linear Case

In this section we deal with the case

$$n_k = pk \tag{4.1}$$

where p is a positive integer. The case $p = 1$ gives the Cesaro sums, so Q_N is exactly the Fejer Kernel. We will see that the behavior of Q_N is not as nice as the Fejer Kernel. The statement

$$\lim_{N\to\infty} Q_N(t) = 0 \tag{4.2}$$

holds uniformly for any fixed $\epsilon > 0 for |t| \geqslant \epsilon$ for $p = 1$ but it is false for $p \geqslant 2$, as there are specific choices of t (for example $\frac{2\pi}{p}$) where $Q_N(t)$ is a nonzero constant. Nevertheless, we will prove that Q_N is an approximate identity, using the average decay (2.17) as a substitute for (4.2).

Lemma 4.1. *For $n_k = pk$ we have*

$$Q_N(t) = \frac{1}{2\pi} \frac{1}{N+1} \frac{sin((\frac{p}{2}N + \frac{1}{2})t) sin(\frac{p}{2}(N+1)t)}{sin(\frac{1}{2}t) sin(\frac{p}{2}t)} \tag{4.3}$$

PROOF. In view of (2.14) it suffices to show

$$\sum_{k=0}^{N} sin(kp + \frac{1}{2}t) = \frac{sin((\frac{p}{2}N + \frac{1}{2})t)sin(\frac{p}{2}(N+1)t)}{sin(\frac{p}{2}t)} \tag{4.4}$$

Now the left side of 4.4 is equal to

$$\begin{aligned}
&\frac{1}{2i}(e^{\frac{i}{2}t}(\frac{e^{i(N+1)pt}-1}{e^{ipt}-1}) - e^{\frac{-i}{2}t}(\frac{e^{-i(N+1)pt}-1}{e^{-ipt}-1}))\\
&= \frac{1}{2i}(\frac{e^{\frac{i}{2}t}e^{i(N+1)\frac{p}{2}t}}{e^{i\frac{p}{2}t}})(\frac{sin(N+1)\frac{p}{2}t}{sin(\frac{p}{2})t}) - \frac{1}{2i}(\frac{e^{\frac{-i}{2}t}e^{-i(N+1)\frac{p}{2}t}}{e^{-i\frac{p}{2}t}})(\frac{sin(N+1)\frac{p}{2}t}{sin(\frac{p}{2})t})\\
&= \frac{1}{2i}(e^{i((\frac{p}{2}N+\frac{1}{2})t)} - e^{-i((\frac{p}{2}N+\frac{1}{2})t)})(\frac{sin(N+1)\frac{p}{2}t}{sin(\frac{p}{2})t})\\
&= \frac{sin((\frac{p}{2}N+\frac{1}{2})t)sin(\frac{p}{2}(N+1)t)}{sin(\frac{p}{2}t)}
\end{aligned}$$

□

But when $p = 2$ we have

$$\begin{aligned}
Q_N(\pi) &= \frac{1}{2\pi}\frac{1}{N+1}\frac{sin(N+\frac{1}{2}\pi)}{sin(\frac{\pi}{2})}\lim_{s\to 0}\frac{sin(N+1)(\pi-s)}{sin(\pi-s)}\\
&= \frac{1}{2\pi}\frac{1}{N+1}(-1)^N(-1)^N\lim_{s\to 0}\frac{sin(N+1)s}{sin(s)}\\
&= \frac{1}{2\pi}
\end{aligned}$$

We have a similar computation for general p

LEMMA 4.2. *Let* $j \leqslant \frac{p}{2}$. *Then*

$$Q_N(\frac{2j\pi}{p}) = \frac{1}{2\pi} \tag{4.5}$$

PROOF. As before,

$$\frac{sin(\frac{p}{2}N+\frac{1}{2})\frac{2j}{p}\pi}{sin(\frac{1}{2}(\frac{2j}{p}\pi))} = \frac{sin(Nj\pi + \frac{j}{p}\pi)}{sin(\frac{j}{p}\pi)} = (-1)^{Nj}$$

and

$$\frac{sin(\frac{p}{2}(N+1)t)}{sin(\frac{p}{2}t)}\Big|_{t=\frac{2}{p}j\pi} = \lim_{s\to 0}\frac{sin(\frac{p}{2}(N+1)(\frac{2}{p}j\pi+s))}{sin(\frac{2}{p}j\pi+s)}$$
$$= \lim_{s\to 0}\frac{sin(j\pi(N+1)+\frac{p}{2}(N+1)s)}{sin(j\pi+\frac{p}{2}s)}$$
$$= (-1)^{Nj}\lim_{s\to 0}\frac{sin(\frac{p}{2}(N+1)s)}{sin(\frac{p}{2}s)}$$
$$= (-1)^{Nj}(N+1)$$

□

THEOREM 1. For $n_k = pk$, Q_n is an approximate identity, so $\widetilde{\sigma_n}(f) \to f$ uniformly for continuous f

PROOF. We need to prove (2.16) and (2.17) since (2.15) is automatic. Note that we can write (4.3) as

$$|Q_N(t)| = \frac{1}{2\pi}\frac{1}{N+1}h_N(t)h_N(pt) \text{ for} \tag{4.6}$$

$$h_N(t) = |\frac{sin(\frac{N+1}{2}t)}{sin(\frac{1}{2}t)}| \tag{4.7}$$

Of course when $p=1$ we obtain the Fejer Kernel which we know satisfies (2.16) and (2.17) by (2.7) and (2.8). By the Cauchy-Schwarz inequality,

$$\int_\epsilon^\pi |Q_N(t)|dt \leqslant \frac{1}{2\pi}(\frac{1}{N+1}\int_\epsilon^\pi h_N(t)^2dt)^{\frac{1}{2}}(\frac{1}{N+1}\int_\epsilon^\pi h_N(pt)^2dt)^{\frac{1}{2}}$$

so it suffices to show that $\frac{1}{2}\int_{-\pi}^{\pi} h_N(pt)^2dt$ is uniformly bounded. But this is equal to $\frac{1}{N+1}\frac{1}{p}\int_{-p\pi}^{p\pi} h_N(t)^2dt$ by a change of variable and h_N is periodic of period 2π, so this is equal to $\frac{1}{N+1}\int_{-\pi}^{\pi} h_N(t)^2dt$, which is uniformly bounded by (2.7).

□

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