VI. The Hahn-Banach Theorem and applications

[Folland] It is not obvious that there are any nonzero bounded functionals on an arbitrary normed vector space. That such functionals exist in great abundance is one of the fundamental theorems of functional analysis.

[Reed & Simon] In dealing with Banach spaces, one often needs to construct linear functionals with certain properties. This is usually done in two steps: first one defines the linear functional on a subspace of the Banach space where it is easy to verify the desire properties; second, one appeals to (or proves) a general theorem which says that any such functional can be extended to the whole space while retaining the desired properties. One of the basic tools the second step is the following theorem,

Theorem VI.1. Let X be a vector space and $p : X \to \mathbb{R}$ such that

(i) $p(\alpha x) = \alpha p(x)$, $\forall \alpha \geq 0$, and

(ii) $p(x + y) \leq p(x) + p(y)$, $\forall x, y \in X$.

If S is a subspace of X and there is a linear functional
$f : S \to \mathbb{R}$ such that $f(s) \leq p(s)$, $\forall s \in S$, then f may be extended to
$F : X \to \mathbb{R}$ with $F(x) \leq p(x)$, $\forall x \in X$, with $F(s) = f(s) \forall s \in S$.

Proof. The idea of the proof is to first show that if $x \in X$ but $x \notin S$, then we can extend f to a functional having all the right properties on the space spanned by x and S. We then use a Zorn's Lemma / Hausdorff Maximality argument to show that this process can be continued to extend f to the whole space X.

(Sketch)

1. Consider the family

$$\mathcal{G} := \{g : D \to \mathbb{R} \vdots\ g \text{ is linear};\ g(x) \leq p(x),\ \forall x \in D;\ g(s) = f(s),\ \forall s \in S\},$$

where D is any subspace of X which contains S. So $\mathcal{G}$ is roughly the collection of "all linear extensions of f which are bounded by p".
Now $\mathcal{G}$ is a poset under

$$g_1 \prec g_2 \quad \iff \quad \text{Dom}(g_1) \subseteq \text{Dom}(g_2) \text{ and } g_2\Big|_{\text{Dom}(g_1)} = g_1.$$

2. Use Hausdorff maximality Principle (or Zorn) to get a maximal linearly ordered subset $\{g_\alpha\} \subseteq \mathcal{G}$ which contains f. Define F on the union of the domains of the $\{g_\alpha\}$ by $F(x) = g_\alpha(x)$ for $x \in \text{Dom}(g_\alpha)$.

3. Show that this makes F into a well-defined linear functional which extends f, and that F is maximal in that $F \prec G \implies F = G$.

4. Show F is defined on all of X using the fact that F is maximal. Do this by showing that a linear functional defined on a *proper* subspace has a *proper* extension. (Hence F must be defined on all of X or it wouldn't be maximal.)

□

Proposition VI.2. (Hausdorff Maximality Principle)
$(A, \prec)$ is a poset $\implies \exists B \subseteq A$ such that B is a maximal *linearly* ordered subset. I.e., if C is linearly ordered, then $B \prec C \prec A \implies C = B$ or $C = A$.

Of course, the HBT is also readily extendable to the complex case:

Theorem VI.3. Let X be a complex vector space and p a real-valued function defined on X satisfying

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \, \forall x, y \in X, \forall \alpha, \beta \in \mathbb{C} \text{ with } |\alpha| + |\beta| = 1.$$

If S is a subspace of X and there is a complex linear functional
$f : S \to \mathbb{R}$ such that $|f(s)| \leq p(s)$, $\forall s \in S$, then f may be extended to
$F : X \to \mathbb{R}$ with $|F(x)| \leq p(x)$, $\forall x \in X$, with $F(s) = f(s) \, \forall s \in S$.

VI.1. **Principle Applications of the HBT.**

Most often, $p(x)$ is taken to be the norm of the Banach space in question.

1. M is a closed subspace of X and $x \in X \backslash M \implies \exists f \in X^*$ such that $f(x) \neq 0, f\big|_M = 0$. In fact, if $\delta = \inf_{y \in M} \|x - y\|$, f can be taken to satisfy $\|f\| = 1$ and $f(x) = \delta$.

   Define f on $M + \mathbb{C}x$ by $f(y + \lambda x) = \lambda\delta$ for $y \in M, \lambda \in \mathbb{C}$. Then
   $$f(x) = f(0 + \cdot x) = 1 \cdot \delta = \delta$$
   but for $m \in M$,
   $$f(m) = f(m + 0) = 0 \cdot \delta = 0.$$
   For $\lambda \neq 0$, we have
   $$|f(y + \lambda x)| = |\lambda|\delta \leq |\lambda| \cdot \|\lambda^{-1}y + x\| = \|y + \lambda x\|$$
   because $\delta = \inf \|y + x\| \leq \|\lambda^{-1}y + x\|$ (putting in λ^{-1} for y). Using $p(x) = \|x\|$, apply the HBT to extend f from $M + \mathbb{C}x$ to all of X.

2. If $x \neq 0, x \in X$, then $\exists f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

   $M = \{0\}$ is trivially a closed subspace, so apply (1) with $\delta = \|x\|$.

3. The bounded linear functionals on X separate points.

   If $x \neq y$, then (2) shows $\exists f \in X^*$ such that $f(x - y) \neq 0$. I.e., $f(x) \neq f(y)$. This result indicates that X^* is BIG.

4. If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$, so $\hat{x} \in X^{**}$. Then $\varphi : x \mapsto \hat{x}$ is a linear isometry from X into X^{**}.

   $$\hat{x}(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \hat{x}(\alpha f) + \hat{x}(\beta g),$$
   so $\hat{x}$ is linear. This verifies that $\hat{x} \in X^{**}$.
   For $\varphi(x) = \hat{x}$, $\varphi(ax + by)$ is defined by
   $$\widehat{ax + by}(f) = f(ax + by) = af(x) + bf(y) = a\hat{x}(f) + b\hat{y}(f),$$

so $\varphi(ax+by) = \widehat{ax+by} = a\hat{x}+b\hat{y} = a\varphi(x)+b\varphi(y)$ shows that $\varphi : x \mapsto \hat{x}$ is linear. Finally,

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \cdot \|x\|$$

shows that

$$\|\hat{x}\| \;=\; \sup_{\|f\|\leq 1} \frac{|\hat{x}(f)|}{\|f\|} \;=\; \sup_{\|f\|\leq 1} \frac{|f(x)|}{\|f\|} \;\leq\; \sup_{\|f\|\leq 1} \frac{\|f\| \cdot \|x\|}{\|f\|} \;=\; \|x\|.$$

To get the reverse inequality, note that (2) provides a function f_0 for which $|\hat{x}(f_0)| = |f_0(x)| = \|x\|$ and $\|f\| = 1$. Then

$$\|\hat{x}\| \;=\; \sup_{\|f\|=1} |\hat{x}(f)| \;\geq\; |\hat{x}(f_0)| \;=\; \|x\|.$$

We saw this for Hilbert spaces, but this example is applicable to general Banach spaces and requires none of the Hilbert space machinery (orthonormal basis, projection theorem, etc.) as the HBT takes care of a lot.