A characterization of simplicial polytopes with $g_2 = 1$

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Abstract

Kalai proved that the simplicial polytopes with $g_2 = 0$ are the stacked polytopes. We characterize the $g_2 = 1$ case.

Specifically, we prove that every simplicial d-polytope ($d \geq 4$) which is prime and with $g_2 = 1$ is combinatorially equivalent either to a join of two simplices whose dimensions add up to d (each of dimension at least 2), or to a join of a polygon with a $(d-2)$-simplex. Thus, every simplicial d-polytope ($d \geq 4$) with $g_2 = 1$ is combinatorially equivalent to a polytope obtained by stacking over a polytope as above. Moreover, the above characterization holds for any homology $(d-1)$-sphere ($d \geq 4$) with $g_2 = 1$.

1 Introduction and results

Let $f_i(K)$ denote the number of i-dimensional faces in a simplicial complex K. In particular, f_0 counts vertices and f_1 counts edges. Let $g_2(K) := f_1(K) - d\, f_0 + \binom{d+1}{2}$ where d is the maximal size of a face of K. (This notation is standard in face-vector theory, see e.g. [23] for details.) The well known Lower Bound Theorem (LBT) proved by Barnette [6, 7, 5], asserts that if K is the boundary complex of a simplicial d-polytope, or more generally a finite triangulation of a connected compact $(d-1)$-manifold without boundary, where $d \geq 3$, then $g_2(K) \geq 0$. Kalai considered several generalizations of this result, including to homology manifolds, and characterized the case of equality [14]. To state his result, define stacked polytopes: a *stacking* is the operation of adding a pyramid over a facet of a given simplicial polytope. A polytope is *stacked* if it can be obtained from a simplex by repeating the stacking operation (finitely many times, may be zero). We will make use of the following result:

Theorem 1.1. *[6, 5] and [14, Theorems 6.2 and 7.1] Let $d \geq 4$, and let K be the boundary complex of a simplicial d-polytope, or more generally a homology $(d-1)$-manifold. Then $g_2(K) \geq 0$ and equality holds iff K is combinatorially isomorphic to the boundary complex of a stacked d-polytope.*

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Kalai's proof is based on results from rigidity theory, to be discussed later; see also Gromov [12] for a proof of the nonnegativity of g_2.

A subset F of the vertices of a simplicial complex K is called a *missing face of* K if $F \notin K$ and all proper subsets of F are in K. A simplicial d-polytope is called *prime* if its boundary complex contains no missing $(d-1)$-faces and is not the simplex. Similarly, prime homology spheres are defined. In [16, Theorem 3.10] Kalai claimed that there exists a function $u(d,b)$ such that if the boundary complex K of a prime d-polytope ($d \geq 4$) satisfies $g_2(K) \leq b$ then $f_0(K) \leq u(d,b)$. We provide a counterexample (Example 1.2). First, some notation: the boundary complex of a simplicial polytope P is denoted by $\partial(P)$, or simply by ∂P. The *join* of two polytopes P, Q, denoted by $P*Q$, is defined as the convex hull of their union when P and Q are embedded in orthogonal spaces with the origin in the interior of both. Indeed, the combinatorial type of $P*Q$ is well defined: its boundary complex is $\partial P * \partial Q$, where the *join* of two simplicial complexes K, L is the collection of disjoint unions $\{A \uplus B \ : \ A \in K,\ B \in L\}$. A direct computation shows:

Example 1.2. Let C_n be a convex 2-polytope with n vertices and let σ^m be the m-simplex. Then for every $d \geq 4$ and any $n \geq 3$, $C_n * \sigma^{d-2}$ is a prime d-polytope with $g_2(\partial(C_n * \sigma^{d-2})) = 1$.

Our main result is that this example is the *only* counterexample for $b = 1$ in Kalai's [16, Theorem 3.10]:

Theorem 1.3. *Let $d \geq 4$, and let K be the boundary complex of a prime d-polytope, or more generally a homology $(d-1)$-sphere. Assume that $g_2(K) = 1$. Then K is combinatorially isomorphic to either the join of boundary complexes of two simplices whose dimensions add up to d (each simplex of dimension at least 2), or the join of the boundary complexes of a convex polygon and a $(d-2)$-simplex.*

Note that any simplicial polytope, can be (uniquely) presented as a connected sum of prime polytopes and simplices, and similarly for homology spheres, and that g_2 of a connected sum is the sum of g_2's of its components. (Recall that the *connected sum* of two disjoint simplicial complexes of equal dimension is the operation of identifying by a bijection the vertices in a facet of one with the vertices in a facet of the other, identifying the faces they form accordingly, and later deleting the identified facet. Thus, the connected sum of homology spheres is a homology sphere, by an easy Mayer-Vietoris argument and Alexander duality. For polytopes, after suitable projective transformations of each, which of course preserve their combinatorial structure, the connected sum, which is gluing along a facet of each, can be made convex too.) Thus, by Theorems 1.1 and 1.3 we conclude that

Corollary 1.4. *Let $d \geq 4$, and let K be the boundary complex of a d-polytope, or a homology $(d-1)$-sphere, with $g_2(K) = 1$. Then K is combinatorially isomorphic to the boundary complex of a polytope obtained by repeated stacking, starting from either the join of two simplices whose dimensions add up to d, or from the join of a polygon and a $(d-2)$-simplex.*

This result can be compared with Perles' characterization of polytopes with $2 \geq \mathrm{g}_1 := \mathrm{f}_0 - (d+1)$ [13, Chapter 6] and with Mani's result that triangulated spheres with $\mathrm{g}_1 \leq 2$ are polytopal [17].

The proof of Theorem 1.3 is based on rigidity theory for graphs, introduced in [1, 2]. Basic properties of polytopes are also used, and in the homology spheres case Alexander duality plays a role.

It would be interesting to know whether the following corrects the above miss-statement of Kalai:

Problem 1.5. *Let $\mathbb{P}$ be the family of prime polytopes, of dimension ≥ 4, which are not the join of a polygon with a simplex. Does there exist a function $u(d,b)$ such that for each $P \in \mathbb{P}$ with $\mathrm{g}_2(P) = b$ and $\dim(P) = d$, $\mathrm{f}_0(P) \leq u(d,b)$?*

This paper is organized as follows: in Section 2 we give the necessary background for polytopes and homology spheres, and develop the needed results in rigidity theory of graphs. In Section 3 we prove Theorem 1.3 and discuss some extensions of it and related open problems.

2 Background

Polytopes and homology spheres. For unexplained terminology we refer to textbooks on polytopes, e.g. [13, 28], and on simplicial homology, e.g. [19]. The *i-skeleton* of a simplicial complex K is $K_{\leq i} = \{F \in K : |F| \leq i+1\}$, and $K_i := \{F \in K : |F| = i+1\}$. The *link* of a face F in a K is $\mathrm{lk}(F) = \mathrm{lk}(F,K) = \{T \in K : \ T \cap F = \emptyset, \ T \cup F \in K\}$, its *closed star* is $\overline{\mathrm{st}(F)} = \overline{\mathrm{st}(F,K)} = \{T \in K : \ T \cup F \in K\}$, its *antistar* is $\mathrm{ast}(F) = \mathrm{ast}(F,K) = \{T \in K : \ T \cap F = \emptyset\}$; they are simplicial complexes as well. The (open) *star* of F is the collection of sets $\mathrm{st}(F) = \mathrm{st}(F,K) = \{T \in K : \ F \subseteq T\}$.

Note that for a vertex v in a simplicial polytope P, its vertex figure P/v satisfies $\partial(P/v) = \mathrm{lk}(v, \partial P)$.

A *homology sphere* is a simplicial complex K such that for every face F in K (including the empty set), and for every $0 \leq i$ there is an isomorphism of reduced homology groups $\tilde{H}_i(\mathrm{lk}(F,K); \mathbf{Z}) \cong \tilde{H}_i(S^{\dim(K)-|F|}; \mathbf{Z})$ where S^m denotes the m-dimensional sphere and $\mathbf{Z}$ the integers (actually any fixed coefficients ring works for Theorem 1.3). In particular, a boundary complex of a simplicial polytope is a homology sphere; however there are many non-polytopal examples of homology spheres, e.g. [15]. Alexander Duality holds for homology spheres, e.g. [19, Chapter 8, §71]:

Theorem 2.1. *(Alexander Duality) Let A be a proper nonempty subcomplex of a homology n-sphere K. Then for every k, $\tilde{H}^k(A) \cong \tilde{H}_{n-k-1}(|K| - |A|)$. (Here $\tilde{H}^k$ denotes reduced cohomology, $|K|$ a geometric realization of K, and $|A|$ is the subset of $|K|$ induced by $A \subseteq K$).*

In particular (we will use only these facts in the sequel), for A an m-homology sphere, $K - A$ is homologic to S^{n-m-1}. If $m = n-1$ then A is the common boundary of the two connected components of $K - A$.

Rigidity. The presentation here is based mainly on Kalai's [14]. Let $G = (V, E)$ be a simple graph. Let $d(a, b)$ denote Euclidian distance between points a and b in Euclidian space. A *d-embedding* is a map $f : V \to \mathbb{R}^d$. It is called *rigid* if there exists an $\varepsilon > 0$ such that if $g : V \to \mathbb{R}^d$ satisfies $d(f(v), g(v)) < \varepsilon$ for every $v \in V$ and $d(g(u), g(w)) = d(f(u), f(w))$ for every $\{u, w\} \in E$, then $d(g(u), g(w)) = d(f(u), f(w))$ for every $u, w \in V$. Loosely speaking, f is rigid if any perturbation of it which preserves the lengths of the edges is induced by an isometry of $\mathbb{R}^d$. G is called *generically d-rigid* if the set of its rigid d-embeddings is open and dense in the topological vector space of all of its d-embeddings. Given a d-embedding $f : V \to \mathbb{R}^d$, a *stress* w.r.t. f is a function $w : E \to \mathbb{R}$ such that for every vertex $v \in V$

$$\sum_{u:\{v,u\}\in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

G is called *generically d-stress free* if the set of its d-embeddings which have a unique stress ($w = 0$) is open and dense in the space of all of its d-embeddings.

Rigidity and stress freeness can be related as follows: Let $V = [n]$, and let $\mathrm{Rig}(G, f)$ be the $dn \times |E|$ matrix associated with a d-embedding f of $V(G)$ defined as follows: for its column corresponding to $\{v < u\} \in E$ put the vector $f(v) - f(u)$ (resp. $f(u) - f(v)$) at the entries of the rows corresponding to v (resp. u) and zero otherwise. G is generically d-stress free if the kernel $\mathrm{Ker}(\mathrm{Rig}(G, f)) = 0$ for a generic f (i.e. for an open dense set of embeddings). G is generically d-rigid if the images $\mathrm{Im}(\mathrm{Rig}(G, f)) = \mathrm{Im}(\mathrm{Rig}(K_V, f)$ for a generic f, where K_V is the complete graph on $V = V(G)$. The dimensions of the kernel and image of $\mathrm{Rig}(G, f)$ are independent of the generic f we choose; $\mathrm{Rig}(G, f)$ is the *rigidity matrix* of G, denoted by $\mathrm{Rig}(G, d)$ for a generic f. For the complete graph, one computes $\mathrm{rank}(\mathrm{Rig}(K_V, d)) = dn - \binom{d+1}{2}$ (see Asimov and Roth [1] for more details). In particular, if G is generically d-rigid then $\mathrm{g}_2(G)$ is the dimension of $\mathrm{Ker}(\mathrm{Rig}(G, d))$. We say that en edge $\{u, v\}$ *participates* in a stress w if $w(\{u, v\}) \neq 0$, and that a vertex v *participates* in w if there exists a vertex u such that the edge $\{u, v\}$ participates in w.

We need the following known results for the proof of Theorem 1.3:

Lemma 2.2. *(Cone Lemma [27, Teorem 5], also [26, Theorem 1.3]) Let $C(G)$ be the graph of the cone over a graph G, i.e. $C(G) = (\{u\} * G)_{\leq 1}$ where $u \notin G$. Then for every $d > 0$, $\mathrm{Ker}(\mathrm{Rig}(C(G), d + 1)) \cong \mathrm{Ker}(\mathrm{Rig}(G, d))$ as real vector spaces. Moreover, u participates in a stress of $C(G)$, provided that* $\mathrm{Ker}(\mathrm{Rig}(G, d)) \neq \{0\}$.

Remark 2.3. The 'moreover part' does not appear explicitly in [27, 26] but is clear for generic embeddings from the isomorphism constructed there.

Lemma 2.4. *(Gluing Lemma [2]) Let $G_i = (V_i, E_i)$ be generically d-rigid graphs, $i = 1, 2$, such that $G_1 \cap G_2$ has at least d vertices. Then $G_1 \cup G_2$ is generically d-rigid.*

By Cauchy's rigidity theorem, (resp. Gluck's rigidity result for triangulated 2-sphere [11]), the following holds:

Lemma 2.5. *Let G be the graph of a convex 3-polytope (or of a homology 2-sphere. They coincide). Then G is generically 3-rigid and 3-stress free.*

Using this fact as the base of induction, and the Cone and Gluing Lemmata for the induction step, Kalai [14] proved:

Lemma 2.6. *Graphs of homology $(d-1)$-spheres are generically d-rigid for $d \geq 3$.*

Lemma 2.7. *([14, Theorems 7.1 and 9.3]) Let $d \geq 4$, and let K be the boundary complex of a d-polytope, or a homology $(d-1)$-sphere. If for every vertex $v \in K$ the link $\mathrm{lk}(v)$ is the boundary of a stacked polytope, then K is the boundary of a stacked polytope.*

The following proposition seems to be new:

Proposition 2.8. *Let $d \geq 4$ and K be the boundary complex of a prime d-polytope, or a prime homology $(d-1)$-sphere. Then every vertex $u \in K$ participates in a generic d-stress of the graph of K.*

Proof. Let $u \in K$ be a vertex. If $\mathrm{lk}(u)$ is not stacked, then by Theorem 1.1 and the Cone Lemma u participates in generic d-stress of the graph of K. Similarly, if there exists an edge in $\mathrm{ast}(u) - \mathrm{lk}(u)$ whose two vertices are in $\mathrm{lk}(u)_0$, then by Lemma 2.6 and the Cone Lemma u participates in generic d-stress of the graph of K.

Thus, assume that $\mathrm{lk}(u)$ is stacked and that $\mathrm{ast}(u) - \mathrm{lk}(u)$ contains no edges with both ends in $\mathrm{lk}(u)_0$. We now show that $\mathrm{ast}(u) - \mathrm{lk}(u)$ contains a vertex and that each vertex in $\mathrm{lk}(u)_0$ is contained in a facet of $\mathrm{ast}(u)$ with a vertex in $\mathrm{ast}(u) - \mathrm{lk}(u)$. Indeed, otherwise there is a $(d-2)$-face F in $\mathrm{ast}(u)$ which is a missing face in $\mathrm{lk}(u)$. Thus, $F \cup \{u\}$ is a missing facet of K, contradicting the assumption that K is prime.

By Lemma 2.6 and the Cone and Gluing Lemmata, $\cup_{v \in (\mathrm{ast}(u) - \mathrm{lk}(u))_0} \overline{\mathrm{st}(v)}$ has a generically d-rigid graph, and by the above, so has $\mathrm{ast}(u)$. Adding the edges with u can increase the rank of the rigidity matrix of $\mathrm{ast}(u)_{\leq 1}$ by at most d. As K is prime, u has at least $d+1$ neighbors, hence the edges with u contribute to the kernel of the rigidity matrix, i.e. u participates in a generic d-stress of $K_{\leq 1}$. □

3 Proof of Theorem 1.3

In each of the following lemmata we first prove the assertion for polytopes, then indicate the needed modification for homology spheres. Theorem 1.3 is then proved for both polytopes and homology spheres in tandem, by induction on dimension. The following proposition allows the inductive step:

Proposition 3.1. *Let $d > 4$ and K be the boundary complex of a prime d-polytope, or homology $(d-1)$-sphere, with $\mathrm{g}_2(K) = 1$. Then, there exists a vertex $u \in K$ such that $\mathrm{lk}(u)$ satisfies:*
(a) $g_2(\mathrm{lk}(u)) = 1$.
(b) $\mathrm{lk}(u)_0 = K_0 - \{u\}$.
(c) $\mathrm{lk}(u)$ *is prime.*

Proof. As $g_2(K) > 0$, by Lemma 2.7 there exists a vertex $u \in K$ whose link is not stacked. By Theorem 1.1, $\mathrm{g}_2(\mathrm{lk}(u)) > 0$. By the Cone Lemma and Lemma 2.6, $\mathrm{g}_2(\mathrm{lk}(u)) = \dim \mathrm{Ker}\,\mathrm{Rig}((\overline{\mathrm{st}(u)})_{\leq 1}, d) \leq \dim \mathrm{Ker}\,\mathrm{Rig}(K_{\leq 1}, d) = \mathrm{g}_2(K) = 1$. Hence $\mathrm{g}_2(\mathrm{lk}(u)) = 1$, proving (a).

By (a) and the Cone Lemma, there is a generic d-stress in K in which only vertices in $\overline{\mathrm{st}(u)}$ participate. As $\mathrm{g}_2(K) = 1$, no other vertex in K participate in any non trivial stress. By Proposition 2.8, $K_0 = (\overline{\mathrm{st}(u)})_0$, proving (b).

Assume by contradiction that $\mathrm{lk}(u)$ is not prime, hence inserting all of its missing facets cuts $\mathrm{lk}(u)$ into at least two parts, one of which is prime and the others must be simplices (it follows from Proposition 2.8, as otherwise one gets two independent generic $(d-1)$-stresses in the graph of $\mathrm{lk}(u)$. Alternatively use the fact that $\mathrm{g}_2(L\sharp Q) = \mathrm{g}_2(L) + \mathrm{g}_2(Q)$ for a connected sum $L\sharp Q$). Hence there exists a missing facet F of $\mathrm{lk}(u)$ whose insertion cuts $\mathrm{lk}(u)$ so that the boundary of a simplex σ forms one side (for homology spheres we conclude the existence of such F by Alexander duality, the separation statement in Theorem 2.1).

Note that $F \notin K$, otherwise $F \cup \{u\}$ would be a missing facet of K. By (b), $\mathrm{ast}(u) - \mathrm{lk}(u) := \{T \in \mathrm{ast}(u) : T \notin \mathrm{lk}(u)\}$ contains no vertices, and by (a) and the fact that $\overline{\mathrm{st}(u)}$ is d-rigid, it contains no edges (otherwise $\mathrm{g}_2(K) \geq 2$, a contradiction). Thus, a $(d-1)$-face S of $\partial\sigma \cap \mathrm{lk}(u)$ is *not* contained in any d-face of $\mathrm{ast}(u)$. But such S must be contained in (exactly) one d-face of $\mathrm{ast}(u)$, a contradiction proving (c). ☐

The following two propositions establish the base of induction, namely the case $d = 4$.

Proposition 3.2. *Let K be the boundary complex of a prime 4-polytope, or a homology 3-sphere, with $g_2(K) = 1$ and with a missing triangle T. Then K is combinatorially isomorphic to the join of the boundary complexes of T and a polygon.*

Proof. Let $T = \{a, b, c\}$. First we show that $\mathrm{lk}(a)_0 = K_0 - \{a\}$. $\mathrm{lk}(a)_{\leq 1}$ is generically 3-rigid, as it is the 1-skeleton of a convex 3-polytope (clearly true for K the boundary complex of a polytope, and note that any homology 2-sphere is a topological 2-sphere and hence can be realized as the boundary of a convex 3-polytope by Steinitz' theorem [24]). This graph together with the edge $\{b, c\}$ has a generic 3-stress. By the Cone Lemma, $G := (\{a\} * (\mathrm{lk}(a) \cup \{b, c\}))_{\leq 1}$ has a generic 4-stress. G is contained in $K_{\leq 1}$. Assume by contradiction the existence of a vertex $v \in K_0 - G_0$. By Proposition 2.8, v participates in a generic 4-stress of $K_{\leq 1}$. Such a stress is independent of the former stress that we found, resulting in $g_2(K) \geq 2$, a contradiction.

By the Cone Lemma and Lemma 2.6, the graph of $\overline{\mathrm{st}(a)}$ is generically 4-rigid and 4-stress free. This graph contains K_0, hence there exists exactly one edge in $\mathrm{ast}(a) - \mathrm{lk}(a)$, which must be $\{b, c\}$.

Let C denote the cycle $\mathrm{lk}(\{b, c\}, K)$, and ΣC denote the suspension of C by b and c, namely $\Sigma C = \{b\} * C \cup \{c\} * C$. Next we show that $\mathrm{lk}(a)$ contains ΣC. We have already seen that $\mathrm{lk}(a)_{\leq 1} \supseteq (\Sigma C)_{\leq 1}$. If a triangle $F \in \Sigma C$ is not contained in $\mathrm{lk}(a)$ then $F \cup \{a\}$ is missing in K, contradicting that K is prime. Thus $\Sigma C \subseteq \mathrm{lk}(a)$, hence $L := \{a\} * \Sigma C \cup_{\Sigma C} \overline{\mathrm{st}(\{b,c\})} = C * \partial(T)$ is the boundary of a 4-polytope such that $L \subseteq K$. Note that by Alexander duality a homology d-sphere cannot *strictly* contain another homology d-sphere, hence $K = L = C * \partial(T)$ for K a homology sphere. □

Proposition 3.3. *If K is the boundary complex of a prime 4-polytope, or a homology 3-sphere, with $g_2(K) = 1$, then K has a missing triangle.*

Proof. Assume by contradiction that K has no missing triangles. As K is prime, all of its missing faces are edges (a complex with this property is called *clique complex*), and K has a missing edge, say $\{u, u'\}$. Note that for every vertex $w \in \mathrm{lk}(u)$, $\mathrm{lk}(w, \mathrm{lk}(u))$ is a cycle of length at least 4, as $\mathrm{lk}(u)$ contains no missing triangle, otherwise K would not be a clique complex. Similarly, there is a vertex in $\mathrm{ast}(w, \mathrm{lk}(u)) - \mathrm{lk}(w, \mathrm{lk}(u))$, otherwise w would form a missing triangle with the vertices of an edge in $\mathrm{ast}(w, \mathrm{lk}(u)) - \mathrm{lk}(w, \mathrm{lk}(u))$.

Let $v \in \mathrm{lk}(u)$ and $I := \mathrm{ast}(v, \mathrm{lk}(u))_0 - \mathrm{lk}(v, \mathrm{lk}(u))_0$. Then $0 < |I| \leq |\mathrm{lk}(u)_0| - 5$. Note that K contains no face of the form $\{v\} \cup F$ where $F \in \mathrm{ast}(v, \mathrm{lk}(u)) - \mathrm{lk}(v, \mathrm{lk}(u))$, as K is a clique complex. Thus $K' := (K - \mathrm{st}(u)) \cup \{v\} * \mathrm{ast}(v, \mathrm{lk}(u))$ has the same topology as K. In fact, K and K' are piecewise linear (PL for short) homeomorphic [21, Theorem 1.4] (and its proof, for the case of 3-homology spheres). If K is any homology 3-sphere then K' is a homology 3-sphere as well. To see this, recall that being a homology sphere is a topological property [20]. Alternatively, use a a Mayer-Vietoris argument.

Next, let us verify that K' is prime. Any face in $K' - K$ contains an edge $\{v, i\}$ for some $i \in I \neq \emptyset$ and is contained in $\mathrm{lk}(u, K)_0$. Together with the fact that K is prime, this implies that all the vertices of a missing tetrahedron of K' must lie in $\mathrm{lk}(u, K)_0$. However, the induced complex in K' on $\mathrm{lk}(u, K)_0$ is a cone (over v), hence contains no missing tetrahedra. In particular, K' is not stacked (clearly K' is not the 4-simplex).

On the other hand, $\mathrm{g}_2(K') = \mathrm{g}_2(K) - |\mathrm{lk}(u, K)_0| + |I| + 4 \leq 1 - 5 + 4 = 0$. This contradicts Theorem 1.1. □

Remark 3.4. Alternatively, Proposition 3.3 can be proved via the Charney-Davis conjecture [9] which was proved for homology 3-spheres by Davis and Okun [10]. It asserts that for a homology clique 3-sphere K, $\mathrm{g}_2(K) - (f_0(K) - 5) + 1 \geq 0$. In our case ($\mathrm{g}_2(K) = 1$) we get $f_0(K) \leq 7$. As K is a clique sphere, it contains at least as many vertices as the octahedral 3-sphere, e.g. [18, Theorem 1.1], i.e. 8 vertices; a contradiction.

Proof of Theorem 1.3: By Propositions 3.2 and 3.3 the assertion holds for $d = 4$. For $d > 4$, by Proposition 3.1 there exists a vertex $u \in K$ such that $K_0 = \{u\} \cup \mathrm{lk}(u)_0$ and the conditions of Theorem 1.3 hold for $\mathrm{lk}(u)$, thus by induction also the conclusion of Theorem 1.3 holds for $\mathrm{lk}(u)$. Clearly, $\mathrm{ast}(u) - \mathrm{lk}(u)$ is nonempty, and any face in $\mathrm{ast}(u) - \mathrm{lk}(u)$ must contain a missing face of $\mathrm{lk}(u)$. By the Cone Lemma, $g_2(\overline{\mathrm{st}(u)}) = 1 = g_2(K)$, hence all the edges in $\mathrm{ast}(u)$ are already in $\mathrm{lk}(u)$. Note that the missing faces in a join are the faces which are missing in one of its components.

Case 1: $\mathrm{lk}(u) = \partial\sigma * C$ for a $(d-3)$-simplex σ and a cycle C. Then $\sigma \in K$. As $\mathrm{lk}(\sigma, K)$ is a cycle and is contained in C, $\mathrm{lk}(\sigma, K) = C$ and $\mathrm{ast}(u, K) = \sigma * C$. Thus, $K = \partial(\sigma \cup \{u\}) * C$.

Case 2: $\mathrm{lk}(u) = \partial\sigma * \partial\tau$ for simplices σ and τ whose dimensions add up to $d-1$. Then $\sigma \in K$ or $\tau \in K$, and we show now that exactly one of them is in K. If $\sigma \in K$, then as $\mathrm{lk}(\sigma, K)$ is a boundary of a $(\dim\tau)$-polytope /a homology $(\dim\tau - 1)$-sphere and is contained in $\partial\tau$ we must have $\mathrm{lk}(\sigma, K) = \partial\tau$. Similarly, if $\tau \in K$ then $\mathrm{lk}(\tau, K) = \partial\sigma$, hence K strictly contains the boundary of a d-polytope, which is in particular a $(d-1)$-homology sphere, namely $\partial(\sigma * \tau) = \sigma * \partial\tau \cup \partial\sigma * \tau$; a contradiction. W.l.o.g. let us assume $\sigma \in K$. Then $K = \partial(\sigma \cup \{u\}) * \partial\tau$. □

Remark 3.5. Let $d \geq 4$ and K be a $(d-1)$-dimensional combinatorial manifold without boundary and with $g_2(K) = 1$. Then K is homeomorphic to a sphere, and hence Theorem 1.3 applies to K.

Proof. If K is not prime then either it has a connected sum decomposition $K = L\#Q$ such that $\mathrm{g}_2(L) = 1$ and $\mathrm{g}_2(Q) = 0$ or it is obtained by handle forming from another combinatorial manifold without boundary K' (i.e. by combinatorially identifying two disjoint closed facets of K' and deleting their interior). Here we used the fact that $d \geq 4$; see [3] for a proof of this fact.

In the first case, by Theorem 1.1 Q is a stacked sphere and by induction on the number of vertices L is homeomorphic to a sphere, and we are done. In the second case, $\mathrm{g}_2(K') = \mathrm{g}_2(K) - \binom{d+1}{2} < 0$ contradicting Theorem 1.1.

Assume that K is prime. If there exists another simplicial complex M which is PL-homeomorphic to K, and with smaller g_2 value, then M is a stacked sphere, hence K is a PL-sphere. Otherwise, Swartz [25] showed that K has at most $d+2$ vertices and hence K is a PL-sphere [4]. □

It is natural to ask for a characterization of (prime) simplicial polytopes with a given g_2. First, observe the following:

Observation 3.6. *Let* g *be the g-vector of a simplicial d-polytope with* $d \geq 4$ *and* $\mathrm{g}_2 > 0$. *Then there exists a prime d-polytope whose g-vector agrees with* g *except maybe in the* g_1 *entry.*

Proof. The connected sum $L\#Q$ of two d-polytopes L and Q satisfies $\mathrm{g}_2(L\#Q) = \mathrm{g}_2(L) + \mathrm{g}_2(Q)$ and $\mathrm{g}_1(L\#Q) = \mathrm{g}_1(L) + \mathrm{g}_1(Q) + 1$ (where $\mathrm{g}_1(L) := f_0(L) - d - 1$). There exists a unique positive integer c such that $\binom{c}{2} < \mathrm{g}_2 \leq \binom{c+1}{2}$. By the sufficiency part of the g-theorem [8] there exists a simplicial polytope P with $\mathrm{g}_1(P) = c$ and $\mathrm{g}_i(P) = \mathrm{g}_i$ for any $2 \leq i$. By the necessity part of the g-theorem [22] any simplicial polytope P with $g_2(P) = \mathrm{g}_2$ satisfies $\mathrm{g}_1(P) \geq c$. In particular, the minimality of c implies that if $P = L\#Q$ then non of L, Q is a simplex, hence w.l.o.g. $\mathrm{g}_1(L), \mathrm{g}_1(Q) > 0$. The necessity part again implies $\mathrm{g}_2(L) \leq \binom{\mathrm{g}_1(L)+1}{2}$ and $\mathrm{g}_2(Q) \leq \binom{\mathrm{g}_1(Q)+1}{2}$, hence $\mathrm{g}_2(P) = \mathrm{g}_2(L) + \mathrm{g}_2(Q) < \binom{\mathrm{g}_1(L)+\mathrm{g}_1(Q)+1}{2} = \binom{c}{2}$, a contradiction. □

Problem 3.7. *Characterize the prime polytopes with* $\mathrm{g}_2 = 2$.

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