Topological Methods in Combinatorics

Margarita Amchislavska

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Contents

Chapter 1

Applications of Borsuk-Ulam Theorem

1.1 Ham-Sandwich Theorem

The name for this theorem comes from the following problem: given any sandwich composed of bread, ham, and cheese, there is a plane that cuts the sandwich into two pieces such that both pieces contain equal amounts of each component. Let's remind ourselves that a finite Borel measure μ on $\mathbb{R}^d$ is a measure such that $0 < \mu(\mathbb{R}^d) < \infty$ and all open sets in $\mathbb{R}^d$ are measurable. The statement of the theorem is as follows.

Theorem 1.1 (Ham-Sandwich Theorem)**.** *Let $\mu_1, \ldots, \mu_d$ be finite Borel measures in $\mathbb{R}^d$ such that $\forall i$ and $\forall$ hyperplanes $H \subseteq \mathbb{R}^d$, $\mu_i(H) = 0$. Then $\exists$ hyperplane $H \subseteq \mathbb{R}^d$ such that $\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}\mu_i(\mathbb{R}^d)$ for every $1 \leq i \leq d$.*

In the theorem above H^+ and H^- denote the halfspaces corresponding to the hyperplane H.

Proof. Let $\mathbf{u} = (u_0, u_1, \ldots, u_d)$ be a point on a d-dimensional sphere S^d. For each such point $\mathbf{u}$ consider the corresponding hyperplanes defined by $H_u^+(\mathbf{u}) = \{x \in \mathbb{R}^d : \sum_1^d u_i x_i \leq u_0\}$ and $H_u^-(\mathbf{u}) = \{x \in \mathbb{R}^d : \sum_1^d u_i x_i \geq u_0\}$. Clearly, $H_{-u}^+ = (H_u)^-$. Note also that $H^+_{(1,0,\ldots,0)} = \mathbb{R}^d$ and $H^+_{(-1,0,\ldots,0)} = \emptyset$. Define a function $f : S^d \to \mathbb{R}^d$ by setting $f(u)_i = \mu_i(H_u^+)$. If we can show that f is continuous, then we would be done since then by Borsuk-Ulam, we would get that $\exists u \in S^d : f(u) = f(-u)$ (note that $u \neq (\pm 1, 0, \ldots, 0)$) and so $\forall i$ $\mu_i(H_u^+) = f(u)_i = f(-u)_i = \mu_i(H_u^-)$. So all that is left to verify is that f is continuous. Let g_n, g be characteristic functions of $H_{u_n}^+$ and H_u^+, respectively, for some sequence $\{u_n\}$ converging to u. Then pointwise we have $\forall x \notin H_u$, $g_n(x) \to g(x)$. But since $\mu_i(H_u) = 0$, we have $g_n(x) \to g(x)$ μ_i-almost everywhere. Since we are only concerned with finite measures, we can integrate both sides and use Lebesgue's Dominated Convergence Theorem to obtain $\mu_i(H_{u_n}^+) = \int g_n d\mu_i \xrightarrow[n\to\infty]{} \int g d\mu = \mu_i(H_u^+)$. Thus, f is continuous. □

Theorem 1.2 (Ham-Sandwich Theorem for point sets)**.** *Let $A_1, \ldots, A_d \subset \mathbb{R}^d$ be finite sets. Then $\exists$ hyperplane $H \subseteq \mathbb{R}^d$ that bisects all A_i's simultaneously.*

In this theorem, by 'bisects' we mean separates the points such that there are $\leq \lfloor |A_i|/2 \rfloor$ points on each open half space $\mathring{H}^{\pm}$. The idea of the proof is to replace each point by a small ball, therefore inducing a Borel measure on the set. First we will derive the result of the theorem for a special case when each set A_i contains an odd number of points, the sets A_i are disjoint and the points are in general position. Then we will generalize the proof by only requiring that $\forall i$ $|A_i|$ are odd, and finally we will prove the theorem in full generality.

Proof.

1. Assume the points are in general position, each $|A_i|$ is odd and $A_i \cap A_j = \emptyset$ for $i \neq j$:
   Then $\exists \epsilon > 0$ such that $\forall x \in \cup A_i$ and all balls $B(x, \epsilon)$, every hyperplane intersects $\leq d$ balls. Define $\mu_i(S) = Vol(S \cap \cup_{x \in A_i} B(x, \epsilon))$, where 'Vol' stands for Lebegue measure volume. Applying Ham-Sandwich Theorem to these balls, says that $\exists$ hyperplane H bisecting all μ_i's. Now since $|A_i|$ are odd, H must intersect ≥ 1 balls for each A_i. But by our choice of ϵ, H intersects $\leq d$ balls. So, H must intersect exactly one ball from each A_i, moreover it must pass through its center in order to divide its volume in half. Thus, H divides the original sets A_i in half as needed.

2. Assume $|A_i|$ is odd $\forall i$:
   Fix $\eta > 0$. Consider $A_{i,\eta}$ to be sets obtained from A_i by perturbation such that the points in A_i move distance $< \eta$ and such that $\cup A_{i,\eta}$ is in the general position. Then by part 1 of the proof, $\exists$ hyperplane H_η bisecting the sets $A_{i,\eta}$ simultaneously. Say $H_\eta = \{x \in \mathbb{R}^d : \langle a_\eta, x \rangle = b_\eta\}$, where a_η is a unit vector and $b_\eta \in \mathbb{R}$. Note that b_η lie in a bounded interval and therefore by compactness $\exists$ cluster point $(a_{\eta_i}, b_{\eta_i}) \to (a, b)$ as $\eta_i \to 0$. But then $H = \{x \in \mathbb{R}^d : \langle a, x \rangle = b\}$ is the desired bisecting hyperplane.

3. General case:
   For each A_i with $|A_i|$ even, remove one point form A_i arbitrarily. Then we can use part 2 of the proof to find a bisecting hyperplane H for these modified sets. But clearly this hyperplane H is also a bisecting hyperplane for our original A_i's, since if $|A_i|$ was even, say $|A_i| = 2k$, then there are $\leq \lfloor (2k-1)/2 \rfloor = k - 1$ points in each $\mathring{H}^{\pm}$, and so after putting the point back in, there are $\leq k = \lfloor 2k/2 \rfloor$ points in each $\mathring{H}^{\pm}$ as needed.

□

From this theorem, we get that each halfspace will contain less than or equal to half the points of each A_i. What we would like to show now is that we can actually get exactly $\lfloor |A_i|/2 \rfloor$ points of A_i on each of its open halfspaces.

Corollary 1.3. *Let $A_1, \ldots, A_d \subseteq \mathbb{R}^d$ be finite sets in general position. Then $\exists$ hyperplane H such that each $\mathring{H}^{\pm}$ contains exactly $\lfloor |A_i|/2 \rfloor$ points of A_i and at most one point of A_i is on H for all $1 \leq i \leq d$.*

If all of the A_i's are odd, this corollary is clear from part 2 of the proof above. If some of the A_i's are even, then we can perform the operation as described in part 3 of the proof above and then perturb the plane slightly so that the point that was on the plane ends up on the opposite side of the arbitrary point that we have just put back.

1.2 Equipartitioning

In the Ham-Sandwich Theorem above we were interested in dividing sets into two 'equal' parts by a hyperplane. A natural follow-up question is: Can we cut a finite measure, μ in $\mathbb{R}^d$ by d hyperplanes into 2^d equal-measure parts? (Note that here again we want to only consider some measure μ, for which $\mu(H) = 0$ for any hyperplane H.)

- In $\mathbb{R}^2$, you can think of this question as follows: Given a pizza, using two cuts, can we slice the pizza into four equal parts. For the two-dimensional case the answer is yes. Let's first try to make one slice that will cut the pizza in half: imagine sliding a hyperplane from infinity towards the set, i.e. $H_b = \{x \in \mathbb{R}^d : \langle a, x \rangle = b\}$ for some fixed $a \in \mathbb{R}$ and b goes from $-\infty$ to $+\infty$. Consider the function $f : \mathbb{R} \cup \{\infty\} \cup \{-\infty\} \to \mathbb{R}$ defined by $b \mapsto \mu(H_b^+)$. Note $f(-\infty) = 0$, $f(\infty) = \mu(\mathbb{R}^2)$ and f is continuous. Thus, $\exists B \in \mathbb{R}$ such that $f(B) = \mu(\mathbb{R}^2)/2$ and so the hyperplane H_B will bisect the measure. Now we can apply the Ham-Sandwich theorem to find a hyperplane H_A that would simultaneously bisect H_B^+ and H_B^-. So we get the desired hyperplanes H_A and H_B that will cut a measure in $\mathbb{R}^2$ into 4 equal-measure parts.

- In $\mathbb{R}^3$, the answer is still yes, but the prove is more involved. This result was proved by Edelsbrunner in 1987 [1].

- In $\mathbb{R}^d$ for $d \geq 5$ the answer is no, i.e. it is in general impossible to cut a set in $\mathbb{R}^d$ into 2^d equal parts by d hyperplanes. To see this, consider the moment curve and the one dimensional Borel measure μ along the curve (i.e. μ is supported on the moment curve). Note that any hyperplane cuts the moment curve in $\mathbb{R}^d$ in at most d points. Therefore, any set of d hyperplanes cuts the moment curve in at most d^2 points, subdividing it into at most $d^2 + 1$ parts. But d hyperplanes determine 2^d disjoint orthants, hence there are $2^d - d^2 - 1$ (which is > 0 for $d \geq 5$) orthants that have measure zero. Therefore, these hyperplanes cannot constitute equipartitioning of the measure in $\mathbb{R}^d$.

- In $\mathbb{R}^4$, the question is still open. [Note that the above argument does not apply in this case since $2^4 - 4^2 - 1 < 0$.]

1.3 Grünbaum Ramos Problem

We will say that (d, h, m) is *admissible* if $\forall m$ masses in $\mathbb{R}^d$, $\exists h$ hyperplanes that cut each mass into 2^h equal parts [2]. By the Ham-Sandwich Theorem, we know that $(d, 1, d)$ is admissible $\forall d$. Also note that if (d, h, m) is admissible then so is $(d+1, h, m)$. This can be seen by thinking of projecting the masses in $\mathbb{R}^{d+1}$ onto a d-dimensional hyperplane, then taking the preimage of h separating hyperplanes in $\mathbb{R}^d$ (which is known to exist since (d, h, m) is admissible) with respect to projection as a separating hyperplane in $\mathbb{R}^{d+1}$. So it makes sense to only consider the minimum d for which (d, h, m) is admissible, i.e. the minimal dimension in which $\forall m$ masses $\exists h$ hyperplanes that cut each mass into 2^h equal parts.

Definition 1.4. *Define* $\Delta(h, m) = \min\{d : (d, h, m) \text{ is admissible}\}$.

As an example, the Ham-Sandwich Theorem says that $\Delta(1, m) = m$.

Theorem 1.5 (Conjecture of Grünbaum)**.** *Let K be a convex body in $\mathbb{R}^2$, area$(K) = 1$, $\forall\ 0 \leq t \leq \frac{1}{4}$, $\exists$ two orthogonal lines partitioning K into parts of areas $t, t, 1/2 - t, 1/2 - t$ in clockwise order.*

- For $t = 0$ the result is clear, since we can find a hyperplane that bisects K (by the same technique as we used to show equipartitioning in $\mathbb{R}^2$) and then we can find another hyperplane perpendicular to this one that does not go through K.

- Let $t = 1/4$. Then we want to divide K into 4 equal parts by two orthogonal hyperplanes. For a given unit vector v, let H_v be a hyperplane with v as its normal such that area$(H_v^+ \cap K) =$ area$(H_v^- \cap K) = \frac{1}{2}$area$(K)$. Consider a map $g : S^1 \to \mathbb{R}$ given by $g(v) = 1/2 - 2a_v$, where a_v denotes the area of the portion of K in the lower quadrant of the partition of K induced by the hyperplanes H_v and $H_{v^\perp}$ (see Figure below).

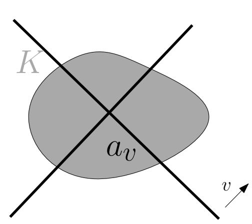

Figure 1.1: Equipartitioning

Clearly, g is continuous. Note also that $g(v^\perp) = 1/2 - 2(1/2 - a_v) = 2a_v - 1/2 = -g(v)$. Thus, by Borsuk-Ulam, $\exists u : g(u) = 0$. So there exists unit vector u, such that $a_u = 1/4$, which by construction of H_u and $H_{u^\perp}$ implies that each quadrant of the partition has area $1/4$, i.e. we have the equipartition of K.

- For $t \in (0, 1/4)$ the conjecture is still open.

1.4 Neckless Partitioning

Imagine that two thieves have stolen a precious necklace made out of platinum with various precious stones on it (such as sapphires, diamonds, etc.). They would like to share the necklace equally, but they are unsure of how much each stone is worth. Hence, they need to make sure that each of them gets an equal share of each type of stone. Also, since platinum is valuable as well, the thieves would like to make the least number of cuts to the necklace as they can. We will always assume that we have even number of stones of each type. This problem gave rise to the following theorem.

Theorem 1.6 (Necklace Theorem). *Every (open) necklace with d kinds of stones can be divided between two thieves using no more than d cuts.*

Note that this is a strict bound, since if we have d types of stones arranged in order, i.e. two diamonds followed by two sapphires, followed by two rubies, etc, we will need exactly d cuts to give each thieve one stone of each type. We can express this pattern of stones, by simply giving numbers to each type of stone, to get $112233\dots dd$. Then the required cut is as follows $1|12|23|34|\dots(d-1)d|d$.

Proof. Consider laying the necklace along a moment curve in $\mathbb{R}^d$. More specifically, think of the necklace as an interval $[0, 1]$ and let the function $f : [0, 1] \to \mathbb{R}^d$ be defined by $t \mapsto (t, t^2, \dots, t^d)$. By the Ham-Sandwich theorem, there exists a hyperplane, H, halfing the d masses. H is cutting the moment curve in at most d points. Note that H does not pass through any stones, since we assumed that there were even number of each type of jewel. $\square$

Another nice application of the Ham-Sandwich Theorem is the following:

Theorem 1.7 (Akiyama-Alon). *Consider pairwise disjoint sets $A_1, A_2, \dots, A_d \subseteq \mathbb{R}^d$, such that $|A_i| = n$ for $1 \leq i \leq d$ and $\bigcup_1^d A_i$ is in general position. Then there exists a partition of $\bigcup_1^d A_i$ into n parts B_j, such that $\bigcup_1^d A_i = \bigcup_1^n B_j$, $|B_j \cap A_i| = 1$, conv(B_j) are pairwise disjoint and $|B_i| = |B_j|$ for all i, j.*

Note that for $d = 2$, this theorem can be easily proved without resorting to the Ham-Sandwich Theorem. Finding the required pairing is equivalent to finding a matching of minimal size (i.e. a matching that minimizes the sum of the lengths of the edges). This can be done by noting that the sum of the diagonals of a quadrilateral is greater than the sum of two of its sides. For $d > 2$ no nontopological proof is known.

Proof. The proof follows by induction on n. If n is even, by the general-position Ham-Sandwich Theorem, there exists a bisecting hyperplane H, that does not pass through any points of $\bigcup_1^d A_i$. By induction, we can find B_j's for H^+ and H^-, the combination of which would give the partition of the whole set $\bigcup_1^d A_i$. Similarly, if n is odd, by the general-position Ham-Sandwich Theorem, there exists a bisecting hyperplane H, that passes through exactly one point from each A_i. We let the points on H form one set and proceed by induction on H^+ and H^-.

□

Bibliography

[1] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, EATCS Monographs in Theoretical Computer Science **10** (1987), 50–51.

[2] B. Matschke, *A Note on Masspartitions by Hyperplanes*, arXiv:1001.0193v1 [math.CO] (2009).