Notes For Algebraic Methods in Combinatorics

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Fall 2008

Contents

Chapter 1

Simplicial Complexes and the Face Ring

1.1 Simplicial Complexes

Definition 1.1. *Δ is called an abstract (finite) simplicial complex if $\Delta \subseteq 2^V$ where $|V| < \infty$ and $F \subseteq T \in \Delta$ implies $F \in \Delta$.*

For a simplicial complex Δ , if $F \in \Delta$, then we define $\dim F = |F| - 1$. The dimension of the complex Δ is defined to be the maximum dimension of any of its elements, $\dim \Delta = \max_{F \in \Delta} \dim F$.

Definition 1.2. *The f-vector of a simplicial complex is $f(\Delta) := (f_{-1}, f_0, f_1, \ldots, f_{\dim \Delta})$ where $f_i = f_i(\Delta) = |\{F \in \Delta : \dim F = i\}|$*

The entry f_{-1} corresponds to the empty set, which is a face of any non-empty complex. Therefore $f_{-1} = 1$ for any non-empty complex. The entry f_0 counts the zero dimensional faces of Δ , which are referred to as the vertices. Similarly, the one dimension faces of Δ are called edges, and the maximal ($\dim \Delta$) dimension faces are called facets.

Later in the chapter we will prove the Kruskal-Katona Theorem, which provides a characterization of the possible f-vectors of simplicial complexes.

In additional to abstract simplicial complexes we also will deal with geometric simplicial complexes. To define geometric simplicial complexes, we first need to recall the definition of a geometric simplex.

Definition 1.3. *A geometric k-simplex Δ^k is the convex hull of $k+1$ affinely independent points $\{p_1, \ldots, p_{k+1}\}$ in $\mathbb{R}^n$ for some $n \geq k$. A face of Δ^k is the convex hull of any subset of $\{p_1, \ldots, p_{k+1}\}$.*

Note that the empty set and Δ^k itself are both faces of Δ^k.

Definition 1.4. *A geometric simplicial complex is a collection $\mathcal{C}$ of geometric simplexes such that*

(1) if $F, G \in \mathcal{C}$ then $G \cap F$ is a face of both F and G.

(2) if $F \in \mathcal{C}$ and G is a face of F then $G \in \mathcal{C}$.

Any geometric simplicial complex is also an abstract simplicial complex on its set of vertices. We can also show that any abstract simplicial complex can be realized by a geometric simplicial complex.

If n is the number of vertices of an abstract simplicial complex Δ, then we can realize Δ as a geometric simplex in $\mathbb{R}^n$ by taking the basis elements of $\mathbb{R}^n$ as the vertices. If d is the dimension of Δ, we can also realize Δ in $\mathbb{R}^{2d+1}$ by taking as vertices n distinct points on the moment curve in $\mathbb{R}^{2d+1}$.

We will write $||\Delta||$ for any geometric realization of an abstract simplicial complex Δ. So $||\Delta||$ is a union of geometric simplexes in some $\mathbb{R}^n$. Therefore $||\Delta||$ inherits a topology from $\mathbb{R}^n$. Any two geometric realizations of Δ are homeomorphic with this induced topology. Therefore the topological type of Δ is well defined, and we are justified in referring to $||\Delta||$ as *the* geometric realization of Δ .

We conclude this section with a few notes about the dimension needed for a geometric realization of an abstract simplicial complex.

First note that for $i, d > 0$, a d-dimensional complex Δ has a realization in $\mathbb{R}^{d+i}$ if and only if it has a realization in S^{d+i}. In the forward direction this follows from taking the one-point compactification of $\mathbb{R}^{d+i}$. The reverse direction follows since some point in S^{d+i} will not be in $||\Delta||$ (since the dimension of $||\Delta||$ is less than that of S^{d+i}, therefore we can remove some point of S^{d+i}, leaving $\mathbb{R}^{d+i}$.

Second we note that the d-skeleton of the $(2d+2)$-simplex can not be realized in S^{2d} or $\mathbb{R}^{2d}$. This can be proved using the Van-Kampen obstruction from algebraic topology. For example, see Matoušek [7]. This shows that in general $2d+1$ dimensions are needed to realize a d-dimensional simplicial complex. For $d = 1$, this corresponds to the fact that not all graphs are planar, and in particular, the complete graph on 5 vertices can not be realized in $\mathbb{R}^2$.

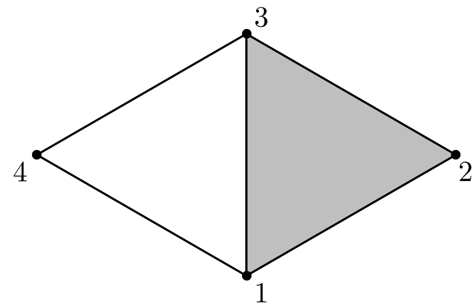
1.2 The Face Ring

Our main goal in this chapter will be to characterize the f-vectors of various classes simplicial complexes. One of our main tools will be the face ring.

Definition 1.5. *Let Δ be a $(d-1)$ dimensional simplicial complex on Ver$(\Delta) = [n]$. Let k be a field. Define $S = k[x_1, \ldots, x_n]$ and define the face ideal I_Δ in S by $I_\Delta = (\prod_{i \in J} x_i : J \subseteq [n], J \notin \Delta)$.*

Then define the face ring of Δ over k as $k[\Delta] := S/I_\Delta$.

Example 1.6.



Let Δ be the complex with vertex set $\{1, 2, 3, 4\}$ and maximal faces {1,2,3}, {1,4}, and {3,4}. Then

$$\begin{aligned} I_\Delta &= (x_2x_4, x_1x_3x_4, x_1x_2x_4, x_2x_3x_4, x_1x_2x_3x_4) \\ &= (x_2x_4, x_1x_3x_4) \end{aligned}$$

We note that in this example and in general, I_Δ is generated by the minimal non-faces of Δ . So $I_\Delta = \left(\prod_{i\in J} x_i : J \text{ is a minimal non-face of } \Delta\ \right)$.

We now turn our attention to the face ring itself. We know that the monomials in S form a basis for S and therefore span the face ring. However, the monomials in I_Δ are not needed in order to span the face ring. We will introduce some new notation and definitions to help us more fully characterize the face ring.

1.3 Grading and Hilbert Series

Throughout the following we use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$. For a vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we define $x^a \in k[x_1, \ldots, x_n]$ by $x^a = \prod_{i=1}^n x_i^{a_i}$. We then define the support of a by $\mathrm{Supp}(a) := \{i : a_i > 0\}$.

Using this notation, we have that the face ring $k[\Delta]$ is spanned by the elements x^a such that $\mathrm{Supp}(a) \in \Delta$. Note that a non-zero linear combination of distinct monomials whose supports are faces of Δ can not be a linear combination of monomials whose supports are not faces of Δ . Therefore the x^a such that $\mathrm{Supp}(a) \in \Delta$ are also independent in $k[\Delta]$, so we can decompose the face ring as a direct sum

$$k[\Delta] = S/I_\Delta = \bigoplus_{a:\mathrm{Supp}(a)\in\Delta} kx^a \tag{1.1}$$

This leads to the idea of a graded ring.

Definition 1.7. *A ring R is $\mathbb{N}$-graded if it can be decomposed as $R = \oplus_{i\in\mathbb{N}} R_i$ where $R_iR_j \subseteq R_{i+j}$ for $i, j \in \mathbb{N}$. Similarly, R is $\mathbb{N}^n$ graded if $R = \oplus_{a\in\mathbb{N}^n} R_a$ where $R_aR_b \subseteq R_{a+b}$ for $a, b \in \mathbb{N}^n$.*

The polynomial ring $S = k[x_1, \ldots, x_n]$ is $\mathbb{N}$-graded by degree, with S_i equal to the set of homogeneous polynomials in S of degree i (polynomials each of whose terms is a monomial of degree i). We can also put a finer, $\mathbb{N}^n$ grading on S based on the powers of each variable appearing in any monomial. In this case $S_{(a_1,\ldots,a_n)} = k \cdot x_1^{a_1} \cdots x_n^{a_n}$.

Note that I_Δ is a homogeneous ideal, i.e. it is generated by homogeneous elements. Therefore I_Δ is also both $\mathbb{N}$ and $\mathbb{N}^n$ graded by degree. So we get an induced $\mathbb{N}$ grading on the quotient $k[\Delta] = S/I_\Delta = \oplus_{i\in\mathbb{N}} S_i/(I_\Delta)_i$. Similarly, we get an induced $\mathbb{N}^n$ grading on $k[\Delta]$. This is the grading that we found in equation (1.1).

For graded algebras we can introduce the idea of the Hilbert Series to keep track of the dimensions of each of the vectors spaces in the grading. For the face ring we will see that the Hilbert Series is determined by the f-vector of our simplicial complex.

Definition 1.8. *Let R be an $\mathbb{N}$-graded k-algebra.*
We define the Hilbert series of R by $\mathrm{Hilb}(R) := \sum_{i\in\mathbb{N}} (\dim_k R_i) t^i$.
Similarly, if R is $\mathbb{N}^n$ graded we define the fine Hilbert Series of R by $\mathrm{Hilb}(R) = \sum_{a\in\mathbb{N}^n} (\dim_k R_a) t^a$*. Here* $t^a = \prod_{i=1}^n t_i^{a_i}$.

So for the polynomial ring S we have $\mathrm{Hilb}(S) = \sum_{i\in\mathbb{N}} \left(\!\binom{n}{i}\!\right) t^i$. Here $\left(\!\binom{n}{i}\!\right)$ is the number of ways of choosing i things from n options with repetition allowed, so $\left(\!\binom{n}{i}\!\right) = \binom{i+n-1}{i}$.

Our next goal will be to compute the Hilbert Series for the face ring of a simplicial complex. We will begin by computing the fine Hilbert Series and then later in the argument we will set all of the t_i equal to t to calculate the Hilbert Series for the $\mathbb{N}$ grading. From (1.1) we have

$$\mathrm{Hilb}(k[\Delta]) = \sum_{a:\mathrm{Supp}(a)\in\Delta} t^a$$

Partitioning by the faces of Δ we get

$$\mathrm{Hilb}(k[\Delta]) = \sum_{F\in\Delta} \left(\sum_{a:\mathrm{Supp}(a)=F} t^a \right)$$

For each $a \in \mathbb{N}^n$ such that $\mathrm{Supp}(a) = F$, t^a must contain $\prod_{i\in F} t_i$ as well as any (non-negative) number of additional powers of t_i for $i \in F$. Therefore we have

$$\mathrm{Hilb}(k[\Delta]) = \sum_{F\in\Delta} \left(\left(\prod_{i\in F} t_i \right) \left(\prod_{i\in F} (1 + t_i + t_i^2 + \cdots) \right) \right) = \sum_{F\in\Delta} \left(\left(\prod_{i\in F} t_i \right) \left(\prod_{i\in F} \frac{1}{1-t_i} \right) \right)$$

At this point we set all the t_i equal to t to consider the Hilbert Series of the $\mathbb{N}$ grading

$$\mathrm{Hilb}(k[\Delta]) = \sum_{F\in\Delta} t^{|F|} \cdot \frac{1}{(1-t)^{|F|}}$$

Partitioning the faces of Δ by dimension, we can write this sum in terms of the face numbers of Δ. If $(d-1)$ is the dimension of Δ we have

$$\mathrm{Hilb}(k[\Delta]) = \sum_{i=0}^{d} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i} \tag{1.2}$$

Factoring out $1/(1-t)^d$ from the right hand side of (1.2) we will be left with a polynomial in t of degree at most d. Write this polynomial as $h_0 + h_1 t + \cdots + h_d t^d$. Then from (1.2), making the substitution $x = 1/t$ we see that the h_i and f_i are related by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} \tag{1.3}$$

Definition 1.9. *Let Δ be a simplicial complex of dimension $(d-1)$. Let $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$ be the f-vector of Δ. Then define $(h_0, h_1, \ldots, h_d)$ to be the h-vector of Δ, where the h_i are determined from the f vector of Δ by (1.3).*

From (1.3) we see that the h_j are completely determined by the f_i and each h_j is an alternating sum of the f_i. Conversely, making the substitution $y = x - 1$ in (1.3) we see that the f_i are determined by the h_j, and the f_i are non-negative linear combinations of the h_j. Therefore the f-vector and h-vector contain the same combinatorial data about the simplicial complex Δ . In some cases we will find it easier to answer combinatorial questions about Δ in terms of the h-vector instead of the f-vector.

We conclude this section with a couple of notes about the relationships between the f and h vectors. From (1.3) with the substitution $y = x-1$ we have $f_{d-1} = \sum_{i=0}^{d} h_i$. So the sum of the elements of the h-vector is the number of facets of Δ. Also from (1.3), we have $h_d = \sum_{i=0}^{d} f_{i-1}(-1)^{d-i} = (-1)^{d-1}\tilde{\chi}(||\Delta||)$, where $\tilde{\chi}(||\Delta||)$ is the reduced Euler characteristic of $||\Delta||$. In particular, if $||\Delta||$ is a sphere we have that $h_d(\Delta) = 1$. For polytopes, this will be a special case of the Dehn-Sommerville equations that we will prove in the next section.

1.4 Shellablility and the Dehn-Sommerville Equations

Our main goal in this section will be to prove the Dehn-Sommerville Equations, which give restrictions on the h-vectors of the boundaries of simplicial polytopes.

Theorem 1.10. *(Dehn-Sommerville Equations)*

If Δ is the boundary complex of a simplicial d-polytope, then for $0 \leq i \leq [d/2]$, $h_i(\Delta) = h_{d-i}(\Delta)$.

Combining this result with equation (1.3) we have the following corollary.

Corollary 1.11. *If Δ is the boundary complex of a simplicial d-polytope, then $\{f_0, f_1, \ldots, f_{[d/2]-1}\}$ determines the entire f vector of Δ .*

In the following we will provide a proof of the Dehn-Sommerville equations for the boundaries of simplicial polytopes. However, the relations also hold for the more general class of all simplicial complexes Δ such that $||\Delta|| \cong S^{d-1}$.

The Dehn-Sommerville equations also hold for an even more general class of simplicial complexes called Eulerian complexes. To define these complexes we first need two definitions that will be useful throughout this section.

Definition 1.12. *A simplicial complex is pure if all of its maximal faces have the same dimension.*

Definition 1.13. *Let Δ be a simplicial complex and let F be a face of Δ . Then the link of F in Δ is* $\operatorname{lk}_\Delta F := \{T \in \Delta : T \cap F = \emptyset, F \cup T \in \Delta\}$.

Definition 1.14. *A pure simplicial complex Δ is Eulerian if the link of every face of Δ (including the empty face) has Euler characteristic equal to the Euler characteristic of a sphere of the same dimension.*

Eulerian complexes, like homology spheres, are a nice class in which to work because they are closed under taking links. The class of simplicial spheres does not have this property.

There is another characterization of Eulerian complexes in terms of their face posets.

Definition 1.15. *The face poset of a simplicial complex Δ is the set of all faces of Δ ordered by inclusion. We write the face poset of Δ as $\mathcal{F}(\Delta)$.*

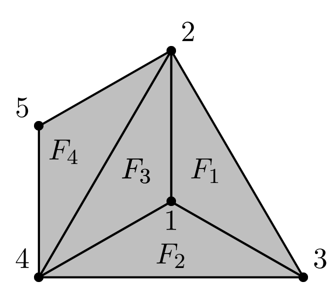
A graded poset with a unique largest and smallest element is called Eulerian if every interval of the poset has equal numbers of odd rank and even rank elements. Given a pure simplicial complex, we can extend its face poset by adding a largest element (the empty set is already the unique smallest element of the poset). If the resulting poset is Eulerian, then the simplicial complex is Eulerian. For more information about Eulerian posets, see for example chapter 3 of [10].

In order to prove the Dehn-Sommerville equations we will use the idea of a shelling of a pure simplicial complex. We will present two definitions of shellability (definitions 1.16 and 1.19); we leave it as an exercise to show that they are equivalent.

Definition 1.16. *A pure simplicial complex Δ is shellable if there exists a total order of the facets of Δ, $F_1, F_2, \ldots, F_t$, such that for the sub-complexes $\Delta_i = \cup_{j \leq i} 2^{F_j}$, there exists a unique minimal element in $2^{F_i} \backslash \Delta_{i-1}$ for every i, $1 \leq i \leq t$. This minimal element is called the restriction of F_i and is denoted $r(F_i)$. We let $r(F_1) = \emptyset$. The order $F_1, F_2, \ldots, , F_t$ is called a shelling order.*

Example 1.17.

For the simplicial complex shown on the right, F_1, F_2, F_3, F_4 is a shelling order. For this order $r(F_1) = \emptyset$, $r(F_2)$ is the vertex $\{4\}$, $r(F_3)$ is the edge $\{2,4\}$, and $r(F_4)$ is the vertex $\{5\}$. Note that F_1, F_4 is *not* the beginning of any shelling order.



From this definition of shellability we see that the face poset of a shellable simplicial complex can be broken up into a disjoint union of intervals

$$\mathcal{F}(\Delta) = \sqcup_{1\leq i\leq t}[r(F_i), F_i]$$

In general, if the face poset of a simplicial complex Δ can be divided into disjoint intervals whose maximal elements are the facets of Δ we say that Δ is partitionable. The following theorem relates the h-vector of a partitionable simplicial complex to the sizes of the minimal elements in the intervals of the partition. In particular, for shellable complexes this gives a relationship between the h-vector and the size of the restriction faces of a shelling order.

Theorem 1.18. *Let Δ be a pure $(d-1)$-dimensional partitionable simplicial complex with $\mathcal{F}(\Delta) = \sqcup_{1\leq i\leq t}[r(F_i), F_i]$. Then $h_i(\Delta) = |\{j : |r(F_j)| = i\}|$.*

Note that this theorem shows that the number of restriction faces of a certain size is independent of the choice of shelling order.

Proof. Starting with equation (1.3) and sorting the faces of Δ by the interval of the partition in which they appear yields

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{j=1}^{t}\left(\sum_{G:r(F_j)\subseteq G\subseteq F_j}(x-1)^{d-|G|}\right)$$

Since F_j is a simplex, the number of simplexes G of size $|r(F_j)|+k$ such that $r(F_j)\subseteq G\subseteq F_j$ is equal to the number of ways of choosing k of the vertices of $F_j\backslash r(F_j)$. So this is equal to $\binom{d-|r(F_j)|}{k}$. Therefore we have

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{j=1}^{t}\left(\sum_{k=0}^{d-|r(F_j)|}\binom{d-|r(F_j)|}{k}(x-1)^{d-|r(F_j)|-k}\right)$$

Then using the binomial theorem we have

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{j=1}^{t} x^{d-|r(F_j)|}$$

Sorting our facets by the size of $r(F_i)$ then yields

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=1}^{d} x^{d-i}\cdot|\{j : |r(F_j)| = i\}|$$

Comparing coefficients we now have the desired equality. □

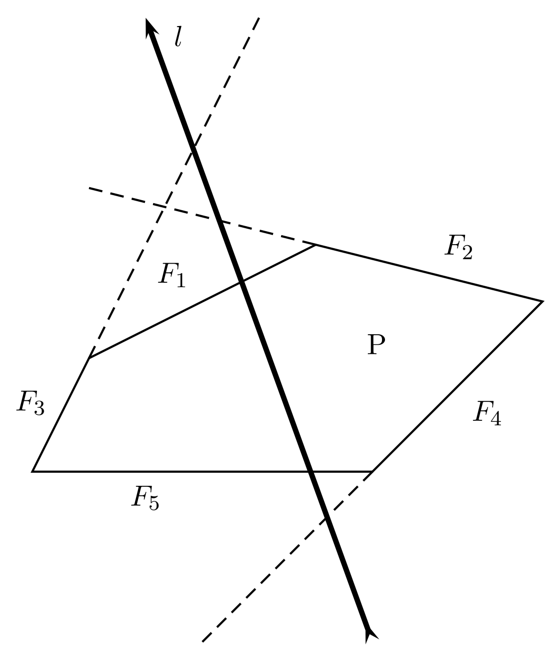
We now introduce our second (equivalent) definition of shellability. The notation will be the same as in the previous definition.

Definition 1.19. *A pure simplicial complex Δ is shellable if there exists a total order of the facets of Δ, $F_1, F_2, \ldots, F_t$, such that for $2 \leq i \leq t$, $2^{F_i} \cap \Delta_{i-1}$ is a pure simplicial complex complex of co-dimension one.*

Our next goal will be to prove that the boundary complex of any polytope is shellable. We will construct a particular type of shelling for these complexes, called a line shelling. This idea was originally due to Bruggesser and Mani [3]. We will give an outline of the construction; for more details see Ziegler's text [11]. (note that Ziegler uses a third definition for shellability).

We begin by taking a generic oriented line l intersecting our polytope. By generic, we require that our line not be parallel to any facet of the polytope, that the line intersects the polytope in the interior of facets, and that the line not pass through the intersection of the supporting hyperplanes of the facets. We then order the facets by the order in which one sees them moving along the line.

The first facet in the shelling is the facet where the line intersects the polytope with the orientation of the line outward with respect to the polytope. The facets are then added one by one in the order in which the line intersects the supporting hyperplanes of the facets as we move away from the polytope. When we have passed all of the intersections of the line with supporting hyperplanes of facets in one direction, we then start at the other end of the line and continue moving along the line toward the facet (we have passed through the point at infinity on the line). Now we are ordering the facets by the order in which they disappear from view, this again being the order in which our line intersects the supporting hyperplanes. Last we add the facet in which the line intersects the polytope with an inward orientation.



We claim that this total order of the facets is in fact a shelling order. We prove this using the second definition of shelling (definition 1.19). We need to show that the intersection of any new facet

F_i with the sub-complex Δ_{i-1} of previous facets is pure. Note that the facet F_i is a simplex in the supporting hyperplane of F_i. Let x be the point where l intersects the supporting hyperplane of F_i.

In the case where F_i was added to the shelling before we passed through infinity on our line, the complex $2^{F_i} \cap \Delta_{i-1}$ is the union of all of the faces of ∂F_i visible from x. This is equal to the union of all of the facets of ∂F_i visible from x and all sub-faces of these facets, and hence is pure. To see this, let G be any face in $2^{F_i} \cap \Delta_{i-1}$ that is not a facet of ∂F_i. Then G must be visible from x. If all of the facets of ∂F_i containing G were not visible from x, then consider any point y in the interior of G (or if G is a vertex, let $y = G$). Then in ∂F_i, a neighborhood of y would not be visible from x, so y itself would not be visible from x and therefore G would not be visible from x. Therefore there must be some facet of ∂F_i containing G that is visible from x, proving the desired claim.

If F_i is added to the shelling after we pass through infinity the situation is reversed and $2^{F_i} \cap \Delta_{i-1}$ is the union of all the faces of ∂F_i that are contained in some face of ∂F_i not visible from x. This is equal to the union of all of the facets of ∂F_i not visible from x and all sub-faces of these facets and is also pure. To see this, again let G be any face in $2^{F_i} \cap \Delta_{i-1}$ that is not a facet of ∂F_i. Then G must be contained in some face of ∂F_i not visible from x, call this face G'. Since G' is not visible from x, all of the facets of ∂F_i that contain G' are also not visible from x. Therefore G' and hence G is contained in a facet of ∂F_i not visible from x, as desired.

Using line shellings, we are now ready to prove the Dehn-Sommerville equations for the boundaries of simplicial polytopes. Let Δ be the boundary of a simplicial d-polytope, and let l be a generic oriented line with respect to this polytope. Let $-l$ be the same line with the reverse orientation. Then l and $-l$ give rise to line shellings of Δ which have the exact opposite shelling orders. Let $G_1, \ldots, G_t$ be any labeling of the facets of Δ. We will write $r_l(G_i)$ and $r_{-l}(G_i)$ for the restriction of G_i with respect to line shellings induced by l and $-l$.

In exercise 1.4.3 you will prove that given a shelling order $F_1, \ldots, F_t$ the restriction of F_i is given by

$$r(F_i) = \{v \in F_i : \exists j < i \text{ such that } (F_i \backslash v) \subseteq F_j\} \tag{1.4}$$

Fix any facet G_i of Δ and any vertex v of G_i. Then since Δ is the boundary of a polytope there is a unique facet $G_j \neq G_i$ such that $G_i \backslash v \subseteq G_j$. This facet G_j will appear before G_i either in the line shelling generated by l or the line shelling generated by $-l$, and after G_i in the other line shelling. So by (1.4) we know that v is either in $r_l(G_i)$ or $r_{-l}(G_i)$. Therefore $r_l(G_i) = G_i \backslash r_{-l}(G_i)$.

Combining this with Theorem 1.18 we have

$$h_i(\Delta) = |\{j : |r_l(G_j)| = i\}| = |\{j : |r_{-l}(G_j)| = d - i\}| = h_{d-i}(\Delta)$$

which are exactly the Dehn-Sommerville equations.

Exercise 1.4.1.

Prove the Dehn-Sommerville equations for any simplicial complex Δ such that $||\Delta|| \cong S^{d-1}$.

Exercise 1.4.2.

Show that the dimension of the span of $\{f(\Delta) : \Delta = \partial P$ for a simplicial polytope $P\}$ is equal to $(d+1) - [(d+1)/2]$, i.e. the Dehn-Sommerville equations are the only linear relations among $f(\Delta)$.

Exercise 1.4.3.

(a) Prove the formula for the restriction of F_i given in equation (1.4).

(b) Show that the two definitions of shellability given in this section (definitions 1.16 and 1.19) are equivalent.

1.5 Krull Dimension

The goal of this section is to develop more of the algebraic tools that will be needed in the proof of the upper bound theorem. We begin with the definition of Krull Dimension.

Definition 1.20. *Let R be a finitely generated k-algebra. The Krull dimension of R is the maximal number of algebraically independent elements of R (over k). We write the Krull dimension of R as* $\dim(R)$.

While this is the definition of Krull dimension that we will use, many other equivalent definitions exist. For some of these, see page 33 of Stanley [9].

We will also use following standard result, known as the Noether Normalization Lemma.

Lemma 1.21. *(Noether Normalization Lemma)*

Let k be a field and let R be a finitely generated $\mathbb{N}$-graded k-algebra, $R = \oplus_{i \geq 0} R_i$.

1. *There exist algebraically independent homogeneous elements $x_1, \ldots, x_m \in R$ all of the same degree such that R is integral over $k[x_1, \ldots, x_m]$. Therefore $\dim(R) = m$. So in particular, the Krull dimension of R is finite.*

2. *If $|k| = \infty$ and R is generated by degree 1 elements, then $x_1, \ldots, x_m$ can be chosen from R_1.*

In order to use part (b) of this lemma we will usually restrict to the case where $|k| = \infty$. We now calculate the Krull dimension of the face ring of a simplicial complex. This dimension is equal to the number of vertices in any maximal dimension face of the complex.

Theorem 1.22. *(Stanley) Let Δ be a $(d-1)$-dimensional simplicial complex and let k be a field with $|k| = \infty$. Then $\dim(k[\Delta]) = d$.*

Proof. If $F \in \Delta$ such that $|F| = d$, then from our decomposition of $k[\Delta]$ in equation (1.1) we know that $\{x_i : i \in F\}$ is algebraically independent. Therefore $\dim(k[\Delta]) \geq d$.

To complete the proof we must show that $k[\Delta]$ has no set of $d+1$ algebraically independent elements.

Assume that there was some set $y_1, \ldots, y_{d+1} \in k[\Delta]$ such that all the y_i are algebraically independent. By the Noether Normalization Lemma we can assume that all the y_i are in $k[\Delta]_1$.

Define $B :=< y_1, \ldots, y_{d+1} > \subseteq k[\Delta]$. So B inherits a grading $B = \oplus_{i \geq 0} B_i$ from the grading of $k[\Delta]$.

Since all of the y_i are algebraically independent, each B_t has a basis consisting of all monomials of degree t in the y_i. Therefore, for large t we have

$$\dim_k B_t = \left(\binom{d+1}{t}\right) \sim t^{d+1}$$

However, using the decomposition in equation (1.1) we have that

$$\dim_k k[\Delta]_t = \sum_{F \in \Delta} \left(\binom{|F|}{t-|F|}\right) \sim t^d$$

where the last step follows from the fact that Δ has a fixed finite number of faces, and the fact that $|F| \leq d$ for all $F \in \Delta$. So for large enough t we have that $\dim_k B_t > \dim_k k[\Delta]_t$. But this is impossible since $B_t \subseteq k[\Delta]_t$. Therefore no set of $d+1$ algebraically independent elements can exist in $k[\Delta]$. □

We conclude this section with two more ideas that we will need for the upper bound theorem.

Definition 1.23. *Let R be a ring with $\dim(R) = d$.*

A homogeneous system of parameters of R (h.s.o.p.) is a sequence $\theta_1, \ldots, \theta_d \in R_+ = \oplus_{i>0} R_i$ of homogeneous elements such that $\dim(R/(\theta_1, \ldots, \theta_d)) = 0$.

Equivalently, $\theta_1, \ldots, \theta_d \in R_+$ is a h.s.o.p. for R iff $\dim(R) = d$ and R is a finitely generated $k[\theta_1, \ldots, \theta_d]$-module.

Definition 1.24. *A linear system of parameters (l.s.o.p) of a ring R is h.s.o.p. of R all of whose elements have degree one.*

1.6 The Upper Bound Theorem

Theorem 1.25. *(Upper Bound Theorem - Stanley)*
Let Δ be a simplicial complex on n vertices with $||\Delta|| \cong S^{d-1}$. Then for $-1 \leq i \leq d-1$, $f_i(\Delta) \leq f_i(\partial C(n,d))$, where $C(n,d)$ is the cyclic polytope of dimension d with n vertices.

Throughout the following we will assume that k is an infinite field. Note that by the Noether Normalization lemma, we know that $k[\Delta]$ has a l.s.o.p. Let θ be any l.s.o.p. of $k[\Delta]$. Then we will write $k(\Delta) = k[\Delta]/(\theta)$.

We define $h(\Delta, t)$ to be the h-vector of Δ written as a polynomial, so $h(\Delta, t) = h_0 + h_1 t + \cdots + h_d t^d$.

The most difficult part of the proof of the Upper Bound Theorem will be the following lemma.

Lemma 1.26. *Let Δ be a simplicial complex with $||\Delta|| \cong S^{d-1}$ and let θ be a l.s.o.p. of $k[\Delta]$. Then* $\mathrm{Hilb}(k[\Delta]/\theta) = h(\Delta, t)$

In this section we will show how to prove the Upper Bound Theorem assuming this lemma.

We first note that $k(\Delta)$ is generated by $n - d$ elements of degree 1. To see this, we start with a l.s.o.p. $\theta_1, \ldots, \theta_d$ of $k[\Delta]$ (by theorem 1.22 we know that this l.s.o.p has d elements). We then extend this l.s.o.p. to a full basis of $k[x_1, \ldots, x_n]_1$, $\theta_1, \ldots, \theta_d, \ldots, \theta_n$. Then $\theta_{d+1}, \ldots, \theta_n$ is a basis for $(k[x_1, \ldots, x_n]/(\theta_1, \ldots, \theta_d))_1$, hence $k[\Delta]/(\theta_1, \ldots, \theta_d)$ is generated by $\theta_{d+1}, \ldots, \theta_n$.

Therefore, assuming lemma 1.26 we have

$$h_i(\Delta) = \dim_k k(\Delta)_i \leq \left(\!\binom{n-d}{i}\!\right) \tag{1.5}$$

Next we compute the h vector of the boundary of the cyclic polytope. From the Dehn-Sommerville equations, we know that it is sufficient to calculate $h_i(\partial C(n, d))$ for $0 \leq i \leq [d/2]$. Also, from lemma 1.26 we know that $h_i(\partial C(n, d)) = \dim_k k(\partial C(n, d))$, so it is sufficient to calculate the latter quantity.

We recall the the cyclic polytope $C(n, d)$ is neighborly; i.e. every set of $[d/2]$ vertices in $C(n, d)$ is a simplex in $C(n, d)$. So up to dimension $[d/2]$, the face ring $k[\partial C(n, d)]$ is the entire polynomial ring $k[x_1, \ldots, x_n]$.

Once again we start with a l.s.o.p. $\theta_1, \ldots, \theta_d$ of $k[\partial C(n, d)]$ and extend it to a full basis $\theta_1, \ldots, \theta_n$ of $k[x_1, \ldots, x_n]_1 = k[\partial C(n, d)]_1$. Then for $0 \leq i \leq [d/2]$ we have

$$\begin{aligned} h_i(\partial C(n, d)) = (\dim_k(k[\partial C(n, d)]/(\theta_1, \ldots, \theta_d))_i &= \dim_k(k[\theta_1, \ldots, \theta_n]/(\theta_1, \ldots, \theta_d))_i \\ &= \dim_k(k[\theta_{d+1}, \ldots, \theta_n])_i \\ &= \left(\!\binom{n-d}{i}\!\right) \end{aligned}$$

Combining this with (1.5) we have $h_i(\Delta) \leq h_i(\partial C(n, d))$ for $0 \leq i \leq [d/2]$. By the Dehn-Sommerville equation, this implies $h_i(\Delta) \leq h_i(\partial C(n, d))$ for $0 \leq i \leq d$. Then since the elements of the f vector of a simplicial complex are non-negative linear combinations of the elements of the h-vector of the complex, we have $f_i(\Delta) \leq f_i(\partial C(n, d))$ for $-1 \leq i \leq d - 1$, which is the desired upper bound theorem.

We see from this proof that equality in the upper bound theorem occurs for any neighborly simplicial sphere. While the cyclic polytope provides an example of such a sphere, many other neighborly simplicial spheres exist, including other neighborly polytopes. For more information about these neighborly polytopes, see for example Shemer's paper [8].

1.7 Cohen-Macaulay Rings and Complexes

In order to prove lemma 1.26 we will define a special class of simplicial complexes called Cohen-Macaulay complexes. We will show that all Cohen-Macaulay complexes satisfy the lemma and then prove that all simplicial complexes Δ with $||\Delta|| \cong S^{d-1}$ are Cohen-Macaulay.

Let R be an $\mathbb{N}$-graded k-algebra, i.e $R = \oplus_{i\geq 0} R_i = k[x_1, \ldots, x_n]/I$ where I is a homogeneous ideal. Let $R_+ = \oplus_{i>0} R_i$, so R_+ is the unique maximal ideal of R.

Recall that an element b is a zero-divisor of a ring A if there exists $a \in A \backslash \{0\}$ such that $ba = 0$. Otherwise b is called a non-zero-divisor.

Definition 1.27. *A collection of elements $\theta_1, \ldots, \theta_l \in R_+$ is called a regular sequence if each θ_i is homogeneous and θ_{i+1} is a non-zero-divisor of $R/(\theta_1, \ldots, \theta_i)$.*
The depth of R is the size of the longest possible regular sequence in R.

Equivalently, $\theta_1, \ldots, \theta_l \in R_+$ is a regular sequence if the θ_i are algebraically independent over k and R is a free $k[\theta_1, \ldots, \theta_l]$-module.

Note that since the Krull dimension of R is the maximal number of algebraically independent elements over k in R, the depth of R is always less than or equal to the Krull dimension of R. In the special case of a k-algebra R where the depth and Krull dimension are equal, R is called Cohen-Macaulay.

Definition 1.28. *A $\mathbb{N}$-graded k-algebra R is called Cohen-Macaulay (CM) if* $\dim(R) = \mathrm{Depth}(R)$.
A simplicial complex Δ is called Cohen-Macaulay over k if its face ring $k[\Delta]$ is a Cohen-Macaulay k-algebra.

Example 1.29.

Let Δ be the simplicial complex with vertices 1,2,3 and a single edge, $\{2, 3\}$. Let k be an infinite field. We know by Theorem 1.22 that the Krull dimension of $k[\Delta]$ is two. The face ring of Δ is $k[\Delta] = k[x_1, x_2, x_3]/(x_1x_2, x_1x_3)$.

We claim that $k[\Delta]$ has no regular sequence of length two. To see this, let $\theta_1, \theta_2 \in k[\Delta]_1$; we need to show that θ_1, θ_2 is not a regular sequence. We know we can form a linear combination of θ_1 and

θ_2 that eliminates x_1; i.e. there exist α, β such that $\alpha\theta_1 + \beta\theta_2 \in k[x_2, x_3]$. If we multiply this linear combination by x_1, then each term in the product $x_1(\alpha\theta_1 + \beta\theta_2)$ will include either x_1x_2 or x_1x_3, both of which are zero in $k[\Delta]$. Therefore R is not a free $k[\theta_1, \theta_2]$-module, so θ_1, θ_2 is not a regular sequence and Δ is not CM.

We now show that lemma 1.26 is true for CM complexes.

Theorem 1.30. *Let Δ be a $(d-1)$ dimensional CM simplicial complex. Let θ be a regular sequence and l.s.o.p of $k[\Delta]$ (so θ has d elements). Then $h(\Delta, t) = \text{Hilb}(k(\Delta))$.*

Proof. We know that $\text{Hilb}(k[\Delta]) = h(\Delta, t)/(1-t)^d$. Since the face ring is a free module over $k[\theta]$, to get a basis for $k[\Delta]$ over k we take products of monomials in $k[\theta]$ and monomials in $k(\Delta)$. Therefore $\text{Hilb}(k[\Delta]) = \text{Hilb}(k(\Delta)) \cdot \text{Hilb}(k[\theta])$. We have previously calculated the Hilbert series of a polynomial ring, so we know that $\text{Hilb}(k[\theta]) = 1/(1-t)^d$. Combining these equalities and canceling the $1/(1-t)^d$ from both sides yields the desired result.

□

1.8 Reisner's Theorem

To complete the proof of the Upper Bound Theorem, we now need to show that for simplicial complexes Δ such that $||\Delta|| \cong S^{d-1}$, Δ is CM. This will follow from Reisner's Theorem, which provides a characterization of CM simplicial complexes without appealing to the face ring.

Theorem 1.31. *(Reisner's Theorem)*

A simplicial complex Δ is CM over k if and only if for all $F \in \Delta$ (including the empty set) and all $i < \dim(\text{lk}_\Delta F)$, $\tilde{H}_i(\text{lk}_\Delta F; k) = 0$.

As an exercise you are asked to prove a purely topological characterization of CM simplicial complexes, involving only $||\Delta||$. From this result we get the following corollary.

Corollary 1.32. *If Δ is a simplicial complex with $||\Delta|| \cong S^{d-1}$, then Δ is CM.*

In order to prove Reisner's Theorem we will use the local cohomology of the face ring. For a finitely generated k-module $R = \oplus_{i \geq 0} R_i$ we will write $H^i(R)$ for the local cohomology of R with respect to the maximal ideal R_+. (In the future, the fact the the local cohomology is with respect to R_+ will always be assumed.) First we will state a pair of theorems about the local cohomology of the face ring and show how these theorems imply Reisner's Theorem. We will then define local cohomology and prove the stated theorems.

The first theorem is a statement about the fine Hilbert series of the local cohomology of the face ring. The fine grading on $H^i(k[\Delta])$ is the fine grading induced by the fine grading of the face ring. It will turn out that this induced fine grading will be a $\mathbb{Z}^n$- grading rather than an $\mathbb{N}^n$-grading.

Theorem 1.33. *(Hochster)*
Let Δ be a simplicial complex with n vertices and let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then

$$\operatorname{Hilb}(H^i(k[\Delta], \lambda)) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-1-|F|}(\operatorname{lk}_\Delta F; k) \cdot \prod_{j \in F} \frac{\frac{1}{\lambda_j}}{1 - \frac{1}{\lambda_j}} \tag{1.6}$$

Our second theorem tells us that local cohomology can 'detect' when an algebra is CM.

Theorem 1.34. *(Grothendieck)*
Let R be a finitely generated k-algebra. Then $H^i(R)$ is equal to zero when $i < \operatorname{Depth}(R)$ and when $i > \dim(R)$, while $H^i(R)$ is not zero for $i = \operatorname{Depth}(R)$ and $i = \dim(R)$.

Assuming these two theorems we are now ready to prove Reisner's theorem for any simplicial complex Δ .

We begin with the reverse implication. Let Δ be a $(d-1)$ dimensional simplicial complex. Then for $F \in \Delta$ and $i < \dim \operatorname{lk}_\Delta F$ we have $\tilde{H}_i(\operatorname{lk}_\Delta F; k) = 0$. We first show that this implies that Δ is pure.

Assume that Δ is not pure. Let K_1 be the sub-complex of Δ consisting of all of the maximal dimension faces of Δ and all sub-faces of these faces. Let K_2 be the sub-complex of Δ consisting all of the maximal faces of Δ (faces that are not contained in any larger face) that do not have maximal dimension, as well as all sub-faces of these faces. Since Δ is not pure, both $K_1 \backslash K_2$ and $K_2 \backslash K_1$ are non-empty. Let G be a maximal face of $K_1 \cap K_2$ (if $K_1 \cap K_2 = \emptyset$, then $G = \emptyset$).

Now consider $\operatorname{lk}_\Delta(G)$. Since G is maximal in $K_1 \cap K_2$ we can write $\operatorname{lk}_\Delta(G)$ as a disjoint union $\operatorname{lk}_{K_1}(G) \sqcup \operatorname{lk}_{K_2}(G)$. We also know that $\dim \operatorname{lk}_\Delta G = \dim \operatorname{lk}_{K_1} G = d - |G| - 1$. Since any face in G is properly contained in some maximal face of non-maximal dimension, $|G| \leq d-2$. Therefore $\dim \operatorname{lk}_\Delta \geq 1$. So by our assumption on Δ, $\tilde{H}_0(\operatorname{lk}_\Delta G; k) = 0$. However, this contradicts the fact that $\operatorname{lk}_\Delta G$ is disconnected. Therefore Δ must be pure.

For Δ pure we know that $\dim(\operatorname{lk}_\Delta F) = d - |F| - 1$, so for $i < d$ we have $\dim_k \tilde{H}_{i-|F|-1}(\operatorname{lk}_\Delta F; k) = 0$. From equation (1.6) this implies that $H^i(k[\Delta], \lambda) = 0$ for $i < d$. Then by Grothendieck's theorem we know that the depth of $k[\Delta]$ is at least d. By Theorem 1.22 we know that the Krull dimension of $k[\Delta]$ is d. Therefore $k[\Delta]$ is CM, so the simplicial complex Δ is CM.

For pure complexes, the steps in this argument can be reversed to show the forward direction of Reisner's theorem. For this direction we must also note that non-zero terms in equation (1.6) can not cancel. All that remains to be proved is that any complex that is not pure is not CM. If Δ is not pure, then as argued in the forward direction we have a face G such that $|G| \leq d-2$ and $\mathrm{lk}_\Delta G$ is not connected, so $\tilde{H}_0(\mathrm{lk}_\Delta G; k) \neq 0$. Then by Hochster's theorem, $H^{|G|+1}(k[\Delta]) \neq 0$. So by Grothendieck's theorem, we know that the depth of $k[\Delta]$ is at most $|G|+1 \leq d-1$. Since $k[\Delta]$ has Krull dimension d, Δ is not CM.

Exercise 1.8.1.

Show that a simplicial complex Δ is CM over k if and only if for all points $p \in ||\Delta||$ and every $i < \dim \Delta$, $\tilde{H}_i(||\Delta||; k) = 0 = H_i(||\Delta||, ||\Delta|| - p; k)$.

1.9 Local Cohomology

In order to define local cohomology we must first define rings of fractions and localizations. For more details on these topics, see for example section 15.4 of Dummit and Foote [4].

Definition 1.35. *Let R be a finitely generated k-algebra and let $S \subseteq R$ be a multiplicatively closed set. Then the ring of fractions $S^{-1}R$ is defined as $\{\left[\frac{r}{s}\right] : r \in R, s \in S\}$ with the equivalence relation $\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \Leftrightarrow \exists t \in S$ such that $t(r_1 s_2 - r_2 s_1) = 0$. Addition and multiplication on $S^{-1}R$ are the same as addition and multiplication of fractions.*

We will be interested in the specific case where $S = \{1, f, f^2, f^2, \ldots\}$ for some element $f \in R$. In this case we will write $S^{-1}R = R_f$ and call this ring R localized at f.

Example 1.36.

Let $R = \mathbb{Z}$ and $f = 2$. Then $R_2 = \mathbb{Z}\left[\frac{1}{2}\right]$, the set of fractions whose denominators are powers of two. If $S = \mathbb{Z} \backslash \{0\}$, then $S^{-1}\mathbb{Z} = \mathbb{Q}$.

We now return to the general case where $R = k[x_1, \ldots, x_n]/I$ for a homogeneous ideal I. For any collection of distinct indicies $F = \{i_1, \ldots, i_l\}$ in the set $\{1, \ldots, n\}$, we define $R_F := R_{x_{i_1} \cdots x_{i_l}}$, the localization of R at the element $x_{i_1} \cdots x_{i_l}$. We then define a chain complex $\mathcal{K}(R)$ by

$$\mathcal{K}(R) := 0 \xrightarrow{\delta_0} R \xrightarrow{\delta_1} \bigoplus_{1 \leq i \leq n} R_{x_i} \xrightarrow{\delta_2} \bigoplus_{1 \leq i < j \leq n} R_{x_i x_j} \xrightarrow{\delta_3} \quad \cdots \quad \xrightarrow{\delta_n} R_{x_1 \cdots x_n} \longrightarrow 0 \tag{1.7}$$

Here the maps δ_i are the natural inclusion maps with signs chosen such that $\delta_{i+1} \circ \delta_i = 0$. More specifically, we consider $\delta_l(0, \ldots, 0, r/s, 0, \ldots, 0)$, where the non-zero entry is in the summand corresponding to some R_F where $F = \{i_1, \ldots, i_l\}$ for l distinct elements in $\{1, \ldots, n\}$, and s is some monomial in

$\{x_i : i \in F\}$. For any set of distinct indicies $T = \{j_1, \ldots, j_{l+1}\}$ the image of $\delta_l(0, \ldots, 0, r/s, 0, \ldots, 0)$ in R_T is defined by

$$\delta_l(0, \ldots, 0, r/s, 0, \ldots, 0)|_{R_T} = \begin{cases} 0 & F \not\subseteq T \\ \left[\frac{r}{s}\right] \cdot \operatorname{sgn}(T \backslash F, F) & F \subseteq T \end{cases}$$

where for F', F disjoint subsets of $\{1, \ldots, n\}$ we define $\operatorname{sgn}(F', F) = (-1)^{|\{(f',f) \in F' \times F : f < f'\}|}$. We then extend δ_l by linearity. The addition of the signs will cause a cancellation of terms when δ_i and δ_{i+1} are applied successively, just as occurs for simplicial co-boundary maps.

Since $\delta_{i+1} \circ \delta_i = 0$, we can discuss the homology of the complex $\mathcal{K}(R)$. This is the local cohomology of R.

Definition 1.37. *Let $R = k[x_1, \ldots, x_n]/I$ for a homogeneous ideal I and let the δ_i be maps in the chain complex (1.7). Then the local cohomology of R is $H^i(R) := (\ker \delta_{i+1})/(\operatorname{im} \delta_i)$.*

We are now ready to prove Hochster's Theorem (Theorem 1.33) and Grothendieck's Theorem (Theorem 1.34).

Proof of Hochster's Theorem

We now specify to the case where R is the face ring of a $(d-1)$-dimensional simplicial complex Δ on $\{1, \ldots, n\}$. First we want to determine $k[\Delta]_F$, the localization of face ring at $x_{i_1} \cdots x_{i_l}$ where $F = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots n\}$. Note that the element $x_F = \prod_{i \in F} x_i$ is invertible in $k[\Delta]_F$.

If $F \notin \Delta$, then we have that $x_F = 0$ in $k[\Delta]$. Therefore we have that 0 is an invertible element of $k[\Delta]_F$. Therefore $k[\Delta]_F = 0$.

We next consider the case where $F \in \Delta$. We define the closed star of F to be $\operatorname{st} F := \{G \in \Delta : G \cup F \in \Delta\}$ (note that G and F need not be distinct in this definition). Let j be any vertex of Δ not in $\operatorname{st} F$. Then we know that $(\{j\} \cup F) \notin \Delta$, so $x_j \cdot x_F$ is zero in $k[\Delta]$. Since x_F is invertible in $k[\Delta]_F$, this implies that $x_j = 0$ in $k[\Delta]_F$. Therefore

$$k[\Delta]_F = k\left[\{x_i : x_i \in \operatorname{st} F\} \cup \{x_i^{-1} : x_i \in F\}\right] = k\left[\{x_i, x_i^{-1} : x_i \in F\} \cup \{x_i : x_i \in \operatorname{lk} F\}\right]$$

Now let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Our goal will be to determine the grade α part of each of the localizations in the complex $\mathcal{K}(R)$. Define $F := \{i : \alpha_i < 0\}$ and $G := \{i : \alpha_i > 0\}$. Then by our above work, in order for $(R_{F'})_\alpha$ to be non-zero we need to have $F' \in \Delta$, $F \subseteq F'$, and $F' \in \operatorname{st}_\Delta G$. Let S_r be the set of all $F' \subset \{1, \ldots, n\}$ such that these three conditions are satisfied and $|F'| = r$. Then we have

$$\left(\bigoplus_{1 \leq i_1 < \ldots < i_r \leq n} \left(R_{x_{i_1} \cdots x_{i_r}} \right) \right)_\alpha = \bigoplus_{F' \in S_r} (R_{F'})_\alpha$$

On the right hand side we have a vector space over k whose basis elements correspond to the elements $F' \in S_r$. If we delete F, we can also think of this as a vector space over k whose basis elements correspond to all the faces of $\operatorname{lk}_{\operatorname{st} G} F$ of size $(r - |F|)$.

Therefore, we can identify the α grade portion of the chain complex (1.7) with the usual augmented oriented simplicial co-chain complex of $\operatorname{lk}_{\operatorname{st} G} F$ except with the dimension shifted by $|F| + 1$ (because we are considering faces of $\operatorname{lk}_{\operatorname{st} G} F$ of size $(r - |F|)$, and hence of dimension $(r - (|F| + 1))$). In order to make this an isomorphism of co-chain complexes we need to choose the correct signs for the basis elements of $\mathcal{K}(R)$.

In $\mathcal{K}(R)$, when $R_{F \cup T}$ maps to $R_{F \cup T \cup v}$, the sign of the map was defined to be $\operatorname{sgn}(v, F \cup T) = \operatorname{sgn}(v, F) \cdot \operatorname{sgn}(v, T)$. In the simplicial co-chain complex, the sign of the map from C_T to $C_{T \cup v}$ is $\operatorname{sgn}(v, T)$. Therefore, we define the sign of each basis element of the α grade portion of $\mathcal{K}(R)$ to be $\operatorname{sgn}(F, T)$ where T is the element of $\operatorname{lk}_{\operatorname{st} G} F$ to which the basis element has been identified. This will give the desired isomorphism of chain complexes.

So now we have

$$H^i(k[\Delta])_\alpha \cong \tilde{H}^{i-1-|F|}(\operatorname{lk}_{\operatorname{st} G} F, k) \cong \tilde{H}_{i-1-|F|}(\operatorname{lk}_{\operatorname{st} G} F, k) \tag{1.8}$$

with the last isomorphism since we are working over a field k.

We now want to calculate $\operatorname{Hilb}(H^i(k[\Delta]), \lambda)$, the fine Hilbert series of $H^i(k[\Delta])$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$. If $G \neq \emptyset$, then $\operatorname{lk}_{\operatorname{st} G} F$ is a cone over G and therefore all of its reduced homology groups vanish. So by (1.8), only the α that contribute to the fine Hilbert series are those for which G is empty. From the definition of G, an empty G corresponds to an α all of whose entries are non-positive. Therefore we have

$$\operatorname{Hilb}(H^i(k[\Delta]), \lambda) = \sum_{\alpha \in \mathbb{Z}^n} \dim_k H^i(k[\Delta])_\alpha \cdot \lambda^\alpha = \sum_{\alpha \leq 0} \dim_k H^i(k[\Delta])_\alpha \cdot \lambda^\alpha$$

Now sorting the α by support and using (1.8) with the fact that $\operatorname{st} \emptyset = \Delta$ we have

$$\begin{aligned}\operatorname{Hilb}(H^i(k[\Delta]), \lambda) &= \sum_{F \in \Delta} \left(\sum_{\alpha \leq 0 : \operatorname{Supp}(\alpha) = F} \dim_k \tilde{H}_{i-1-|F|}(\operatorname{lk}_\Delta F, k) \cdot \lambda^\alpha \right) \\ &= \sum_{F \in \Delta} \left(\dim_k \tilde{H}_{i-1-|F|}(\operatorname{lk}_\Delta F, k) \cdot \sum_{\alpha \leq 0 : \operatorname{Supp}(\alpha) = F} \lambda^\alpha \right)\end{aligned}$$

The last sum can be simplified by the same type of power series argument we used to calculate the Hilbert Series of the face ring. This yields Hochster's theorem:

$$\operatorname{Hilb}(H^i(k[\Delta], \lambda)) = \sum_{F \in \Delta} \dim_k \tilde{H}_{i-1-|F|}(\operatorname{lk}_\Delta F; k) \cdot \prod_{j \in F} \frac{\frac{1}{\lambda_j}}{1 - \frac{1}{\lambda_j}}$$

Proof of Grothendieck's Theorem

Here we will prove a special case of Grothendieck's theorem that will be sufficient for our purposes. The specialized version has the advantage of not needing large amounts of algebraic machinery for its proof; rather we use the $\mathbb{Z}$-grading on $H^i(R)$.

Theorem 1.38. *(Special Case of Grothendieck)*
Let Δ be a simplicial complex and let $R = k[\Delta]/\theta$ for some regular sequence θ. Then $H^t(R)$ is non-zero when $t = \text{Depth}(R)$ and zero when $t < \text{Depth}(R)$.

In fact, our proof of Reisner's theorem only used the fact that $H^t(R) \neq 0$ when $t = \text{Depth}(R)$. However, we will see that in order to prove this result we will get the stronger result of Theorem 1.38 anyway.

Our proof will be by induction on the depth of R. We begin with the inductive step. Assume that $\text{Depth}(R) = t > 0$ and that the result is known for rings as above with depth less than t. Let x be non-zero-divisor of R (since $\text{Depth}(R) > 0$ we know such an x exists). Then multiplication by x gives a map from R to itself, so we get a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{K}(R) \xrightarrow{\cdot x} \mathcal{K}(R) \longrightarrow \mathcal{K}(R/x) \longrightarrow 0$$

By the zigzag lemma, this induces a long exact sequence on the homology of the chain complexes

$$\cdots \longrightarrow H^{i-1}(R/x) \longrightarrow H^i(R) \xrightarrow{\cdot x} H^i(R) \longrightarrow H^i(R/x) \longrightarrow \cdots \tag{1.9}$$

$$\cdots \longrightarrow H^{t-1}(R) \xrightarrow{\cdot x} H^{t-1}(R) \longrightarrow H^{t-1}(R/x) \longrightarrow H^t(R) \longrightarrow \cdots$$

Since R/x has depth $t-1$, by the inductive hypothesis we know that $H^{t-1}(R/x) \neq 0$ while $H^i(R/x) = 0$ for $i < t-1$. Therefore, if we can show that $H^{t-1}(R) = 0$, then by the exactness of (1.9) we will have that $H^t(R) \neq 0$. In fact, we will show that all of the $H^i(R) = 0$ for $i < t$, thereby completing the inductive step.

From the inductive hypothesis and (1.9) we know that all of the multiplication by x maps from $H^i(R)$ to itself are injective for $i < t$. We will show that if $H^i(R)$ is not zero then multiplication by x must have a non-zero kernel, which implies the desired result.

Using Hochster's theorem, we can obtain the Hilbert series for the $\mathbb{Z}$ grading of $H^i(k[\Delta])$ by setting all of the λ_j to be equal to one single t in equation (1.6). From this we see that the Hilbert series only has non-positive powers t, so that $(H^i(k[\Delta]))_j$ is zero for $j > 0$.

We want to show that $(H^i(R))_j$ is also zero for sufficiently large j. We obtain R from $k[\Delta]$ by taking the quotient of $k[\Delta]$ by a regular sequence. Therefore, we will show that if S is a ring with $(H^i(S))_j = 0$

for $j > l$ (where l is some fixed integer) and x is a non-zero-divisor of S, then $(H^i(S/x))_j = 0$ for $j > l+1$. This will imply that $(H^i(R))_j$ is zero for $j > \text{Depth}\, k[\Delta] - \text{Depth}\, R$.

So let S be a ring with $(H^i(S))_j = 0$ for $j > l$ and let x be a non-zero-divisor of S. Then we have a long exact sequence for S and x as in (1.9). Note that multiplication by x adds one to the grade of any homogeneous element, while the quotient map is grade preserving. Then from the zigzag lemma, the connecting homomorphism will reduce any homogeneous element's grade by one.

Consider the case $j > l+1$. Fix any $i \in \mathbb{N}$. Then $(H^i(S))_j = 0$ by assumption, so the connecting homomorphism from $(H^i(S/x))_j$ to $(H^{i+1}(S))_{j-1}$ must be an injection. Also by assumption, $(H^{i+1}(S))_{j-1} = 0$. Therefore we must have $(H^i(S/x))_j = 0$, which proves our claim. So we know that $(H^i(R))_j$ is zero for $j > \text{Depth}\, k[\Delta] - \text{Depth}\, R$.

Now fix any $i < t$ and assume that $H^i(R)$ is not zero. From the previous claim we know that there exists some j such that $(H^i(R))_j \neq 0$ while $(H^i(R))_{j+1} = 0$. Therefore $(H^i(R))_j$ must be in the kernel of the multiplication by x map. By the previous argument, this completes the inductive step.

It remains to prove the base case of Grothendieck's theorem. The base case will follow from the following theorem (Corollary 3.2 in Eisenbud [6]).

Theorem 1.39. *Let R be a Noetherian Ring, let $M \neq 0$ be a finitely generated R-module, and let I be an ideal of R that contains only zero-divisors of M. Then there exists $m \in M$, $m \neq 0$, such that $mI = 0$.*

In our case, we will let $M = R$ and I be the maximal ideal $(x_1, \ldots, x_n)$. Since we are in the base case where R has depth zero, all of the elements of I must be zero divisors. So the theorem gives us an element m such that $mI = 0$. Then m must be zero in R_{x_i} for every i. Therefore $m \in \ker \delta_1$, so $m \in H^0(R)$, and $H^0(R)$ is non-zero, as desired.

Theorem 1.39 follows from the two following results. For the proofs of these results, see Eisenbud [6].

Lemma 1.40. *(Prime Avoidance, [6] Lemma 3.3)*
Let $I_1, \ldots, I_n, J$ be ideals of a ring R with I_i prime for all i and $J \not\subseteq I_i$ for all i. Then there exists an element $a \in J$ such that $a \notin \cup_{i=1}^n I_i$.

Definition 1.41. *Let R be a ring and M be an R-module. The set of prime ideals associated to M is the set of prime ideals P of R such that there exists a non-zero element $m \in M$ such that the annihilator of m is P. We write* Ass(M) *for the set of prime ideals associated to M.*

Lemma 1.42. *([6] Theorem 3.1)*
Let R be a Noetherian Ring and let $M \neq 0$ be a finitely generated R-module. Then $\text{Ass}(M)$ *is non-empty and finite. The union of all the prime ideals associated to M is the set of zero-divisors of M along with zero.*

Given these two results we now prove theorem 1.39. Let R be a Noetherian Ring and let $M \neq 0$ be a finitely generated R-module. Let J be an ideal of R that contains only zero-divisors of M. Then by Lemma 1.42 we can write $\text{Ass}(M) = \{I_1, \dots, I_n\}$. If $J \subseteq I_i$ for some i, then the non-zero element m whose annihilator is I_i satisfies the theorem so we are done. If $J \not\subseteq I_i$ for any i then by Lemma 1.40 we know there is an element $a \in J$ such that $a \notin \cup_{i=1}^{n} I_i$. Then this element a is not a zero-divisor, contradicting the definition of J.

1.10 The Kruskal-Katona Theorem and Macaulay's Theorem

From theorem 1.30 we know that for Δ a CM simplicial complex, $h(\Delta, t) = \text{Hilb}(k(\Delta))$. In particular, this implies that the $h_i(\Delta)$ are all non-negative. Also, from the bound on the number of monomials of degree i in h_1 variables we have

$$h_i(\Delta) \leq \left(\binom{h_1}{i}\right) = \binom{h_1 + i - 1}{i}$$

Our eventual goal will be a complete characterization of the h-vectors of CM simplicial complexes. In this section we will state the Kruskal-Katona theorem, which characterizes the f-vectors of simplicial complexes. We also state a theorem of Macaulay that characterizes the Hilbert series of standard rings and also the h-vectors of multicomplexes (which will be defined below). In order to state these theorems we need the following lemma.

Lemma 1.43. *For all integers $n, k > 0$ there exists a unique representation*

$$n = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_j}{j}$$

where $n_k > n_{k-1} > \ldots > n_j \geq j \geq 1$.

The decomposition given by this lemma is called the k-canonical representation of n.

Proof. To construct the decomposition, we first choose $n_k = \max\{n' : n \geq \binom{n'}{k}\}$. Then we choose the remaining n_{k-i} for $i > 0$ by

$$n_{k-i} = \max\left\{n' : n \geq \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_{k-i+1}}{k-i+1} + \binom{n'}{k-i}\right\} \tag{1.10}$$

until we get exact equality (if we reach n_1 then we know that we can get equality since the last binomial coefficient in 1.10 will be $\binom{n'}{1}$). To see that the n_i are decreasing as i decreases, note that if $n_l \leq n_{l-1}$ then we would have

$$\binom{n_l+1}{l} = \binom{n_l}{l} + \binom{n_l}{l-1} \leq \binom{n_l}{l} + \binom{n_{l-1}}{l-1}$$

Therefore we did not choose n_l to be the maximum in (1.10).

Next we prove uniqueness. Assume that we have a second decomposition

$$n = \binom{m_k}{k} + \binom{m_{k-1}}{k-1} + \cdots + \binom{m_j}{j}$$

where $m_k > m_{k-1} > \ldots > m_j \geq j \geq 1$. Let l be the largest index such that $n_l \neq m_l$. First note that since binomial coefficients are non-negative, each of our choices of n_i is the maximum possible (given the choices of the n_l with $l > i$). Therefore we must have $m_l < n_l$. Using this along with the fact that the m_i are decreasing as i decreases we have

$$\begin{aligned} n &= \binom{m_k}{k} + \cdots + \binom{m_{i+1}}{i+1} + \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j} \\ &\leq \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i - 1}{i} + \binom{n_i - 2}{i-1} + \cdots + \binom{n_i - i - 1 + j}{j} \end{aligned}$$

Then possibly adding additional non-negative terms to the right hand side we get

$$n \leq \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i - 1}{i} + \binom{n_i - 2}{i-1} + \cdots + \binom{n_i - i}{1}$$

Now using a standard binomial coefficient identity (see for example p. 44 of [10]) we have

$$n < \binom{n_k}{k} + \cdots + \binom{n_{i+1}}{i+1} + \binom{n_i}{i} \leq n$$

which is a contradiction. ☐

With this lemma we can now make the following pair of definitions.

Definition 1.44. *Let $n, k > 0$ and let the k-canonical representation of n be given by*

$$n = \binom{n_k}{k} + \cdots + \binom{n_j}{j}$$

Then we define

$$\begin{aligned} n^{(k)} &:= \binom{n_k}{k+1} + \cdots + \binom{n_j}{j+1} \\ n^{<k>} &:= \binom{n_k+1}{k+1} + \cdots + \binom{n_j+1}{j+1} \end{aligned}$$

We let $0^{(0)} = 0^{<0>} = 0$.

With this definition we are now ready to state the Kruskal-Katona Theorem

Theorem 1.45. *(Kruskal-Katona)*
$(f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{N}^{d+1}$ *is the* f*-vector of a simplicial complex if and only if* $f_{-1} = 1$ *and* $0 \leq f_{i+1} \leq f_i^{(i+1)}$ *for* $0 \leq i \leq d-2$.

Before we can state Macaulay's theorem we need the idea of a multicomplex.

Definition 1.46. *A multicomplex* Γ *on* $\{x_1, \ldots, x_n\}$ *is a collection of monomials in the* x_i *such that when* $m \in \Gamma$ *every divisor of* m *is also in* Γ. *Define the* h*-vector of* Γ *by* $h(\Gamma) = (h_0, h_1, \ldots)$ *where* $h_i = |\{m \in \Gamma : \deg m = i\}|$. *A sequence is called an* M*-vector if it is the* h*-vector of some non-empty multicomplex.*

Theorem 1.47. *(Macaulay)*
A sequence $(h_0, h_1, h_2, \ldots)$ *is an* M*-vector*
$\Leftrightarrow$ *it is the Hilbert series of a C.M. graded algebra* $R = R_0 \oplus R_1 \oplus \cdots$ *generated by* R_1
$\Leftrightarrow$ $h_0 = 1$ *and* $0 \leq h_{i+1} \leq h_i^{<i>}$ *for* $i \geq 1$.

Note that in both of these theorems, the arithmetic condition only requires that we check the relationship between consecutive levels of either the f-vector or the h-vector.

In order to prove the Kruskal-Katona theorem we need to introduce the ideas of compression and shifting.

1.11 Compression and Shifting

We begin this section by defining the reverse lexicographic order both on sets and on monomials (which can also be thought of as multi-sets). We first give an example of the order on 3-sets and monomials of degree 3, and then provide the formal definition.

Example 1.48.

For sets of size three in the positive integers, the reverse lex ordering begins

$$123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < \cdots$$

For monomials of degree three in the variables $x_1, x_2, \ldots$ the reverse lex ordering begins

$$x_1^3 < x_1^2x_2 < x_1x_2^2 < x_2^3 < x_1^2x_3 < x_1x_2x_3 < x_2^2x_3 < x_1x_3^2 < x_2x_3^2 < x_3^3 < x_1^2x_4 < \cdots$$

Definition 1.49. *Let* S *and* T *be* k*-sets. Then* $S <_{rl} T$ *if the maximum element of the symmetric difference of* S *and* T *is in* T, *i.e.* $\max \{S \Delta T\} \in T$.

Definition 1.50. *Let $a, b \in \mathbb{N}^k$ and let x^a and x^b be monomials of the same degree. Then $x^a <_{rl} x^b$ if $\max\{i : x_i | (\mathrm{lcm}(x^a, x^b)/\gcd(x^a, x^b))\}$ satisfies $x_i | (x^b/\gcd(x^a, x^b))$.*

Using reverse lexicographic ordering, we can now make the following definitions.

Definition 1.51. *Let Δ be a simplicial complex. Define the compression of Δ by*

$$C(\Delta) := \cup_{i \geq -1}\{ \text{ first } f_i(\Delta) \ (i+1)\text{-sets with respect to the reverse lex ordering } \}$$

Similarly, if Γ is a multicomplex, or equivalently an order ideal of monomials, define the compression of Γ by

$$C(\Gamma) := \cup_{i \geq 0}\{ \text{ first } h_i(\Gamma) \text{ monomials of degree } i \text{ with respect to the reverse lex ordering } \}$$

The key to the idea of compression is that $C(\Delta)$ is also a simplicial complex, and $C(\Gamma)$ is also a multicomplex. Our next goal will be to prove this fact for any simplicial complex Δ. To do this we will use the idea of combinatorial shifting. We start by defining a partial order of k-sets.

Definition 1.52. *Let $S = \{s_1 < s_2 < \ldots < s_k\}$ and $T = \{t_1 < t_2 < \ldots < t_k\}$ Then we say $S \leq_p T$ if $s_i \leq t_i$ for $1 \leq i \leq k$.*

Note that the reverse lexicographic order is an extension of this partial order, as is the usual lexicographic order. Using this partial order we now define what it means for a collection of sets to be shifted.

Definition 1.53. *A collection of k-sets $F \subseteq \binom{[n]}{k}$ is shifted if $S <_p T \in F$ implies $S \in F$.*

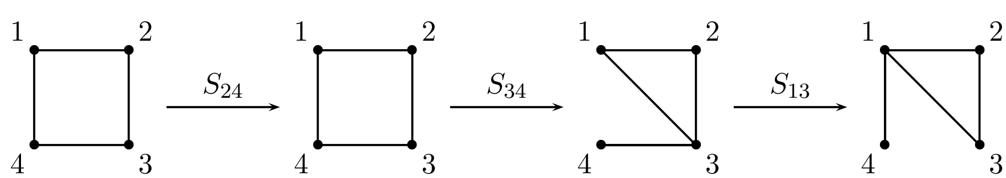
Now let Δ be simplicial complex with vertex set $[n]$. Fix $i, j \in [n]$ with $i < j$. For any face $F \in \Delta$ we define the shift $S_{ij}(F)$ by

$$S_{ij}(F) = \begin{cases} (F \backslash \{j\}) \cup \{i\} & \text{if } j \in F,\ i \notin F, \text{ and } (F \backslash \{j\}) \cup \{i\} \notin \Delta \\ F & \text{otherwise} \end{cases}$$

Then we define the shift of the entire simplicial complex Δ by $S_{ij}(\Delta) = \cup_{F \in \Delta} S_{ij}(F)$.

Example 1.54.

The diagram below shows a series of shifts applied to a simplicial complex.



Note that this final complex is shifted. Any shift applied to this final complex will result in no change.

In general, given any simplicial complex Δ we can perform a series of shifts to get to a shifted collection of sets $C(\Delta)$. So in order to prove that $C(\Delta)$ is a simplicial complex, it is enough to show that any shift $S_{ij}(\Delta)$ of a simplicial complex Δ is a new simplicial complex. In particular, we need to show that if $F' \notin S_{ij}(\Delta)$ and $T' \supseteq F'$ then $T' \notin S_{ij}(\Delta)$. This argument is left as an exercise.

We note that the process of applying S_{ij} does not alter the f-vector of our simplicial complex. Therefore we will be able to reduce the Kruskal-Katona theorem to the case of shifted complexes.

Proof of Kruskal-Katona Theorem

For the reverse direction, given a sequence $(f_{-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{N}^{d+1}$ such that $f_{-1} = 1$ and $0 \leq f_{i+1} \leq f_i^{(i+1)}$ for $0 \leq i \leq d-2$ we need to create a simplicial complex with this f-vector. We define Δ on $[f_0]$ to be the first f_i $(i+1)$-subsets of $[f_0]$ for $i = -1, 0, 1, \ldots, d-1$ using the reverse lex ordering. One can then check that the arithmetic condition on the f_i ensures that Δ is a simplicial complex. We leave this check as an exercise.

For the forward direction, we need to show that the f-vector of any simplicial complex satisfies the arithmetic conditions of the theorem. Since we have shown that for any simplicial complex there is a shifted simplicial complex with the same f-vector, it is enough to prove this direction for shifted simplicial complexes.

Given the standard notation for the k-canonical representation of n, we define

$$\partial_k(n) = \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \cdots + \binom{n_j}{j-1}$$

Then to prove Kruskal-Katona we must show that for any shifted simplicial complex $\partial_{i+1}(f_i) \leq f_{i-1}$ for $1 \leq i \leq d-1$.

Define $\Delta_i := \{$ dimension i sets in Δ $\}$. Define the shadow of a face F by $\partial F := \{T \subseteq F : |T| = |F| - 1\}$. Then we define the shadow of Δ_i by $\partial\Delta_i := \cup_{F \in \Delta_i} \partial F$. Equivalently, if $\Delta_0 = [n]$ then we have $\partial\Delta_i = \{T \in \binom{[n]}{i} : \exists S \in \Delta_i \text{ such that } T \subseteq S\}$. Define the open star of a vertex $v \in \Delta$ by $\mathring{\mathrm{st}}(v) := \{G \in \Delta : v \in G\}$ and define the anti-star of v by $\mathrm{ast}(v) := \{G \in \Delta : v \notin G\} = \Delta \backslash \mathring{\mathrm{st}}(v)$.

What we will prove is that for shifted simplicial complexes, $|\partial\Delta_i| \geq \partial_{i+1}(f_i)$ for $1 \leq i \leq d-1$. For any simplicial complex Δ we know that $\partial\Delta_i \subseteq \Delta_{i-1}$, so $|\Delta_{i-1}| \geq |\partial\Delta_i|$. Therefore the claim will prove the Kruskal-Katona theorem. We will prove the claim by induction on the number of vertices in the simplex. Note that for the base case of this induction, a one vertex simplicial complex, the claim is trivial. We also note that in the case $i = 0$, for any non-empty simplicial complex, $\partial_1(f_0) \leq 1 = f_{-1}$, and for the empty simplex $\partial_1(f_0) = 0 = f_{-1}$, so our claim is valid.

Let the vertex set of Δ be $[n]$. Then we have $\Delta_i = \mathring{\mathrm{st}}(\{1\})_i \sqcup \mathrm{ast}(\{1\})_i$. So when we take the shadow of Δ_i we get

$$|\partial\Delta_i| \geq |\partial(\mathrm{lk}(\{1\}))_{i-1}| + \max\{|\,\mathrm{lk}(\{1\}_{i-1})|, |\partial(\mathrm{ast}(\{1\})_i)|\} \tag{1.11}$$

The first term on the right hand side corresponds to all of the elements of $\partial\Delta_i$ which contain $\{1\}$; these are in one-to-one correspondence with shadow of the dimension $i-1$ elements of the of link of $\{1\}$. The second term is the maximum of two different subsets of the elements of $\partial\Delta_i$ that do not contain $\{1\}$.

Now consider the case where Δ is shifted. Let $S \in \partial(\mathrm{ast}(\{1\})_i)$. Then there exists a $T \in \mathrm{ast}(\{1\})_i \subseteq \Delta$ such that $T = S \cup v$ for some vertex v. Since Δ is shifted we have $S \cup \{1\} \in \Delta$. Since $1 \notin S$, $S \cup \{1\}$ has dimension i. Therefore $S \in \mathrm{lk}(\{1\})_{i-1}$. So we have shown that $\partial(\mathrm{ast}(\{1\})_i) \subseteq \mathrm{lk}(\{1\})_{i-1}$. So in the shifted case, (1.11) reduces to

$$|\partial\Delta_i| \geq |\partial(\mathrm{lk}(\{1\}))_{i-1}| + |\,\mathrm{lk}(\{1\})_{i-1}| \tag{1.12}$$

Now let the $i+1$ canonical representation of $f_i = |\Delta_i|$ be

$$f_i = |\Delta_i| = \binom{n_{i+1}}{i+1} + \cdots + \binom{n_j}{j}$$

We claim that in the shifted case we have the inequality

$$|\,\mathrm{lk}(\{1\})_{i-1}| \geq \binom{n_{i+1}-1}{i} + \cdots + \binom{n_j-1}{j-1} \tag{1.13}$$

We prove this claim by contradiction, so assume (1.13) is not true. We know that $|\,\mathrm{ast}(\{1\})_i| = |\Delta_i| - |\mathring{\mathrm{st}}(\{1\})_i| = |\Delta_i| - |\,\mathrm{lk}(\{1\})_{i-1}|$. Therefore we have

$$|\,\mathrm{ast}(\{1\})_i| > \left[\binom{n_{i+1}}{i+1} + \cdots + \binom{n_j}{j}\right] - \left[\binom{n_{i+1}-1}{i} + \cdots + \binom{n_j-1}{j-1}\right]$$

Using the basic identity for binomial coefficients we get

$$|\,\mathrm{ast}(\{1\})_i| > \binom{n_{i+1}-1}{i+1} + \cdots + \binom{n_j-1}{j} \tag{1.14}$$

Note that $\mathrm{ast}(\{1\})$ is a simplicial complex on fewer vertices than Δ. So combining our inductive hypothesis applied to $\mathrm{ast}(\{1\})$ with (1.14) yields

$$|\partial\,\mathrm{ast}(\{1\})_i| \geq \partial_{i+1}(|\,\mathrm{ast}(\{1\})_i|) > \binom{n_{i+1}-1}{i} + \cdots + \binom{n_j-1}{j-1} > |\,\mathrm{lk}(\{1\})_{i-1}| \tag{1.15}$$

with the last inequality coming from the assumption we aim to prove false. However, we have already shown that for a shifted complex $\partial\,\mathrm{ast}(\{1\})_i \subseteq \mathrm{lk}(\{1\})_{i-1}$, which contradicts (1.15). Therefore the inequality (1.13) must be valid.

We now complete our claim. From (1.12) and the inductive hypothesis applied to lk({1}) (this is where we may use the $i = 0$ case) we have

$$|\partial\Delta_i| \geq |\partial(\mathrm{lk}(\{1\}))_{i-1}| + |\,\mathrm{lk}(\{1\})_{i-1}| \geq \partial_i(|\,\mathrm{lk}(\{1\})_{i-1}|) + |\,\mathrm{lk}(\{1\})_{i-1}|$$

Then using (1.13) we have

$$|\partial\Delta_i| \geq \left[\binom{n_{i+1}-1}{i-1} + \cdots + \binom{n_j - 1}{j-2}\right] + \left[\binom{n_{i+1}-1}{i} + \cdots + \binom{n_j-1}{j-1}\right]$$

Applying the binomial coefficient identity we have

$$|\partial\Delta_i| \geq \binom{n_{i+1}}{i} + \cdots + \binom{n_j}{j-1} = \partial_{i+1}(f_i)$$

This completes the desired claim.

Both the Kruskal-Katona theorem and Macaulay's theorem are generalized by the following result of Clements and Lindstrom [5].

Theorem 1.55. *(Clements and Lindstrom)*
Let $1 \leq k_1 \leq k_2 \leq \ldots \leq k_n \leq \infty$. Let $F := \{x^a : 0 \leq a_i \leq k_i$ for $1 \leq i \leq n\}$.
For $H \subseteq F$ and $v \in \mathbb{N}$, let $H_v := \{x^a \in H : a_1 + a_2 + \cdots + a_n = v\}$.
Let $H_v^c :=$ { the first $|H_v|$ monomials in F_v using the reverse lex order }. Let ∂x^d be all monomials dividing x^d with degree one less than the degree of x^d. Define $\partial H := \cup_{x^d \in H} \partial x^d$.
Then for all $H \subseteq F$ and all $v \in \mathbb{N}$ we have $\partial(H_v^c) \subseteq (\partial H_v)^c$.

The case where all of the k_i are equal to one gives simplicial complexes. The case where all of the k_i are infinity gives multi-complexes.

Exercise 1.11.1.

Show that for any simplicial complex Δ on $[n]$ and any $i, j \in [n]$ with $i < j$, the collection of sets $S_{ij}(\Delta)$ is a simplicial complex.

Exercise 1.11.2.

Prove the reverse direction of the Kruskal-Katona Theorem.

1.12 h-vectors of Cohen-Macaulay Complexes

We now return to the question of characterizing the h-vectors of Cohen-Macaulay complexes. The following theorem provides a characterization of the h-vectors of both Cohen-Macaulay complexes and shellable complexes, and also shows that all shellable complexes are Cohen-Macaulay.

Theorem 1.56. *(Stanley)*

1. *If Δ is a CM simplicial complex then $h(\Delta)$ is an M-vector (see definition 1.46).*

2. *If h is an M-vector, then there exists a shellable complex Δ such that $h(\Delta) = h$.*

3. *If Δ is shellable, then Δ is CM.*

4. *If Δ is a $(d-1)$ dimensional shellable complex with shelling order $F_1, \dots, F_t$, $r(F_i)$ is the restriction face of F_i, and θ is a l.s.o.p. for $k[\Delta]$, then $\{x^{r(F_i)}\}_{i=1}^{t}$ is a basis for $k(\Delta) = k[\Delta]/(\theta)$.*

Proof. We leave the proof of statement 2 as an exercise. First we prove statement 1. Let Δ be a CM simplicial complex and let θ be a l.s.o.p. for $k[\Delta]$. Then we know by theorem 1.30 that $\mathrm{Hilb}(k(\Delta)) = h(\Delta, t)$. So we need to find a multicomplex whose h-vector is equal to $\mathrm{Hilb}(k(\Delta))$.

We know that the quotient $k(\Delta)$ is generated by degree one elements, say $y_1, \dots, y_k$. Therefore the set of all monomials in the y_i spans $k(\Delta)$. For each $i \geq 0$, order all of the monomials in the y_i using the reverse lex order. Then pick a basis of $k(\Delta)_i$ from these monomials greedily, i.e. add an element to the basis if and only if it is not a linear combination of elements before it in the reverse lex order. Call this basis B. We claim that B is a multicomplex. Once we prove this claim, since the h-vector of B is equal to $\mathrm{Hilb}(k(\Delta))$ we will have completed the proof of claim 1.

So assume that $y^a \notin B$ for some $a \in \mathbb{N}^k$. We need to show that $y_i \cdot y^a \notin B$ for $1 \leq i \leq k$. Since $y^a \notin B$ we can write it as a linear combination of smaller monomials in the reverse lex order

$$y^a = \sum_{y^b <_{rl} y^a} \alpha_b y^b$$

Then multiplying by any variable y_i we have

$$y_i \cdot y^a = \sum_{y^b <_{rl} y^a} \alpha_b \cdot y_i \cdot y^b \tag{1.16}$$

Since $y^b <_{rl} y^a$ implies $y_i \cdot y^b <_{rl} y_i \cdot y^a$, equation (1.16) shows that $y_i \cdot y^a$ is a linear combination of elements smaller than $y_i \cdot y^a$, so $y_i \cdot y^a \notin B$, as desired.

We now prove statements 3 and 4. Statement 3 can be proved using simplicial homology and Reisner's theorem with the use of Mayer-Vietoris sequences. Statement 3 can also be proved using Mayer-Vietoris on the level of rings. Here we will provide a more combinatorial proof. We will prove statements 3 and 4 together.

Let $(\theta_1, \dots, \theta_d)$ be a l.s.o.p. of $k[\Delta]$. We know that products of basis elements of $k(\Delta)$ and $k[\theta_1, \dots, \theta_d]$ span $k[\Delta]$. Therefore we have the term by term inequality

$$\mathrm{Hilb}\, k[\Delta] \leq \mathrm{Hilb}(k(\Delta)) \cdot \mathrm{Hilb}(k[\theta_1, \dots, \theta_d]) \tag{1.17}$$

We have equality if and only if $k[\Delta]$ is a free module over $k[\theta_1, \ldots, \theta_d]$, i.e. if and only if Δ is C.M.

From the definition of the h-vector and our previous calculation of the Hilbert series of $k[\theta_1, \ldots, \theta_d]$ we can rewrite (1.17) as

$$\frac{h(\Delta, t)}{(1-t)^d} \leq \frac{1}{(1-t)^d} \cdot \mathrm{Hilb}(k(\Delta)) \qquad \Rightarrow \qquad h(\Delta, t) \leq \mathrm{Hilb}(k(\Delta))$$

Since Δ is shellable, from Theorem 1.18 we know

$$h_i(\Delta) = |\{F \in \Delta_{d-1} : |r(F)| = i\}|$$

Combining these last two results, if we can show that the set $B := \{x^{r(F)} : F \in \Delta_{d-1}\}$ spans $k(\Delta)$, then B must be a basis for $k(\Delta)$ and we must have equality in (1.17) and therefore Δ must be C.M. Therefore statement 3 and 4 both follow from showing that B spans $k(\Delta)$.

To prove that B spans $k(\Delta)$ we will use induction on t. In the base case $t = 1$ we have just a simplex. Therefore $k[\Delta] = k[x_1, \ldots, x_d]$. Taking $x_1, \ldots, x_d$ as a l.s.o.p. for $k[\Delta]$ we see that $k(\Delta) \cong k$. In this case the restriction face of the one facet in the shelling is the empty set, so $x^{r(F)} = 1$, which spans k.

We now prove the inductive step. We define the socle of $k(\Delta)$ by $\mathrm{soc}(k(\Delta)) := \{s \in k(\Delta) : (x_1, \ldots, x_n)s = 0\}$. We claim that $x^{r(F_t)} \in \mathrm{soc}(k(\Delta))$. Note that it is enough to show that $x_i x^{r(F_t)} = 0$ for $1 \leq i \leq n$.

We first consider the case where $i \notin F_t$. Then since $r(F_t)$ is a new face when F_t is added to complex, $r(F_t) \cup \{i\}$ can not be a face in Δ. Therefore $x_i x^{r(F_t)} = 0$ in $k[\Delta]$ and hence also in $k(\Delta)$.

Now we consider the case $i \in F_t$. Let $(\theta_1, \ldots, \theta_d)$ be our l.s.o.p. for $k[\Delta]$. Then using Gaussian elimination, since there are d elements in our l.s.o.p and $d-1$ elements in $F_t \backslash \{i\}$ we have a linear combination

$$\sum_{l=1}^{d} \beta_j \theta_j = x_i + \sum_{j \notin F_t} \alpha_j x_j$$

Therefore we have

$$x_i x^{r(F_t)} = \left(\sum_{l=1}^{d} \beta_j \theta_j - \sum_{j \notin F_t} \alpha_j x_j \right) \cdot x^{r(F_t)} \tag{1.18}$$

From the previous case we know that $x_j \cdot x^{r(F_t)}$ is zero in $k[\Delta]$ and $k(\Delta)$ for $j \notin F_t$. We also know that all of the θ_j are zero in $k(\Delta)$. Therefore the entire right-hand side of equation (1.18) is zero, so $x_i x^{r(F_t)} = 0$ in this case. Therefore we have $x^{r(F_t)} \in \mathrm{soc}(k(\Delta))$.

Now define $S' = k(\Delta)/(x^{r(F_t)})$. Also define $\Delta' = \cup_{i=1}^{t-1} 2^{F_i}$. Then since $r(F_t)$ is the unique minimal new face when F_t is added to Δ' we have $k[\Delta'] = k[\Delta]/(x^{r(F_t)})$. So if θ is a l.s.o.p. for $k[\Delta]$ then it is also a l.s.o.p. for $k[\Delta']$. Therefore we have $S' = k[\Delta']/(\theta)$.

So by the inductive hypothesis we know that $\{x^{r(F_i)}\}_{i=1}^{t-1}$ spans S'. Then since $S' = k(\Delta)/(x^{r(F_t)})$ and $x^{r(F_t)} \in \mathrm{soc}(k(\Delta))$ we know that $\{x^{r(F_i)}\}_{i=1}^{t}$ spans $k(\Delta)$, completing the inductive step.

□

Exercise 1.12.1.

Show that if h is an M-vector, then there exists a shellable complex Δ such that $h(\Delta) = h$.

1.13 Shellable, Partitionable, and Cohen-Macaulay Complexes

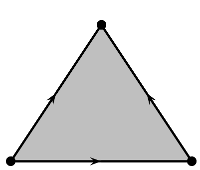
Our goal in this section is to examine the relationships between shellable, partitionable, and Cohen-Macaulay complexes. In the section on shellability we showed that every shellable complex is partitionable. In the previous section we showed that every shellable complex is CM. It turns out that both of these inclusions are strict.

As an example of a complex that is CM but not shellable consider the dunce cap. It is a standard algebraic topology question to show that this space is contractible. Let Δ be a triangulation of the dunce cap. Using the topological version of Reisner's theorem (Exercise 1.8.1) we know that Δ is C.M. Also we know that $\tilde{\chi}(\Delta) = 0$. As we noted at the end of section 1.3, this implies that $h_3(\Delta) = 0$.

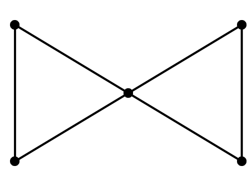
So assume that Δ were shellable, with shelling order $F_1, \ldots, F_t$. Since every edge of Δ is in at least two facets, when we add in the final facet F_t of the shelling, all of the faces in the boundary of F_t must already have been added to the complex. Therefore $r(F_t) = F_t$, which means that $h_3(\Delta) \geq 1$, contradicting our remark above.

We can also find partitionable complexes that are not shellable. An example is shown in the second complex in the figure.

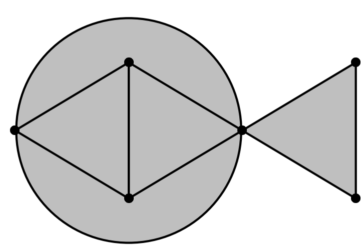
This leaves the question about the relationship between partitionable complexes and CM complexes. The third complex in the figure, due to Björner, is an example of a partitionable complex that is not CM.



Dunce Cap



Partitionable
Not Shellable



Partitionable
Not CM

The remaining implication, if CM complexes are partitionable, remains an open question, though Stanley has conjectured that all CM complexes are partitionable.

1.14 The g-theorem

In this section we state the g-theorem, which gives a complete characterization of the f-vectors of simplicial polytopes. The characterization is in terms of the g-vector of a polytope, which we now define.

Definition 1.57. *Let Δ be a triangulation of the $(d-1)$ sphere. Define $g_0(\Delta) = 1$ and $g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta)$ for $1 \leq i \leq [\frac{d}{2}]$. The sequence $(g_0, g_1, \ldots, g_{[d/2]})$ is called the g-vector of Δ.*

Note that because of the Dehn-Sommerville equations, we can recover the entire h-vector of Δ from the g-vector. Therefore we can also determine the f-vector of Δ from the g-vector.

Theorem 1.58. *(g-theorem)*
A sequence g in $\mathbb{Z}^{d+1}$ is an M-sequence if and only if there exists a simplicial d-polytope P such that $g(\partial P) = g$.

The g-theorem was originally conjectured by McMullen. The forward direction of the theorem was established by a construction of Billera and Lee [2]. The reverse direction was first proved by Stanley. We will outline his proof below; more details can be found in his text [9]. The theorem was later proved using a different method by McMullen.

The main idea of Stanley's proof is to construct an isomorphism between $k(\Delta)_i$ and $k(\Delta)_{d-i}$. This isomorphism is given by multiplication by w^{d-2i} for a particular element $w \in k[x_1, \ldots, x_n]_1$. We can choose a particular l.s.o.p. $\theta = (\theta_1, \ldots, \theta_d)$ such that this map is an isomorphism, and then generically this will also be an isomorphism.

Once we have constructed this isomorphism, we note that since it is an isomorphism for all $0 \leq i \leq [\frac{d}{2}]$, the multiplication by w map from $k(\Delta)_{i-1}$ to $k(\Delta)_i$ must be injective for $1 \leq i \leq [\frac{d}{2}]$. Combining this with the fact that $\dim_k(k(\Delta))_i = h_i$ we know that for $1 \leq i \leq [\frac{d}{2}]$

$$g_i = h_i - h_{i-1} = \dim_k(k(\Delta))_i - \dim_k(k(\Delta))_{i-1} = \dim_k \left(\frac{k(\Delta)}{(w)}\right)_i$$

So applying Macaulay's theorem to the ring $k[\Delta]/(w)$ we have that the g-vector is an M-sequence.

The construction of the isomorphism relies on the hard Lefschetz theorem. We will not discuss that result here. Instead we will just describe how to choose w and the the l.s.o.p. θ that is used to show that the multiplication map is an isomorphism.

First we are required to alter our polytope so that its vertices are in $\mathbb{Z}^d$. To do this, we first perturb the vertices so that they all have rational coordinates. Since we have a simplicial polytope, this can be done without changing the combinatorial type of the complex. Since we have a finite number of vertices we can than blow up our complex by a sufficiently large factor such that all of the vertices will have integer coordinates. Again, this will not alter our combinatorial type.

We then write out the coordinates of each of our vertices as $v_i = (\alpha_{i1}, \ldots, \alpha_{id})$ for $1 \leq i \leq n$. Then the elements of our l.s.o.p. are defined by $\theta_i = \sum_{j=1}^{n} \alpha_{ji} x_j$. We take w to be the sum of the variables, $w = \sum_{j=1}^{n} x_j$.

One of the major open questions in this area is if the g-theorem can be extended beyond polytopes. Classes of complexes for which the g-theorem is conjectured to hold include triangulated spheres, piecewise linear spheres, and homology spheres. In general, these conjectures are called the g-conjecture for the appropriate class of complexes. There are two versions of the g-conjecture for each class. The numerical g-conjecture is the fact that the g-vectors of complexes of that class are M-sequences. The algebraic g-conjecture is the fact that $k(\Delta)$ has the hard Lefschetz property for some l.s.o.p. Θ, i.e. the existence of an element w that induces the isomorphisms we saw in the sketch of the proof of the g-theorem.

Another class of simplexes whose g-vectors might all be M-sequences are doubly CM complexes.

Definition 1.59. *A dimension $(d-1)$ simplicial complex Δ is doubly Cohen-Macaulay (also called 2-CM) if Δ is CM and for every vertex p of Δ the sub-complex $\Delta \backslash \{p\} = \{F \in \Delta : p \notin F\}$ is a $(d-1)$ dimensional CM complex.*

Using Reisner's theorem we can see that spheres are 2-CM while balls are not.

There are some situations where parts of the g-conjecture are known to be valid. Let Δ be a $(d-1)$-dimensional sphere. Then if Δ is generically d-rigid, we know that

$$g_2 = h_2 - h_1 = f_1 - df_0 + \binom{d+1}{2} \geq 0.$$

So we at least have non-negativity of g_2. In fact, it has shown that when Δ is generically d-rigid there is an injection from $k(\Delta)_1$ to $k(\Delta)_2$ given by multiplication by a degree one element w. By the argument given above, this implies that (g_0, g_1, g_2) is an M-sequence. Some examples of classes of complexes known to be generically d-rigid include homology spheres and doubly CM complexes.

Beyond g_2 very little is known. It is not even known if g_3 is non-negative for PL-5-spheres.

If Δ is a homology sphere, or even just CM, then it is known that the g-vector of the barycentric subdivision of Δ is an M-sequence. However, it should be noted that taking the barycentric subdivision significantly alters the f-vector of a complex. For example, the g vector of a CM complex may have negative coefficients, while the g vector of its barycentric subdivision is an M-sequence.

Taking the connected sum of simplicial complexes preserves both versions of the g-conjecture. On the enumerative level, this corresponds to taking the disjoint union of two multicomplexes. Taking the join of simplicial complexes also preserves both versions of the g-conjecture.

Further, the stellar subdivision operation preserves the hard Lefschetz property. Given a simplicial complex Δ, a face $F \in \Delta$, and a point v in the interior of F, we define a new complex Δ' formed from Δ by stellar subdivision of F as follows. Faces $G \in \Delta$ that do not contain F as a face are left unchanged in Δ'. For a face $H \in \Delta$ that does contain F as a face we consider the complex B consisting of all the simplexes in ∂H that do not contain F as a face. In Δ' we replace B with the cone over B and vertex v. Note that this includes replacing the simplex F itself. For more information about stellar subdivision see for example Armstrong [1].

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