# Tube Formulas and Complex Dimensions of Self-Similar Tilings

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# Contents

# Abstract

We extend some aspects of the theory of fractal strings and their complex dimensions from the real line to general Euclidean spaces. This is accomplished by using the explicit formulas and techniques of [**La-vF4**].

We use the self-similar tilings constructed in [**Pe1**] to define a zeta function which encodes the scaling properties of the tiling. This allows us to define a generating function for the geometry of the tiling as a more complicated geometric zeta function. The complex dimensions of the tiling are defined to be the poles of the geometric zeta function. We obtain a tube formula for a self-similar tiling in $\mathbb{R}^d$ as a power series in $\varepsilon$.

The Steiner formula gives the volume of the $\varepsilon$-neighbourhood of a compact convex subset $A \subseteq \mathbb{R}^d$, as a polynomial in $\varepsilon$ with coefficients given by the curvature measures of $A$, summed over $i = 0, 1, \ldots, d-1$. We obtain a fractal extension of this, in which the sum is not only taken over the integers $0, 1, \ldots, d-1$, but also has a term for each *complex* dimension and thus has infinitely many terms in general. This provides further justification for the term "complex dimension". It also extends to higher dimensions and sheds new light on the tube formula for fractals strings obtained in [**La-vF4**].



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CHAPTER 1

# Introduction

In [**Pe1**], the second author has shown that a self-similar tiling $\mathcal{T}$ is canonically associated with any self-similar system, i.e., any finite collection $\Phi = \{\Phi_j\}$ of contractive similarity transformations. Such a tiling $\mathcal{T}$ is essentially a decomposition of the complement of the unique self-similar set associated with $\Phi$, and is reviewed in greater detail in §2.

At the heart of this paper is the geometric zeta function $\zeta_{\mathcal{T}}(s)$ of a self-similar tiling $\mathcal{T}$. It will take some work before we are able to describe this meromorphic distribution-valued function precisely in §6. The function $\zeta_{\mathcal{T}}$ is a generating function for the geometry of a self-similar tiling: it encodes the density of geometric states of a tiling, including curvature and scaling properties. The poles of $\zeta_{\mathcal{T}}$ are the complex dimensions $\mathcal{D}_{\mathcal{T}}$ of the tiling, and we obtain a tube formula for $\mathcal{T}$ given as a sum of the residues of $\zeta_{\mathcal{T}}$ over $\mathcal{D}_{\mathcal{T}}$. By a tube formula for $A \subseteq \mathbb{R}^d$, we mean an explicit expression for the $d$-dimensional volume of the inner $\varepsilon$-neighbourhood of $A$, i.e.,

$$V_A(\varepsilon) = \text{vol}_d\{x \in A : dist(x, \partial A) \leq \varepsilon\}. \tag{1.1}$$

Thus, we are able to obtain the following key result, stated fully (and in more generality) in Thm. 6.8:

Theorem 1.1. *The $d$-dimensional volume of the inner tubular neighbourhood of $\mathcal{T}$ is given by the following distributional explicit formula:*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} \text{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right). \tag{1.2}$$

The first ingredient of $\zeta_{\mathcal{T}}$ is a *scaling zeta function* $\zeta_{\mathfrak{s}}(s)$ which encodes the scaling properties of the tiling. This comparatively simple zeta function is the Mellin transform of a discrete *scaling measure* $\eta_{\mathfrak{s}}$ which encodes the combinatorics of the scaling ratios of a self-similar tiling. More precisely, the measure $\eta_{\mathfrak{s}}$ is a sum of Dirac masses, where each mass is located at a reciprocal scaling ratio of some composition of the similarity transformations $\{\Phi_j\}$. Such a mass is weighted by the multiplicity of the corresponding scaling ratio. The scaling zeta function $\zeta_{\mathfrak{s}}$ coincides with the zeta functions studied in [**La-vF4**]. The function $\zeta_{\mathfrak{s}}$ also allows us to define the complex dimensions of a self-similar set in $\mathbb{R}^d$ (as the poles of $\zeta_{\mathfrak{s}}$), and we find these dimensions to have the same structure as in the 1-dimensional case. The definition and properties of the scaling measure $\eta_{\mathfrak{s}}$ and zeta function $\zeta_{\mathfrak{s}}$ is the subject of §3.

The next ingredient of $\zeta_{\mathcal{T}}$ is a *generator tube formula* $\gamma_G$. In [**Pe1**], it is shown that certain tiles $G_1, \ldots, G_Q$ of $\mathcal{T}$ are generators. More precisely, any tile $R_n$ of $\mathcal{T}$

is the image of a generator under some composition of the mappings $\Phi_j$:

$$R_n = \Phi_{n_k} \circ \ldots \circ \Phi_{n_1}(G_q),$$

for some $G_q$ and some $n_1, \ldots, n_k$. In §4, we discuss the role of the generators and introduce the function $\gamma_G$ which gives the inner tube formula for a generator. Moreover, appropriately parameterizing $\gamma_G$ yields the inner tube formula for a scaled generator. Therefore, by integrating $\gamma_G$ against $\eta_{\mathfrak{s}}$, one obtains the total contribution of $G_q$ (and its iterates under $\Phi$) to the final tube formula $V_{\mathcal{T}}$. This is elaborated upon in §4.2.

At last, the geometric zeta function of the tiling $\zeta_{\mathcal{T}}$ is assembled from the scaling zeta function, the tiling, and the terms appearing in $\gamma_G$. In some precise sense, $\zeta_{\mathcal{T}}$ is a generating function for the geometry of the self-similar tiling. Using $\zeta_{\mathcal{T}}$, and following the distributional techniques and explicit formulas of [**La-vF4**], we are able to obtain an explicit distributional tube formula for self-similar tilings. This is accomplished by exploiting the self-similar nature of the tiling and the scaling properties of various components.

The primary object of study in [**La-vF4**] is a *fractal string*, a countable collection $L = \{L_n\}_{n=1}^{\infty}$ of disjoint open intervals which form a bounded open subset of $\mathbb{R}$. Due to the exceedingly simple geometry of such intervals, this reduces to studying the lengths of these intervals $\mathcal{L} = \{\ell_n\}_{n=1}^{\infty}$, and the sequence $\mathcal{L}$ is also referred to as a fractal string. The tube formula for a fractal string $\mathcal{L}$ (and in particular, for a self-similar tiling in $\mathbb{R}^1$) is defined to be[1] $V_{\mathcal{L}}(\varepsilon) := V_L(\varepsilon)$ and is shown to be essentially given by a sum of the form

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}} \cup \{0\}} c_\omega \varepsilon^{1-\omega} \tag{1.3}$$

in [**La-vF4**, Thm. 8.1]. Here, the sum is taken over the set of complex dimensions $\mathcal{D}_{\mathcal{L}} = \{\text{poles of } \zeta_{\mathcal{L}}\}$, where the coefficients $c_\omega$ are given in terms of the residues of $\zeta_{\mathcal{L}}(s)$, the geometric zeta function of $\mathcal{L}$.

In this paper, we will also show that the self-similar tiling constructed in [**Pe1**] is the appropriate higher-dimensional analogue of a self-similar fractal string. The tube formula for tilings not only extends the 1-dimensional tube formula for fractal strings (1.3), but is also a fractal extension of the renowned Steiner formula

$$V_A(\varepsilon) = \sum_{i=0}^{d-1} \mu_{d-i}(B^{d-i})\mu_i(A)\varepsilon^{d-i} = \sum_{i \in \{0,1,\ldots,d-1\}} c_i \varepsilon^{d-i}. \tag{1.4}$$

Here, for $i = 0, 1, \ldots, d$, the $\mu_i$ are the invariant/intrinsic measures of dimension $i$ (i.e., which are homogeneous of degree $i$). Also, $A$ is a $d$-dimensional convex body (that is, a nonempty convex compact set), and $B^i$ is the $i$-dimensional unit ball.

The $\varepsilon$-neighbourhood considered in the Steiner formula includes all points exterior to the set, but within $\varepsilon$ of the set. We consider the inner $\varepsilon$-neighbourhood, which consists of those points inside the set and within $\varepsilon$ of its boundary. This is quite different from Steiner's neighbourhood, but the obvious similarities between the tube formulas is striking. In fact, we show in Thm. 6.13 that for a self-similar

[1] The definition $V_{\mathcal{L}}(\varepsilon) := V_L(\varepsilon)$ is justified because, as is shown in [**LaPo1**], $V_L$ depends exclusively on $\mathcal{L}$.

tiling $\mathcal{T}$, we have the following tube formula (see Rem. 6.16):

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D} \cup \{0,1,\ldots,d-1\}} c_\omega \varepsilon^{d-\omega}. \tag{1.5}$$

Our tube formula is a power series in $\varepsilon$, rather than just a polynomial in $\varepsilon$ (as in Steiner's formula). Moreover, our series is summed not just over the 'integral dimensions' $\{0, 1, \ldots, d-1\}$, but also over the countable set $\mathcal{D}$ of complex dimensions. The coefficients $c_\omega$ of the tube formula are expressed in terms of the 'curvatures' and the inradii of the generators of the tiling. They can also be written as in Thm. 6.13 (and in its extension Thm. 6.8) as the residues of the meromorphic distribution-valued function $\zeta_{\mathcal{T}}$.

The rest of this paper is organized as follows. §2 contains a quick overview of the background material concerning self-similar tilings. §3 defines the scaling and geometric measures, the scaling zeta function, and complex dimensions of a self-similar tiling. §4 develops the tube formula for the generators of a self-similar tiling, and establishes the general form of $V_{\mathcal{T}}(\varepsilon)$ in terms of this. §5 reviews the explicit formulas for fractal strings which will be used in the proof of the main results. §6 defines the geometric zeta function of the tiling, and states and proves the tube formula for fractal sprays (a generalization of a tiling) given in Thm. 6.8, from which the tube formula for self-similar tilings follows readily. §7 discusses several examples illustrating the theory, and §8 contains some concluding remarks. Appendix A contains the technical definition of *languid* from [**La-vF4**], which is used in the proof of Thm. 6.8 and in Appendices B–C. Appendix B verifies the validity of the definition of the geometric zeta function $\zeta_{\mathcal{T}}$. Appendix C verifies the distributional error term and its estimate, from Thm. 6.8.

Remark 1.2 (A note on the references). The primary reference for this paper is the research monograph "Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and spectra of fractal strings" by Lapidus and van Frankenhuijsen [**La-vF4**]. This volume is essentially a revised and much expanded version of "Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions" [**La-vF1**], by the same authors. The present paper cites [**La-vF4**] almost exclusively, so we provide the following partial correspondence between chapters for the aid of the reader, as [**La-vF4**] has not yet appeared at the time of this writing.

| [**La-vF1**] | Ch. 2 | Ch. 3 | Ch. 4 | Ch. 6 |
|---|---|---|---|---|
| [**La-vF4**] | Ch. 2–3 | Ch. 4 | Ch. 5 | Ch. 8 |

Remark 1.3. Throughout, we reserve the symbol $\mathbb{i} = \sqrt{-1}$ for the imaginary number.

CHAPTER 2

# The Self-Similar Tiling

This section provides an overview of the necessary background material concerning self-similar tilings. Further details may be found in [**Pe1**].

DEFINITION 2.1. A *self-similar system* is a family $\Phi := \{\Phi_j\}_{j=1}^J$ (with $J \geq 2$) of contraction similitudes

$$\Phi_j(x) := r_j A_j x + a_j, \quad j = 1, \dots, J.$$

For $j = 1, \dots, J$, we have $0 < r_j < 1, a_j \in \mathbb{R}^d$, and $A_j \in O(d)$, the orthogonal group of rigid rotations in $d$-dimensional Euclidean space $\mathbb{R}^d$. The numbers $r_j$ are referred to as the *scaling ratios* of $\Phi$. For convenience, we may take the scaling ratios in nonincreasing order, i.e., reindex $\{\Phi_j\}$ so that

$$1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0. \tag{2.1}$$

The metric space $(\mathcal{C}(\mathbb{R}^d), \delta)$ of nonempty compact subsets of $\mathbb{R}^d$ is complete for the Hausdorff metric $\delta$, and it is well known that there is a unique nonempty compact subset $F \subseteq \mathbb{R}^d$ satisfying the fixed-point equation

$$F = \Phi(F) := \bigcup_{j=1}^J \Phi_j(F).$$

See [**Hut**], as described in [**Fal1**] or [**Kig**], for example. This set $F$ is called the self-similar set associated with $\Phi$, or the attractor of $\Phi$. Note that different self-similar systems may give rise to the same self-similar set. In this paper, we will place the emphasis on the self-similar system and its corresponding dynamical system, rather than on just the self-similar set.

Define $C := [F]$ as the convex hull of $F$, and $T := \operatorname{relint} C$, that is, the relative interior of $C$. Denote the image of the hull $C$ under iterations of the mappings $\Phi_j$ by

$$C_k := \Phi^k(C) = \bigcup_{w \in \mathcal{W}_k} \Phi_w(C), \tag{2.2}$$

where $w$ is a *word* in $\mathcal{W}_k := \{1, 2, \dots, J\}^k$. Such a word $w = w_1 \dots w_k$ is used to write a composition of mappings

$$\Phi_w(x) := \Phi_{w_k} \circ \dots \circ \Phi_{w_2} \circ \Phi_{w_1}(x). \tag{2.3}$$

We say that the system satisfies the *tileset condition* iff

$$T \nsubseteq \bigcup_{j=1}^J \Phi_j(C). \tag{2.4}$$

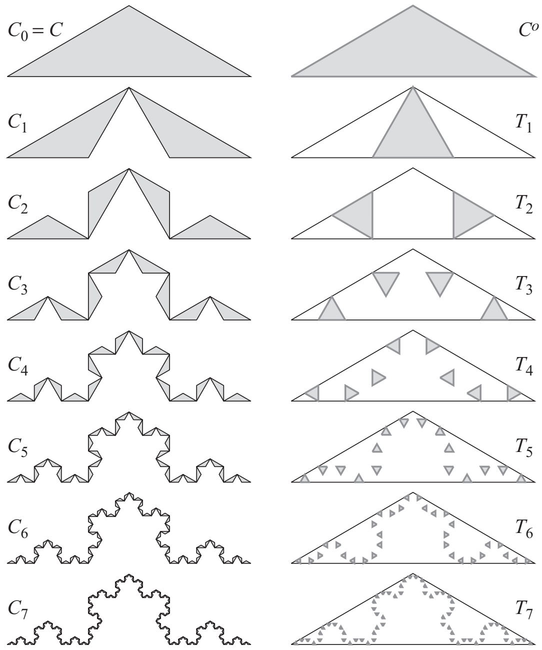


FIGURE 1. Construction of the Koch tiling $\mathcal{K}$. This example is discussed further in §7. The tiling $\mathcal{K}$ has the single generator $T_1 = G_1$.

This is a restriction on the overlap of the images of the mappings, comparable to the open set condition. For any system satisfying the tileset condition,

$$T_1 := T - C_1 \neq \varnothing \tag{2.5}$$

is well-defined and nonempty, and hence so is

$$T_k := \text{int}(C_{k-1} - C_k) = \Phi^k(T_1) \tag{2.6}$$

The second equality in (2.6) is a theorem of [**Pe1**]. As a bounded open set, $T_k$ can be written as a (countable) disjoint union of connected open sets. We are primarily interested in

$$T_1 = G_1 \cup G_2 \cup \cdots \cup G_Q. \tag{2.7}$$

DEFINITION 2.2. The connected components of $T_1$, i.e., the disjoint open sets $\{G_q\}$ in (2.7), are called *generators*. The number $Q$ of generators depends on the $\{\Phi_j\}$. In general, the number of connected components of an open subset of $\mathbb{R}^d$ may be countable; however, in this paper we assume $Q < \infty$.

In [**Pe1**], it is shown that given a self-similar system, there exists a natural tiling of $C - F$ which is produced by the system $\Phi$. The construction of the tiling of a well-known example, the Koch curve, is shown in Fig. 1 of this section.

DEFINITION 2.3. The *self-similar tiling* of $\Phi$ is

$$\mathcal{T} := \left(\{\Phi_j\}_{j=1}^J, \{G_q\}_{q=1}^Q\right). \tag{2.8}$$

We may also abuse the notation a little, and use $\mathcal{T}$ to denote the set of corresponding tiles:

$$\mathcal{T} = \{R_n\}_{n=1}^\infty = \{\Phi_w(G_q) : w \in W, q = 1, \ldots, Q\}, \tag{2.9}$$

where the sequence $\{R_n\}$ is an enumeration of the $\{\Phi_w(G_q)\}$, and $\Phi_w$ is as defined in (2.3) for some word $w \in \mathcal{W} := \bigcup_{k=1}^{\infty} \mathcal{W}_k$. Note that $\mathcal{W}$ is the set of all *finite* words $w$ over the alphabet $\{1, 2, \ldots, J\}$.

We say $\mathcal{T}$ is a tiling of $C - F$ because

$$C = \bigcup_n \overline{R_n}, \quad F \subseteq \bigcup_n \partial R_n, \quad \text{and} \quad R_{n_1} \cap R_{n_2} = \partial R_{n_1} \cap \partial R_{n_2}.$$

In other words, the tiles $R_n$ have disjoint interiors and $F$ does not intersect the interior of any $R_n$. It may help the reader to look ahead to the depiction of the Koch tiling in Fig. 1 of §7, and some of the other illustrations in that section.

CHAPTER 3

# The Geometric Zeta Function of a Self-Similar Tiling

In [**La-vF4**],[1] a *fractal string* is defined to be a bounded open subset of $\mathbb{R}$, that is, a countable collection of disjoint open intervals, $L = \bigcup_{n=1}^{\infty} L_n$, with lengths $\mathcal{L} = \{\ell_n\}_{n=1}^{\infty}$. The geometric zeta function of such an object is

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s, \tag{3.1}$$

and can be used to study the geometry of $\mathcal{L}$ and of its (presumably fractal) boundary $\partial\mathcal{L} := \partial L$.

Note that $\zeta_{\mathcal{L}}(s)$ is the Mellin transform of the measure

$$\eta_{\mathcal{L}} = \sum_{n=1}^{\infty} \delta_{1/\ell_n}, \tag{3.2}$$

where $\delta_x$ denotes the Dirac mass (or Dirac measure) at $x$:

$$\zeta_{\mathcal{L}}(s) = \int_0^{\infty} x^s \, d\eta_{\mathcal{L}}(x). \tag{3.3}$$

Here, and throughout the remainder of the paper, we denote by $\zeta_{\mathcal{L}}$ the meromorphic extension of (3.1) or (3.3).

In the case when $\partial\mathcal{L}$ is self-similar, i.e., when $\mathcal{L}$ is a self-similar string, the intervals of $\mathcal{L}$ form a tiling of the convex hull (which is just an interval in this case), in the sense of Definition 2.3. Since higher-dimensional tiles are more complicated than intervals, we will need something more than "length" to define the higher-dimensional zeta function.

## 3.1. The inradius

The inradius of a set $A$ (known to geometers as the radius of the largest ball contained in $A$) replaces length in the higher-dimensional theory.

Definition 3.1. Given $\varepsilon > 0$, we define the *inner $\varepsilon$-neighbourhood* of a set $A \subseteq \mathbb{R}^d$, $d \geq 1$, as

$$A_\varepsilon := \{x \in A : dist(x, \partial A) < \varepsilon\},$$

and denote its $d$-dimensional Lebesgue measure by

$$V_A(\varepsilon) := \text{vol}_d(A_\varepsilon). \tag{3.4}$$

The *inradius* $\rho$ of a set $A$ is

$$\rho = \rho(A) := \sup\{\varepsilon > 0 : V(A_\varepsilon) < V(A)\}. \tag{3.5}$$

[1]See also [**LaPo1**–2,**LaMa**,**La2**–3,**La-vF1**–3,**HaLa**], along with [**La1**, Exm. 5.1 and App. C].

The inradius is characterized [**Pe1**, Prop. 4.4] by

$$\begin{aligned}\rho(A) &= \sup\{dist(x, \partial A) : x \in A\} \\ &= \sup\{\varepsilon > 0 : \exists x \text{ with } B(x, \varepsilon) \subseteq A\}.\end{aligned}$$

This well-known concept from Riemannian geometry is key to our discussion because: (a) it indicates whether or not a set is completely contained in the inner $\varepsilon$-neighbourhood of its boundary, and (b) it behaves well under the action of the self-similar system (as in §2):

$$\rho(R_n) = \rho(\Phi_w(G_q)) = r_1^{e_1} \dots r_J^{e_J} g_q, \tag{3.6}$$

where $r_j$ is the scaling ratio of $\Phi_j$, and the exponent $e_j \in \mathbb{N}$ indicates the multiplicity of $j$ in $w \in \mathcal{W}$. Here and henceforth, $g_q$ is the inradius of the $q^{\text{th}}$ generator $G_q$ of $\mathcal{T}$:

$$g_q := \rho(G_q). \tag{3.7}$$

Throughout the remainder of this paper,[2] we assume that $\mathcal{T}$ is a self-similar tiling associated with a given self-similar system (and having inradii $g_q$), as described in §2–§3.1 and in further detail in [**Pe1**]. For convenience, we may take the generators in nonincreasing order, i.e., index the generators so that

$$g_1 \geq g_2 \geq \dots \geq g_Q. \tag{3.8}$$

### 3.2. Measures and zeta functions

Definition 3.2. The *scaling measure* encodes all products of scaling ratios as a sum of Dirac masses:

$$\eta_{\mathfrak{s}}(x) := \sum_{w \in \mathcal{W}} \delta_{1/r_w}(x). \tag{3.9}$$

From this, we get the *geometric measure*

$$\eta_{\mathfrak{g}}(x) := \left[\sum_{w \in \mathcal{W}} \delta_{1/g_1 r_w}(x), \dots, \sum_{w \in \mathcal{W}} \delta_{1/g_Q r_w}(x)\right] \tag{3.10}$$

$$= \left[\eta_{\mathfrak{s}}(x/g_1), \dots, \eta_{\mathfrak{s}}(x/g_Q)\right], \tag{3.11}$$

a $Q$-vector of horizontally dilated (or 'predilated') scaling measures which encodes the multiplicity and inradius of the tiles of $\mathcal{T}$. More precisely, the geometric measure is supported on the reciprocal inradii, and the weight associated to each Dirac mass corresponds to the multiplicity of tiles with that inradius. Both of the measures $\eta_{\mathfrak{s}}$, $\eta_{\mathfrak{g}}$ are defined on $(0, \infty)$. The decomposition (3.10) also shows the term corresponding to each $G_q$ for $q = 1, \dots, Q$; the $q^{th}$ *geometric measure*

$$\eta_{\mathfrak{g}q}(x) := \eta_{\mathfrak{s}}(x/g_q), \tag{3.12}$$

where $g_q$ is given by (3.7). All of these measures are defined on $(0, \infty)$.

Definition 3.3. Correspondingly, we also have the *scaling zeta function* $\zeta_{\mathfrak{s}} : \mathbb{C} \to \mathbb{C}$ which encodes the scaling factors of successively iterated maps and is given by

$$\zeta_{\mathfrak{s}}(s) := \sum_{w \in \mathcal{W}} r_w^s = \sum_{k=0}^{\infty} \sum_{w \in \mathcal{W}_k} r_w^s. \tag{3.13}$$

[2] Except for the discussion surrounding Thm. 6.8, wherein $\mathcal{T}$ is taken to be a fractal spray. See the introduction to §6.

More precisely, "scaling zeta function" is understood to refer to the meromorphic continuation of (3.13) throughout the remainder of the paper. Note that $\zeta_{\mathfrak{s}}$ is the Mellin transform (as defined in (3.3)) of the scaling measure $\eta_{\mathfrak{s}}$; see Rem. 3.8.

REMARK 3.4. We also have the *geometric zeta function of the tiling* or *tiling zeta function* $\zeta_{\mathcal{T}}$, which encodes the density of geometric states of the tiling. However, further discussion of this object is postponed to Def. 6.5 because more tools are required before we can give the precise definition.

The following theorem is the higher-dimensional counterpart of [**La-vF4**, Thm. 2.4] and can, in fact, be viewed as a corollary of it (see Rem. 3.8).

THEOREM 3.5. *The scaling zeta function of a self-similar system is*

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - \sum_{j=1}^{J} r_j^s}. \tag{3.14}$$

*This remains valid for the meromorphic extensions of $\zeta_{\mathfrak{s}}$ to all of $\mathbb{C}$.*

PROOF OF (3.14). As in [**La-vF4**, Thm. 2.4], apply the geometric series formula to $\sum_{j=1}^{J} r_j^{\operatorname{Re} s}$ to obtain the result for $s$ such that $1 - \sum_{j=1}^{J} r_j^{\operatorname{Re} s} > 0$ (i.e., for $\operatorname{Re} s > D$), then extend meromorphically to all of $\mathbb{C}$. □

DEFINITION 3.6. We can now define the *(scaling) complex dimensions* of a tiling $\mathcal{T}$ as the poles of the scaling zeta function:

$$\mathcal{D}_{\mathfrak{s}} := \{\omega \in \mathbb{C} : \zeta_{\mathfrak{s}}(s) \text{ has a pole at } \omega\}. \tag{3.15}$$

REMARK 3.7. Thus by (3.14), $\mathcal{D}_{\mathfrak{s}}$ consists of the set of complex solutions of the equation

$$\sum_{j=1}^{J} r_j^s = 1 \tag{3.16}$$

which is studied in detail in [**La-vF2**] and [**La-vF4**, Chap. 2–3]. In particular, the complex dimensions lie in a horizontally bounded strip of the form $D_l \leq \operatorname{Re} s \leq D$, where $D$ is the unique real solution of (3.16).

The positive number $D$ is called the *similarity dimension* of $F$ and coincides with the abscissa of convergence of $\zeta_{\mathfrak{s}}$. If the self-similar system defining $F$ satisfies the 'open set condition' (see, e.g., [**Hut**], as described in [**Fal1**] or [**Kig**]), then $D$ coincides with the Hausdorff and Minkowski dimensions of $F$. We will sometimes refer to $D$ as the (real) dimension of the tiling.

REMARK 3.8 (Comparison with [**La-vF4**]). Although the measures and zeta function defined in Def. 3.2 and Def. 3.3 above correspond to fractal subsets of $\mathbb{R}^d$, it is crucial to note that they are also formally identical to the objects $\eta, \zeta_\eta$ studied in [**La-vF4**]. To be precise, the scaling and geometric measures each meet the criteria for being a generalized fractal string in the sense of [**La-vF4**, Def. 4.1]; see Def. 5.1 below.

Moreover, they are actually self-similar strings of the sort studied in [**La-vF4**, Chap. 2–3]. In the terminology of [**La-vF4**], $\zeta_{\mathfrak{s}}$ is just the geometric zeta function of a self-similar string with scaling ratios $\{r_j\}_{j=1}^{J}$ and a single *gap*, which has been normalized so as to have $\ell_1 = 1$, where $\ell_1$ is the first length in the string. (The term "gap" of [**La-vF4**] has been replaced by "generator" in the present paper.) Thus, all of the explicit formulas developed in [**La-vF4**] are applicable to the measures and zeta functions described in this paper. This will be useful in the proof of Thm. 6.8.

One can also check (as in [**La-vF4**, §5.1]) that $\zeta_{\mathfrak{s}}$ satisfies

$$\eta_{\mathfrak{s}}(x) = \frac{1}{2\pi\mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} x^{s-1} \zeta_{\mathfrak{s}}(s)\, ds, \quad \text{and} \quad \zeta_{\mathfrak{s}}(s) = \int_0^\infty x^{-s}\, \eta_{\mathfrak{s}}(dx), \tag{3.17}$$

for some real constant $c > D$.

Furthermore, another consequence of the above observation is that the Structure Theorem for complex dimensions [**La-vF4**, Thm. 3.6] holds for the set defined in (3.15) and consisting of the solutions to (3.16), above. In particular, one distinguishes two complementary cases (see [**La-vF4**, §2.4] for further discussion of the lattice/nonlattice dichotomy):

- In the *lattice case*, i.e., when the logarithms of the underlying scaling ratios are rationally dependent, the complex dimensions lie periodically on finitely many vertical lines (including the line $\operatorname{Re} s = D$). In this case, there are infinitely many complex dimensions with real part $D$.
- In the *nonlattice case*, the complex dimensions are quasiperiodically distributed and $s = D$ is the only complex dimension with real part $D$. Moreover, there exists an infinite sequence of complex dimensions approaching the line $\operatorname{Re} s = D$ from the left. See [**La-vF4**, Chap. 2–3 and Chap. 8] for details, including a discussion of quasiperiodicity.

REMARK 3.9. In [**La-vF4**, §8.3–8.4], it is shown that a self-similar fractal string (i.e., a 1-dimensional self-similar tiling) is Minkowski measurable if and only if it is nonlattice. Gatzouras showed in [**Gat**] that nonlattice self-similar subsets of $\mathbb{R}^d$ are Minkowski measurable, thereby extending to higher dimensions a result in [**La3**], [**Fal2**] and partially proving the geometric part of [**La3**, Conj. 3.3, p. 163]. The present paper shows that self-similar tilings in $\mathbb{R}^d$ are nonlattice if and only if they are Minkowski measurable (Cor. 6.14), in virtue of Thm. 6.8 and Rem. 3.8. With the exception of Rem. 7.4, each of the examples discussed in §7 below is lattice and hence not Minkowski measurable. Our results, however, apply to nonlattice tilings as well.

CHAPTER 4

# The Role of the Generators

## 4.1. The tube formula for a generator

The simplest tilings have just one generator, i.e., for each $n$, $R_n = \Phi_w(G)$ for some $w \in \mathcal{W}$. Let $\mu_i$ denote $i$-dimensional invariant measure (as discussed further in Rem. 4.4 below), and let $xG$ be a homothetic image of $G$, scaled by $x > 0$. Then define

$$\gamma_G(x, \varepsilon) := V_{(1/xg)G}(\varepsilon), \quad \text{for } 1/x > \varepsilon, \tag{4.1}$$

so that $\gamma_G(1, \varepsilon)$ denotes the volume of the inner $\varepsilon$-neighbourhood of the generator $G$ scaled to have inradius 1, and $\gamma_G(x, \varepsilon)$ is the volume of a tile which is similar to $G$, but has inradius $1/x > \varepsilon$. The reason for defining $\gamma_G$ to correspond to a tile with inradius $1/x$ (rather than $x$) will become clear in (4.14); the general motivation for definition (4.1) becomes apparent in (4.7) and (7.1). In the case of multiple generators, $\gamma_q$ will be used to indicate the tube formula of the $q^{\text{th}}$ generator. In general, the computation of $\gamma_G(x, \varepsilon)$ may be nontrivial. However, it is possible to obtain or approximate such a formula (depending on the specific system involved). This is the subject of [**LaPe2**], as it is beyond the scope of the present discussion. For now, we make the following assumptions.

(i) For $\varepsilon < g$, we can write $V_G$ as

$$V_G(\varepsilon) = \gamma_G(\tfrac{1}{g}, \varepsilon) = \sum_{i=0}^{d-1} \kappa_i(G)\varepsilon^{d-i}, \tag{4.2}$$

for some real coefficients $\kappa_i(G)$ defined for $i = 0, \dots, d-1$.

(ii) Each $\kappa_i(G)$ is homogeneous of degree $i$, so that for $x > 0$,

$$\kappa_i(xG) = \kappa_i(G)\, x^i. \tag{4.3}$$

(iii) Each $\kappa_i(G)$ is rigid motion invariant, so that

$$\kappa_i(T(G)) = \kappa_i(G), \tag{4.4}$$

for any (affine) isometry $T$ of $\mathbb{R}^d$.

(iv) If the functions $\kappa_i(G) = \kappa_i(G; \varepsilon)$ depend on $\varepsilon$, they are measurable and bounded for $\varepsilon \in (0, g)$.

Up to this point, $\kappa_i$ has only been defined for $i = 0, 1, \dots, d-1$, and for $\varepsilon < g$. To rectify this, put $\kappa_i(G; \varepsilon) = 0$ for $\varepsilon \geq g$, and define

$$\kappa_d(G; \varepsilon) := \begin{cases} 0, & \varepsilon < g, \\ -\mu_d(G), & \varepsilon \geq g. \end{cases} \tag{4.5}$$

where $\mu_d$ is Lebesgue measure on $\mathbb{R}^d$, so that $\kappa_i(G;\varepsilon)$ is bounded on all of $(0,\infty)$ for $i=1,\dots,d$. Now when (i)–(iv) are satisfied, (4.1) may be expressed as

$$\gamma_G(x,\varepsilon) = \begin{cases} \sum_{i=0}^{d-1} \kappa_i(G) x^{-i} \varepsilon^{d-i}, & \varepsilon < 1/x, \\ -\kappa_d(G) x^{-d}, & \varepsilon \geq 1/x. \end{cases} \tag{4.6}$$

The function $\gamma_G(x,\varepsilon)$ gives the volume of the $\varepsilon$-neighbourhood of a tile with inradius $1/x$. Although it may not be immediately obvious from (4.6) that $\gamma_G$ is continuous at $x=\frac{1}{\varepsilon}$, it becomes clear after consideration of (4.1) and (4.2) and the fact that $\gamma_G(\frac{1}{g},\varepsilon)=\mu_d(G)$, with $g=\rho(G)$, in virtue of their geometric interpretations. The value $x=\frac{1}{\varepsilon}$ just corresponds to the point where the volume of a set is equal to the volume of its inner $\varepsilon$-neighbourhood, i.e., where the set becomes contained in its inner $\varepsilon$-neighbourhood. Note, however, that $\gamma_G$ is generally not differentiable at $x=\frac{1}{\varepsilon}$.

Definition 4.1. We refer to a formula like (4.2) which satisfies (i)–(iv) above as a *Steiner-like formula*, and we describe sets (especially generators) whose inner tube formula satisfies these conditions as being *Steiner-like*.

Remark 4.2. Any tile $R_n$ is the image of a generator $G_q$ under some composition of mappings, i.e., $R_n=\Phi_w(G_q)$. Properties (4.3)–(4.4) above allow us to use the equality

$$V_{R_n}(\varepsilon) = V_{\Phi_w(G_q)}(\varepsilon) = V_{r_1^{e_1}\dots r_J^{e_J} G_q}(\varepsilon) = V_{(\rho_n/g)G_q}(\varepsilon) = \gamma_q(1/\rho_n,\varepsilon), \tag{4.7}$$

where $\rho_n=\rho(R_n)$ is the inradius of $R_n$, as given by (3.6). Thus, the usefulness of having Steiner-like generators is that it suffices to know $V_G(\varepsilon)$ in order to find any $V_{R_n}(\varepsilon)$. See, for example, (4.13)–(4.14) below.

Remark 4.3. The final provision (iv) is for the most general case. When the generators are not convex, the quantity $\kappa_{qi}(G)$ will depend on $\varepsilon$; in fact, it is Federer's notion of *reach* (see [**Fed**]) which will be important here. For such cases, the inner tube formula will be obtained in [**LaPe2**] via the more general formulas of [**Sta**] and [**HLW**]. In this case, additivity may be lost or subject to conditions on $\varepsilon$. Note that (i)–(iv) are automatically satisfied if $\kappa_i$ is independent of $\varepsilon$ or, as is frequently the case, if $\kappa_i$ is piecewise constant. In particular, this is true whenever the generators are convex, as in the examples of §7. Since all examples of the functions $\kappa_i$ discussed in the sequel are independent of $\varepsilon$, we let (depending on whether or not there are multiple generators)

$$\kappa_i := \kappa_i(G), \quad \text{or} \quad \kappa_{qi} := \kappa_i(G_q). \tag{4.8}$$

The hypotheses (i)–(iv) will be useful in §4.2, as well as in the proof of Thm. 6.8. The extent to which they hold will be studied further in [**LaPe2**]. However, they are definitely what one might expect from geometric measure theory, especially Federer's generalizations in [**Fed**] of the tube formulas of Weyl [**We**] and Steiner [**Schn2**, Chap. 4]. Although Weyl's formula is for smooth submanifolds of $\mathbb{R}^d$, and both Weyl's and Steiner's formulas are for exterior $\varepsilon$-neighbourhoods, this does not prevent obvious similarities to the results of the present paper.

Remark 4.4. Caution: the description of $\kappa_i(G)$ given in conditions (i)–(iv) above is intended to suggest that $\kappa_i(G)$ bears a remarkable resemblance to the $i^{\text{th}}$ curvature measure of $G$ (as in [**Schn2**]); however, $\kappa_i(G)$ may be signed (even when

$G$ is convex and $i = d-1, d$) and is in general a more complicated object. By way of comparison, recall that the Steiner formula gives the $d$-dimensional volume of the outer $\varepsilon$-neighbourhood of a compact convex subset of $\mathbb{R}^d$ (i.e., the measure of all points within $\varepsilon$ of the set, which lie outside the set itself, a set quite different from the inner $\varepsilon$-neighbourhood) as discussed in §1:

$$V_A(\varepsilon) = \sum_{i=0}^{d-1} \mu_i(A)\mu_{d-i}(B^{d-i})\varepsilon^{d-i}, \tag{4.9}$$

where for $i = 0, \dots, d-1$, $\mu_i$ is the $i$-dimensional invariant measure and $B^i$ is the $i$-dimensional unit ball. When $A$ is a convex body, the relation between the invariant measures $\mu_i$, the curvature measures $C_i$ and the curvatures $\kappa_i$ is given by

$$\kappa_i(A) = d \cdot \mu_i(A)\mu_{d-i}(B^d) = \binom{d}{i} C_i(A). \tag{4.10}$$

In this case, the homogeneity and invariance of $\kappa_i$ (as expressed in (4.3)–(4.4)) follow directly from the corresponding properties of the $\mu_i$. More precisely,

(1) each $\mu_i$ is homogeneous of degree $i$, so that for $x > 0$,

$$\mu_i(xA) = \mu_i(A)\, x^i, \text{ and} \tag{4.11}$$

(2) each $\mu_i(A)$ is rigid motion invariant, so that

$$\mu_i(T(A)) = \mu_i(A), \tag{4.12}$$

for any (affine) isometry $T$ of $\mathbb{R}^d$.

The second equality of (4.10) is always true; the first may not be true if $A$ is not convex. The quantity $C_i(A)$ is sometimes called the *total curvature of* $A$ and is a special case of the generalized curvature measure

$$C_i(A) := C_i(A, \mathbb{R}^d) = \Theta_i(A, \mathbb{R}^d \times S^{d-1}).$$

Here, $\Theta_i$ is defined on $U(\mathbb{K}^d) \times \mathcal{B}(\Sigma)$, where $U(\mathbb{K}^d)$ is the ring of polyconvex sets[1] of dimension not exceeding $d$, and $\mathcal{B}(\Sigma)$ is the $\sigma$-algebra of Borel sets of $\Sigma = \mathbb{R}^d \times S^{d-1}$, the normal bundle of $d$-dimensional Euclidean space, as in [**Schn2**, §4.2].

REMARK 4.5. The Steiner–Weyl–Federer tube formula has been extended in various directions by a number of researchers in integral geometry and geometric measure theory, including [**Schn1**–2], [**Zä1**–2], [**Fu1**–2], [**Sta**], and most recently (and most generally) in [**HLW**]. The books [**Gr**] and [**Schn2**] contain extensive endnotes with further information and many other references.

## 4.2. Tilings with one generator

Suppose we have a tiling $\mathcal{T}$ with just one generator $G$. Then the inner tube formula of $\mathcal{T}$ is given by

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \sum_{n=1}^{\infty} V_{R_n}(\varepsilon) \\ &= \sum_{\rho_n \geq \varepsilon} V_{R_n}(\varepsilon) + \sum_{\rho_n < \varepsilon} V_{R_n}(\varepsilon), \end{aligned} \tag{4.13}$$

[1]A *polyconvex set* is a finite union of nonempty convex compact subsets of $\mathbb{R}^d$.

as in [**LaPo2**, Eqn. (3.2)]. Recall that $\rho_n$ is the inradius of the tile $R_n$. For $R_n = \Phi_w(G_q)$, invariance under rigid motions allows us to use the equality (4.7) to rewrite the sums in (4.13) as integrals with respect to $\eta_{\mathfrak{g}}$:

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \sum_{\rho_n^{-1} \leq 1/\varepsilon} V_{R_n}(\varepsilon) + \sum_{\rho_n^{-1} > 1/\varepsilon} V_{R_n}(\varepsilon) \\ &= \int_0^{1/\varepsilon} V_{(1/xg)G}(\varepsilon)\, d\eta_{\mathfrak{g}}(x) + \mu_d(G) \int_{1/\varepsilon}^{\infty} x^{-d}\, d\eta_{\mathfrak{g}}(x) \qquad (4.14) \\ &= \int_0^{\infty} \gamma_G(x,\varepsilon)\, d\eta_{\mathfrak{g}}(x) \\ &= \langle \eta_{\mathfrak{g}}, \gamma_G \rangle, \qquad (4.15) \end{aligned}$$

where $\gamma_G$ is the 'test function' defined by (4.1) and given explicitly by (4.6). Recall that $\gamma_G(x,\varepsilon)$ gives the volume of a tile which is similar to $G$, but has inradius $1/x$. Further recall that $\gamma_G$ is a continuous function of $\varepsilon$, but may not be differentiable.

## 4.3. Tilings with multiple generators

Upon replacing $G$ by $G_q$, we use the notation $V_q, \gamma_q, \kappa_{qi}$, etc., to refer to the corresponding quantity for the $q^{\text{th}}$ generator. For example, $\gamma_G(x,\varepsilon)$ is replaced by $\gamma_q(x,\varepsilon) = \gamma_{G_q}(x,\varepsilon)$, the volume of the $\varepsilon$-neighbourhood of a tile which is similar to $G_q$ and has inradius $x$.

Again using the notation $\langle \cdot, \cdot \rangle$ as defined in (4.15) above, the key concept lies in the formula

$$V_{\mathcal{T}}(\varepsilon) = \langle \eta_{\mathfrak{g}}, \gamma_G \rangle = \sum_{q=1}^{Q} \langle \eta_{\mathfrak{g}q}, \gamma_q \rangle, \qquad (4.16)$$

where we consider $\eta_{\mathfrak{g}}$ to be the density of geometric states as in [**La-vF4**, §5.1.1 and §6.3.1]. Note that the contribution to $V_{\mathcal{T}}(\varepsilon)$ resulting from one generator $G_q$ and its successive images is

$$V_q(\varepsilon) := \langle \eta_{\mathfrak{g}q}, \gamma_q \rangle. \qquad (4.17)$$

CHAPTER 5

# Distributional Explicit Formulas for Fractal Strings

We now recall some key results from [**La-vF4**] which will be needed for the proof of Thm. 6.8.

DEFINITION 5.1. In [**La-vF4**], a *generalized fractal string* is defined to be a local positive measure on $(0, \infty)$ and is denoted by $\eta$. Here, *local* means locally bounded with support bounded away from 0.

REMARK 5.2. It is clear that the scaling and geometric measures $\eta_{\mathfrak{s}}$ and $\eta_{\mathfrak{g}}$ introduced in Def. 3.2 are special cases of generalized fractal strings. As in (3.9)–(3.10), each corresponds to a sum of Dirac masses:

$$\eta_{\mathfrak{s}} = \sum_{w \in \mathcal{W}} \delta_{1/r_w}, \qquad \eta_{\mathfrak{g}} = \sum_{\rho} \delta_{1/\rho}.$$

DEFINITION 5.3. The string $\eta$ is said to be *languid* if its associated zeta function $\zeta_\eta$ (defined more precisely in (6.1) below) satisfies certain mild polynomial growth conditions on horizontal lines and a vertical contour in $\mathbb{C}$. The vertical contour is called the *screen* and denoted $S$; the region to the right of it is called the window $W$. Also, we define $\sup S := \sup\{\operatorname{Re} s : s \in S\}$ and $\inf S := \inf\{\operatorname{Re} s : s \in S\}$, and require that both of these be finite. These notions are precisely defined in Appendix A.

The function $\zeta_\eta$ is assumed to have a meromorphic continuation to a neighbourhood of $W$. The poles lying in the window are called the *visible complex dimensions* and the set of such poles is denoted $\mathcal{D}_\eta = \mathcal{D}_\eta(W)$. See [**La-vF4**, §5.3] for a full discussion.

Taking [**La-vF4**, Thm. 5.26 and Thm. 5.30] at level $k = 0$ gives the following distributional explicit formula for the action of a fractal string $\eta$ on a test function $\varphi \in C^\infty(0, \infty)$. While $\varphi$ may not have compact support, it must satisfy decay properties as described in (5.1)–(5.2).

THEOREM 5.4 (Extended distributional explicit formula). [**La-vF4**, Thm. 5.26] *Let $\eta$ be a generalized fractal string which is languid of order $M$.*[1] *Let $\varphi \in C^\infty(0, \infty)$ with $n^{th}$ derivative satisfying, for some $\delta > 0$, and every $0 \leq n \leq N = [M] + 2$,*

$$\varphi^{(n)}(x) = O\left(x^{-n-D-\delta}\right) \qquad \text{as } x \to \infty, \quad \text{and} \tag{5.1}$$

$$\varphi^{(n)}(x) = \sum_{\alpha} a_\alpha^{(n)} x^{-\alpha-n} + O\left(x^{-n-\inf S+\delta}\right) \qquad \text{as } x \to 0^+. \tag{5.2}$$

[1] "*Languid of order $M$*" refers to the fact that $M$ is the exponent appearing in conditions **L1** and **L2** of Def. A.2. Also, $[M]$ is the integer part of $M$.

*Then we have the following distributional explicit formula with error term for $\eta$:*[2]

$$\langle \eta, \varphi \rangle = \sum_{\omega \in \mathcal{D}_\eta} \operatorname{res}\left(\zeta_\eta(s)\widetilde{\varphi}(s); \omega\right) + \sum_{\alpha \in W - \mathcal{D}_\eta} a_\alpha \zeta_\eta(\alpha) + \langle \mathcal{R}, \varphi \rangle, \tag{5.3}$$

*where the error term $\mathcal{R}(x)$ is the distribution given by*

$$\langle \mathcal{R}, \varphi \rangle = \tfrac{1}{2\pi \mathbb{i}} \int_S \zeta_\eta(s)\widetilde{\varphi}(s)\, ds \tag{5.4}$$

*and estimated by*

$$\mathcal{R}(x) = O\left(x^{\sup S - 1}\right), \qquad \text{as } x \to \infty. \tag{5.5}$$

Here, $\widetilde{\varphi}$ is the Mellin transform of the function $\varphi$, defined by

$$\widetilde{\varphi}(s) := \int_0^\infty x^{s-1}\varphi(x)\, dx. \tag{5.6}$$

Note: the sum in (5.2) is over finitely many complex exponents $\alpha$ with $\operatorname{Re}\alpha > -\sigma_l + \delta$. This condition is described by saying that $\varphi$ has an asymptotic expansion of order $-\sigma_l + \delta$ at 0. Further, the order of the distributional error term, as in (5.5), is defined in Def. C.6 of Appendix C.

Remark 5.5. In the following, we may also use the more suggestive way of writing (5.3):

$$\eta = \sum_{\omega \in \mathcal{D}_\eta} \operatorname{res}\left(x^{s-1}\zeta_\eta(s); \omega\right) + \sum_{\alpha \in W - \mathcal{D}_\eta} \tau_\alpha(x)\zeta_\eta(\alpha) + \mathcal{R}(x), \tag{5.7}$$

where the distribution $\tau_\alpha$ is defined by $\langle \tau_\alpha, \varphi \rangle := a_\alpha$ (as in (5.2)).

Remark 5.6. If $\eta$ is *strongly languid* (which implies that $W = \mathbb{C}$, in particular), then it follows from the explicit formula [**La-vF4**, Thm. 5.27] that formula (5.3) has an analogue without error term. I.e., (5.3) holds with $\mathcal{R} \equiv 0$ for the appropriate test functions. As stated in the conclusion of Thm. 5.7, the same comment applies to formula (5.8) below; see also Rem. 6.11. The precise definition of strongly languid is given in Def. A.3 of Appendix A. A full discussion of the strongly languid case may be found in [**La-vF4**, Def. 5.3].

Theorem 5.7 (Tube formula for fractal strings). [**La-vF4**, Thm. 8.1] *Let $\eta = \eta_\mathcal{L}$ be a languid fractal string with geometric zeta function $\zeta_\eta$. The volume of the (one-sided) tubular neighbourhood of radius $\varepsilon$ of the boundary of $\eta$* [3] *is given by the following distributional explicit formula for test functions $\varphi \in C_c^\infty(0,\infty)$, the space of $C^\infty$ functions with compact support contained in $(0,\infty)$:*

$$V_\eta(\varepsilon) = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\frac{\zeta_\eta(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega\right) + \{2\varepsilon\zeta_\eta(0)\} + \mathcal{R}(\varepsilon). \tag{5.8}$$

*Here the term in braces is only included if $0 \in W - \mathcal{D}_\eta(W)$, and $\mathcal{R}(\varepsilon)$ is the error term, given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_S \frac{\zeta_\eta(s)(2\varepsilon)^{1-s}}{s(1-s)}\, ds \tag{5.9}$$

[2] Here, $\mathcal{D}_\eta := \mathcal{D}_\eta(W)$ and $W - \mathcal{D}_\eta$ is the complement of $\mathcal{D}_\eta$ in the window $W$. Further, $\operatorname{res}(g(s); \omega)$ denotes the residue at $\omega$ of the meromorphic function $g$.

[3] When $\eta = \eta_\mathcal{L}$ corresponds to an ordinary fractal string $\mathcal{L}$, as in (3.2) above, then $V_\eta(\varepsilon) = V_L(\varepsilon)$ as in Def. 3.1, where $L$ is the bounded open set defining $\mathcal{L}$.

*and estimated by*

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{1-\sup S}), \qquad \text{as } \varepsilon \to 0^+. \tag{5.10}$$

*In the case when* $\eta$ *is strongly languid,* $W = \mathbb{C}$ *and the error term vanishes, i.e.,* $\mathcal{R}(\varepsilon) \equiv 0$.

The order of the distributional error term, as in (5.10), is defined in Def. C.6 of Appendix C.

CHAPTER 6

# The Tube Formula for Self-Similar Tilings

In this chapter, we present the main result of the paper, a higher-dimensional analogue of Thm. 5.7. While the proof parallels that of [**La-vF4**, Thm. 8.1], it is significantly more involved; especially if Appendices B and C are taken into account. The current work introduces the proper conceptual framework and provides new insight, particularly with regard to the geometric interpretation of the terms of the formula. Indeed, the origin of the term $\{2\varepsilon\zeta_{\mathcal{L}}(0)\}$ in (5.8) is now understood to come from a Steiner-like formula (akin to (4.2)) for the unit interval; see §6.3. In fact, all terms coming from the third sum in [**La-vF4**, Thm. 5.26] are now understood to be related to a Steiner-type formula, and are naturally included in the first sum. In the proof of [**La-vF4**, Thm. 8.1], the calculation (6.33) shows how this unification may be accomplished.

Although our primary goal in this paper is to obtain a tube formula for self-similar tilings, we state our main result for the more general class of fractal sprays, as we expect it to be useful in the study of other fractal structures and tilings. We hope to investigate this further in forthcoming work. The important special case of self-similar tilings will be stated in Thm. 6.13 of §6.2.

## 6.1. The tube formula for fractal sprays

In [**La-vF4**, §1.4] (following [**LaPo2**]), a *fractal spray* is defined to be given by a nonempty bounded open set $B \subseteq \mathbb{R}^d$ (called the *basic shape* or *generator*), scaled by a fractal string $\eta$. That is, a fractal spray is a bounded open subset of $\mathbb{R}^d$ which is the disjoint union of open sets $\Omega_n$ for $n = 1, 2, \ldots$, where $\Omega_n$ is congruent to $\ell_n B$ (the homothetic of $\Omega$ by $\ell_n$) for each $\ell_n$. Thus, a fractal string is a fractal spray on the basic shape $B = (0, 1)$, the unit interval.

In the context of the current paper, a *self-similar tiling* is a special type of fractal spray with one or more generators. More precisely, a self-similar tiling is a union of fractal sprays on each of the basic shapes $G_1, \ldots, G_Q$, and scaled by a *self-similar* string. In our work, a general fractal spray may have multiple generators, but they are all scaled by the same measure $\eta$. Throughout this section, we continue to assume that each $G_q$ is Steiner-like, and that $Q < \infty$.

Definition 6.1. Define the *scaling zeta function of a fractal spray* by

$$\zeta_\eta(s) = \int_0^\infty x^s \, d\eta(x), \tag{6.1}$$

and the *visible scaling dimensions of a fractal spray* by

$$\mathcal{D}_\eta(W) := \{\omega \in W : \omega \text{ is a pole of } \zeta_\eta\}. \tag{6.2}$$

Note that $\mathcal{D}_\eta(W)$ is a discrete subset of $W \subseteq \mathbb{C}$, and hence is at most countable.

Just below, Def. 6.2–6.4 describe the components of the geometric zeta function of a fractal spray (or of a self-similar tiling). These notions are the ingredients of the geometric zeta function as presented in Def. 6.5.

DEFINITION 6.2. The $Q$-dimensional vector of generating inradii is

$$\mathfrak{g}(s) := = \left[g_1^s, g_2^s, \ldots, g_Q^s\right]. \tag{6.3}$$

Note that $g_q^s \zeta_\eta(s)$ is the Mellin transform of the $q^{\text{th}}$ geometric measure $\eta_{\mathfrak{g}q}(x) := \eta(x/g_q)$ (defined exactly as for tilings in (3.12)) and thus encodes the sizes (via inradii) of the homothetics of the generator $G_q$, as scaled by each of the scaling ratios described by $\eta$.

DEFINITION 6.3. The *curvature matrix* $\boldsymbol{\kappa}$ is a $Q \times (d+1)$ matrix with entries

$$\boldsymbol{\kappa} := [\kappa_{qi}(\varepsilon)] = \begin{bmatrix} \kappa_{10} & \kappa_{11} & \ldots & \kappa_{1d} \\ \kappa_{20} & \kappa_{21} & \ldots & \kappa_{2d} \\ \vdots & \vdots & & \vdots \\ \kappa_{Q0} & \kappa_{Q1} & \ldots & \kappa_{Qd} \end{bmatrix}, \tag{6.4}$$

Recall from §4.1 that $\kappa_{qd} := -\mu_d(G_q)$ for $\varepsilon \geq g$, and that for $i = 1, \ldots, d-1$ and $\varepsilon < g$, we define $\kappa_{qi} = \kappa_i(G_q)$ as the $i^{\text{th}}$ coefficient of the Steiner-like formula

$$V_{G_q}(\varepsilon) = \sum_{i=0}^{d-1} \kappa_{qi} \varepsilon^{d-i}.$$

For other values of $\varepsilon$, we set $\kappa_{qi}(\varepsilon) = 0$ for $i = 1, \ldots, d$.

DEFINITION 6.4. The $(d+1)$-vector $\mathcal{E}(\varepsilon, s)$ of 'boundary terms' is given by

$$\mathcal{E}(\varepsilon, s) := \left[\frac{\varepsilon^{d-s}}{s-i}\right]_{i=0}^{d} = \left[\frac{1}{s}, \frac{1}{s-1}, \ldots, \frac{1}{s-d}\right] \varepsilon^{d-s}. \tag{6.5}$$

We are now ready to define the geometric zeta function of a fractal spray, also the geometric zeta function of a self-similar tiling.

DEFINITION 6.5. Define the *geometric zeta function of a fractal spray* by the matrix product (or bilinear form)

$$\zeta_{\mathcal{T}}(\varepsilon, s) := \langle \mathfrak{g}\zeta_\eta, \mathcal{E}\rangle_{\boldsymbol{\kappa}} = \left(\mathfrak{g}^\top \boldsymbol{\kappa} \mathcal{E}\right) \zeta_\eta, \tag{6.6}$$

where $\mathfrak{g}^\top$ is the transpose of $\mathfrak{g}$. The *geometric zeta function of a self-similar tiling* or *tiling zeta function* is similarly defined, except that $\zeta_\eta$ is replaced by $\zeta_{\mathfrak{s}}$, the scaling zeta function of a self-similar tiling described in §3.2. The action of $\zeta_{\mathcal{T}}$ on a test function $\varphi$ is given by

$$\langle \zeta_{\mathcal{T}}(\varepsilon, s), \varphi(\varepsilon)\rangle = \int_0^\infty \zeta_{\mathcal{T}}(\varepsilon, s) \varphi(\varepsilon)\, d\varepsilon. \tag{6.7}$$

The zeta function given by the product (6.6) can also be written as

$$\left[g_1^s \; g_2^s \; \ldots \; g_Q^s\right] \begin{bmatrix} \kappa_{10} & \kappa_{11} & \ldots & \kappa_{1d} \\ \kappa_{20} & \kappa_{21} & \ldots & \kappa_{2d} \\ \vdots & \vdots & & \vdots \\ \kappa_{Q0} & \kappa_{Q1} & \ldots & \kappa_{Qd} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s-1} \\ \vdots \\ \frac{1}{s-d} \end{bmatrix} \varepsilon^{d-s} \zeta_\eta(s). \tag{6.8}$$

The concrete form of (6.11) can be found in (6.36) or, more precisely, in (B.11) of App. B. It turns out that $\zeta_{\mathcal{T}}$ is a meromorphic distribution-valued function. This verification is given in Appendix B; see Def. B.6 and Thm. B.8.

REMARK 6.6. The geometric zeta function defined in (6.6) differs from the presentation given in [**La-vF4**], wherein there is no real distinction between scaling and geometric zeta functions. For several reasons, it behooves one to think of $\zeta_{\mathcal{T}}$ as the geometric zeta function most naturally associated with the spray (or tiling), especially as pertains to the tube formula:

(i) The function $\zeta_{\mathcal{T}}$ encodes all the geometric information of $\mathcal{T}$.

(ii) Using $\zeta_{\mathcal{T}}$ leads to the natural unification of expressions which previously appeared unrelated. This will be seen by comparing (6.37) to (6.39) of Cor. 6.15.

(iii) The function $\zeta_{\mathcal{T}}$ arises naturally in the expression of the tube formula for the tiling as will be seen in Thm. 6.8 and Thm. 6.13.

(iv) It is the poles of $\zeta_{\mathcal{T}}(\varepsilon, s)$ that naturally index the sum appearing in $V_{\mathcal{T}}$, and the residues of $\zeta_{\mathcal{T}}$ that give the actual volume.

It is especially interesting that the unification mentioned in (ii) leads to a geometric interpretation of the term $\{2\varepsilon\zeta_\eta(0)\}$ in (5.8). This term is the inner Steiner formula for an interval, and can be dissected as

$$2\varepsilon\zeta_\eta(0) = \kappa_0(G)\varepsilon^{1-0}\zeta_\eta(0) = (-2)\mu_i(G)\varepsilon^{d-i}\zeta_\eta(i), \tag{6.9}$$

where $i = 0$ and $d = 1$. Note that $\mu_0(G) = -1$ is the Euler characteristic of an open interval. Some other interesting facts about the special case $d = 1$ are discussed in §6.3. For example, the geometric zeta function of a string, considered as a spray in the present context, is given by the formula occurring in (5.8)–(5.9):

$$\zeta_{\mathcal{T}}(\varepsilon, s) = \zeta_\eta(s)\frac{(2\varepsilon)^{1-s}}{s(1-s)}.$$

DEFINITION 6.7. The *visible complex dimensions of a fractal spray* are

$$\mathcal{D}_{\mathcal{T}}(W) := \mathcal{D}_\eta(W) \cup \{0, 1, \ldots, d-1\}, \tag{6.10}$$

where $\mathcal{D}_\eta(W)$ is as in Def. 6.1. Thus, $\mathcal{D}_{\mathcal{T}}(W)$ consists of the visible complex dimensions and the "integral dimensions" of the spray. Note that $\mathcal{D}_\eta(W)$ and $\mathcal{D}_{\mathcal{T}}(W)$ are discrete subsets of $W \subseteq \mathbb{C}$, and hence are at most countable. Also, it is clear from (6.8) that the poles of $\zeta_{\mathcal{T}}$ are contained in $\mathcal{D}_{\mathcal{T}}$.

THEOREM 6.8 (Tube formula for fractal sprays). *Let $\eta$ be a fractal spray on generators $\{G_q\}_{q=1}^Q$, with generating inradii $g_q = \rho(G_q) > 0$. Assume that $\zeta_\eta$ is languid on a screen $S$ which avoids the dimensions $\mathcal{D}_{\mathcal{T}}(W)$, and that each generator is Steiner-like (as in Def. 4.1). Then for test functions in $C_c^\infty(0,\infty)$, the d-dimensional volume of the inner tubular neighbourhood of the spray is given by the following distributional explicit formula:*

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right) + \mathcal{R}(\varepsilon), \tag{6.11}$$

*where the sum ranges over the set* (6.10) *of integral and visible complex dimensions dimensions of the spray. Here, the error term* $\mathcal{R}(\varepsilon)$ *is given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_S \zeta_{\mathcal{T}}(\varepsilon, s)\, ds, \tag{6.12}$$

*and estimated by*

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{d-\sup S}), \qquad \text{as } \varepsilon \to 0^+. \tag{6.13}$$

The order of the distributional error term is defined in Def. C.6. Due to their technical and specialized nature, we leave the proofs of (6.12) and (6.13) to Appendix C. The concrete form of (6.11) can be found in (6.36) or, more precisely, in (B.11) of App. B.

Proof of Thm. 6.8. Fix $q \in \{1, \dots, Q\}$, put $\eta_{\mathfrak{g}q}(x) := \eta(x/g_q)$ and let $\gamma_q = \gamma_{G_q}(x, \varepsilon)$ as in §4.3. This will allow us to calculate an explicit formula $V_q(\varepsilon)$ for the contribution of the fractal spray on one generator $G_q$, scaled by $\eta$. At the very end of the proof, in (6.36), we will sum over $q = 1, \dots, Q$ to obtain the volume formula for the entire spray.

Recall that we understand $V_{\mathcal{T}}(\varepsilon)$ as a distribution,[1] so we understand $V_q(\varepsilon) = \langle \eta_{\mathfrak{g}q}, \gamma_q \rangle$ by computing $\big\langle \langle \eta_{\mathfrak{g}q}, \gamma_q \rangle, \varphi \big\rangle$, the action of $V_q(\varepsilon)$ on a test function $\varphi = \varphi(\varepsilon) \in C_c^\infty(0, \infty)$, i.e., a smooth function with compact support contained in $(0, \infty)$:

$$\begin{aligned} \langle V_q(\varepsilon), \varphi \rangle = \langle \langle \eta_{\mathfrak{g}q}, \gamma_q \rangle, \varphi \rangle &= \int_0^\infty \left( \int_0^\infty \gamma_q(x, \varepsilon) \eta_{\mathfrak{g}q}(dx) \right) \varphi(\varepsilon)\, d\varepsilon \\ &= \int_0^\infty \int_0^\infty \gamma_q(x, \varepsilon) \varphi(\varepsilon)\, d\varepsilon\, d\eta_{\mathfrak{g}q}(x) \\ &= \langle \eta_{\mathfrak{g}q}, \langle \gamma_q, \varphi \rangle \rangle . \end{aligned} \tag{6.14}$$

Now, we use (4.6) to compute $\langle \gamma_q, \varphi \rangle$ as follows:

$$\begin{aligned} \int_0^\infty \gamma_q \varphi(\varepsilon)\, d\varepsilon &= \sum_{i=0}^{d-1} \int_0^{1/x} \kappa_{qi}(\varepsilon) x^{-i} \varepsilon^{d-i} \varphi(\varepsilon)\, d\varepsilon - \int_{1/x}^\infty \kappa_{qd}(\varepsilon) x^{-d} \varphi(\varepsilon)\, d\varepsilon \\ &= \sum_{i=0}^{d} \varphi_{qi}(x) \end{aligned} \tag{6.15}$$

where, for $x > 0$, we have introduced

$$\varphi_{qi}(x) := \begin{cases} x^{-i} \int_0^{1/x} \kappa_{qi}(\varepsilon) \varepsilon^{d-i} \varphi(\varepsilon)\, d\varepsilon, & 0 \le i \le d-1, \\ x^{-i} \int_\infty^{1/x} \kappa_{qi}(\varepsilon) \varphi(\varepsilon)\, d\varepsilon, & i = d, \end{cases} \tag{6.16}$$

in the last line. Caution: $\varphi_{qi}$ is a function of $x$, whereas $\varphi$ is a function of $\varepsilon$. Putting (6.15) into (6.14), we obtain

$$\langle V_q(\varepsilon), \varphi \rangle = \left\langle \eta_{\mathfrak{g}q}, \sum_{i=0}^{d} \varphi_{qi} \right\rangle = \sum_{i=0}^{d} \langle \eta_{\mathfrak{g}q}, \varphi_{qi} \rangle . \tag{6.17}$$

To apply Thm. 5.4, we must first check that the functions $\varphi_{qi}$ satisfy the hypotheses (5.1)–(5.2). Recall that $\varphi \in C_c^\infty(0, \infty)$.

[1] Indeed, $V_{\mathcal{T}}(\varepsilon)$ is clearly continuous and bounded (by the total volume of the spray), hence it defines a locally integrable function on $(0, \infty)$.

For $i < d$, (5.1) is satisfied because for large $x$, the integral is taken over a set outside the (compact) support of $\varphi$. This gives $\varphi_{qi}(x) = 0$ for sufficiently large $x$, and it is clear that, *a fortiori*,

$$\varphi_{qi}^{(n)}(x) = O(x^{-n-D-\delta}) \quad \text{for } x \to \infty, \forall n \geq 0. \tag{6.18}$$

To see that (5.2) is satisfied, note that $\varphi$ vanishes for $x$ sufficiently large and thus we have

$$\varphi_{qi}(x) = x^{-i} \int_0^\infty \kappa_{qi}(\varepsilon)\varepsilon^{d-i}\varphi(\varepsilon)\, d\varepsilon \quad \text{for } x \approx 0,$$

i.e., $\varphi_{qi}(x) = a_{qi}x^{-i}$ for small enough $x > 0$, where $a_{qi}$ is the constant

$$a_{qi} := \int_0^\infty \kappa_{qi}(\varepsilon)\varepsilon^{d-i}\varphi(\varepsilon)\, d\varepsilon = \lim_{x\to 0} x^i \varphi_{qi}(x). \tag{6.19}$$

Thus, the expansion (5.2) for the test function $\varphi_{qi}$ consists of only one term, and for each $n = 0, 1, \ldots, N$,[2]

$$\varphi_{qi}^{(n)}(x) = \tfrac{d^n}{dx^n}\left[a_{qi}x^{-i}\right] = O(x^{-n-i}) \quad \text{for } x \to 0^+, \forall n \geq 0. \tag{6.20}$$

A key point is that since $\varphi$ is smooth, (6.18) and (6.20) will hold for each $n = 0, 1, \ldots, N$, as required by Thm. 5.4. Since the expansion of $\varphi_{qi}$ has only one term, the only $\alpha$ in the sum is $\alpha = i$. Thus $a_{qi}$ is the constant corresponding to $a_\alpha$ in (5.2).

Applying Thm. 5.4 in the case when $i < d$, (5.3) becomes

$$\langle \eta_{\mathfrak{g}q}, \varphi_{qi}\rangle = \sum_{\omega\in\mathcal{D}_\eta(W)} \operatorname{res}\left(g_q^s\zeta_\eta(s)\widetilde{\varphi}_{qi}(s);\omega\right) \;+\; \{a_{qi}g_q^s\zeta_\eta(i)\}_{i\in W-\mathcal{D}_\eta}$$
$$+\; \tfrac{1}{2\pi\mathbb{i}}\int_S g_q^s\zeta_\eta(s)\widetilde{\varphi}_{qi}(s)\, ds, \tag{6.21}$$

where the term in braces is to be included iff $i \in W - \mathcal{D}_\eta$.

The case when $i = d$ is similar (or antisimilar). The compact support of $\varphi$ again gives

$$\kappa_{qd}(x) = -x^{-d}\int_0^\infty \kappa_{qd}(\varepsilon)\varphi(\varepsilon)\, d\varepsilon, \quad \text{for } x \to \infty, \tag{6.22}$$

so that for some positive constant $c$, and for all sufficiently large $x$, we have $\kappa_{qd}(x) = cx^{-d}$. Hence

$$\varphi_{qd}^{(n)}(x) = O(x^{-n-d}) \quad \text{for } x \to \infty, \forall n \geq 0, \tag{6.23}$$

and (5.1) is satisfied. For very small $x$, the integral in the definition of $\kappa_{qd}(x)$ is taken over an interval outside the support of $\varphi$, and hence $\kappa_{qd}(x) = 0$ for $x \approx 0$. Then clearly (5.2) is satisfied:

$$\varphi_{qd}^{(n)}(x) = 0 \quad \text{for } x \to 0^+, \forall n \geq 0. \tag{6.24}$$

An immediate consequence of (6.24) is that for $i = d$ in (6.19), the constant term is

$$a_{qd} = \lim_{x\to 0} x^d\varphi_{qd}(x) = 0, \tag{6.25}$$

---

[2] Recall from Thm. 5.4 that $N = [M] + 2$ and that $\eta$ is languid of order $M$.

and compared with (6.21) we have one term less in

$$\langle \eta_{\mathfrak{g}q}, \kappa_{qd} \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(g_q^s \zeta_\eta(s) \widetilde{\varphi}_{dq}(s); \omega\right) + \tfrac{1}{2\pi\mathbb{i}} \int_S g_q^s \zeta_\eta(s) \widetilde{\varphi}_{qd}(s)\, ds. \tag{6.26}$$

As in (5.6), denote the Mellin transform of $\psi$ by $\widetilde{\psi}$ and compute

$$\begin{aligned}
\widetilde{\varphi}_{qi}(s) = \int_0^\infty x^{s-1} \varphi_{qi}(x)\, dx &= \int_0^\infty x^{s-i-1} \int_0^{1/x} \kappa_{qi}(\varepsilon) \varepsilon^{d-i} \varphi(\varepsilon)\, d\varepsilon\, dx \\
&= \int_0^\infty \left( \int_0^{1/\varepsilon} x^{s-i-1}\, dx \right) \kappa_{qi}(\varepsilon) \varepsilon^{d-i} \varphi(\varepsilon)\, d\varepsilon \\
&= \frac{1}{s-i} \int_0^\infty \varepsilon^{i-s} \kappa_{qi}(\varepsilon) \varepsilon^{d-i} \varphi(\varepsilon)\, d\varepsilon \\
&= \frac{1}{s-i} (\widetilde{\kappa_{qi}\varphi})(d-s+1)
\end{aligned} \tag{6.27}$$

and

$$\begin{aligned}
\widetilde{\varphi}_{qd}(s) = \int_0^\infty x^{s-1} \varphi_{qd}(x)\, dx &= \int_0^\infty x^{s-d-1} \int_\infty^{1/x} \kappa_{qd}(\varepsilon) \varphi(\varepsilon)\, d\varepsilon\, dx \\
&= \int_0^\infty \left( \int_\infty^{1/\varepsilon} x^{s-d-1}\, dx \right) \kappa_{qd}(\varepsilon) \varphi(\varepsilon)\, d\varepsilon \\
&= \frac{1}{s-d} \int_0^\infty \varepsilon^{d-s} \kappa_{qd}(\varepsilon) \varphi(\varepsilon)\, d\varepsilon \\
&= \frac{1}{s-d} (\widetilde{\kappa_{qd}\varphi})(d-s+1),
\end{aligned} \tag{6.28}$$

where in both (6.27) and (6.28),

$$(\widetilde{\kappa_{qi}\varphi})(s) = \int_0^\infty \varepsilon^{s-1} \kappa_{qi}(\varepsilon) \varphi(\varepsilon)\, d\varepsilon. \tag{6.29}$$

Note that for $0 \leq i < d-1$, (6.27) is valid for $\operatorname{Re} s > i$, and for $i = d$, (6.28) is valid for $\operatorname{Re} s < i$. Thus both are valid in the strip $d-1 < \operatorname{Re} s < d$, and hence by analytic (meromorphic) continuation, they are valid everywhere in $\mathbb{C}$. Indeed, by Cor. B.5, $(\widetilde{\kappa_{qi}\varphi})$ is entire for each $q$ and $i = 0, \dots, d$.

We return to the evaluation of (6.17), applying Thm. 5.4 to find the action of $\eta_{\mathfrak{g}q}$ on the test function $\varphi_{qi}$, for $i = 0 \dots, d$. Substituting (6.27) and (6.28) into (6.21) gives

$$\begin{aligned}
\langle \eta_{\mathfrak{g}q}, \varphi_{qi} \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} &\operatorname{res}\left(g_q^s \zeta_\eta(s) \tfrac{1}{s-i} (\widetilde{\kappa_{qi}\varphi})(d-s+1); \omega\right) \\
&+ \{a_{qi} g_q^s \zeta_\eta(i)\}_{i \in W - \mathcal{D}_\eta} + \langle \mathcal{R}_{qi}, \varphi \rangle,
\end{aligned} \tag{6.30}$$

where $\mathcal{R}_{qi}$ is defined by

$$\langle \mathcal{R}_{qi}, \varphi \rangle := \tfrac{1}{2\pi\mathbb{i}} \int_S g_q^s \zeta_\eta(s) (\widetilde{\kappa_{qi}\varphi})(s)\, ds. \tag{6.31}$$

Substituting (6.30) into (6.17), we obtain

$$\langle V_q(\varepsilon), \varphi\rangle = \sum_{i=0}^{d} \left( \sum_{\omega\in\mathcal{D}_\eta(W)} \operatorname{res}\left( g_q^s \zeta_\eta(s) \tfrac{1}{s-i} (\widetilde{\kappa_{qi}\varphi})(d-s+1); \omega \right) \right.$$
$$\left. + \{a_{qi} g_q^s \zeta_\eta(i)\}_{i\in W-\mathcal{D}_\eta} + \langle \mathcal{R}_{qi}(\varepsilon), \varphi(\varepsilon)\rangle \right),$$

which, by Rem. 5.5, may also be written as the distribution

$$V_q(\varepsilon) = \sum_{\omega\in\mathcal{D}_\eta(W)} \operatorname{res}\left( \sum_{i=0}^{d} g_q^s \zeta_\eta(s) \tfrac{\varepsilon^{d-s}}{s-i} \kappa_{qi}(\varepsilon); \omega \right)$$
$$+ \sum_{i=0}^{d-1} \left\{ g_q^s \zeta_\eta(i) \kappa_{qi}(\varepsilon) \varepsilon^{d-i} \right\} + \mathcal{R}_q(\varepsilon), \tag{6.32}$$

where $\mathcal{R}_q(\varepsilon) := \sum_{i=0}^{d} \mathcal{R}_{qi}(\varepsilon)$ and the braces indicate that the terms of the second sum are only included for $i \in W - \mathcal{D}_\eta(W)$.

Recall from (6.25) that the $d^{\text{th}}$ term is $a_{qd} = 0$, so it is left out of the second sum. Since the terms of the second sum are only included for $i \in W - \mathcal{D}_\eta(W)$, at each such $i$ we have a residue

$$\operatorname{res}\left( g_q^s \zeta_\eta(s) \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i}; i \right) = \lim_{s\to i} g_q^s \zeta_\eta(s) \kappa_{qi}(\varepsilon) \varepsilon^{d-s}$$
$$= g_q^s \zeta_\eta(i) \kappa_{qi}(\varepsilon) \varepsilon^{d-i}. \tag{6.33}$$

Thus we can put

$$\mathcal{D}_{\mathcal{T}}(W) := \mathcal{D}_\eta(W) \cup \{0, 1, \ldots, d-1\} \tag{6.34}$$

and combine the two sums of (6.32) without losing or duplicating terms:

$$V_q(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left( \sum_{i=0}^{d} g_q^s \zeta_\eta(s) \frac{\varepsilon^{d-s}}{s-i} \kappa_{qi}(\varepsilon); \omega \right) + \mathcal{R}_q(x). \tag{6.35}$$

Now sum (6.35) over $q = 1, \ldots, Q$ and then interchange the resulting sums over $\omega$ and $q$, using the linearity of the residue and the notation of (6.6) and the identity (6.8). Since $V_{\mathcal{T}}(\varepsilon) = \sum_{q=1}^{Q} V_q(\varepsilon)$, as indicated at the start of the proof, we obtain

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left( \sum_{i=0}^{d} \sum_{q=1}^{Q} g_q^s \zeta_\eta(s) \frac{\varepsilon^{d-s}}{s-i} \kappa_{qi}(\varepsilon); \omega \right) + \mathcal{R}(\varepsilon) \tag{6.36}$$
$$= \sum_{\omega\in\mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left( \zeta_{\mathcal{T}}(\varepsilon, s); \omega \right) + \mathcal{R}(\varepsilon),$$

where $\mathcal{R}(\varepsilon) := \sum_{q=1}^{Q} \mathcal{R}_q(\varepsilon)$.

This completes the proof of (6.11). All that remains is the verification of the expression (6.12) for the error term, and error estimate (6.13). As these issues are of a more technical, and of a somewhat different nature, we postpone them to Appendix C. □

REMARK 6.9 (To $d$ or not to $d$). The reader may wonder why some sums include a $d^{\text{th}}$ summand and others do not. The explanation is as follows: the residue of $\zeta_{\mathcal{T}}$ at $d$ does not appear in the formula, for the reasons given in (6.25) and near (6.33). The essential reason for the absence of the $d^{\text{th}}$ residue in (6.34) is that $a_{qd} = 0$, as in (6.25). However, the $d^{\text{th}}$ term is necessary in the definition of $\zeta_{\mathcal{T}}$ itself, as evinced by (6.36). When the residue of $\zeta_{\mathcal{T}}$ is taken at any complex dimension (including $\{0, 1, \ldots, d-1\}$), all terms of $\zeta_{\mathcal{T}}$ must be included in the evaluation of the residue. Intuitively, if this sum neglected the $d^{\text{th}}$ term, the volumes of all the small tiles with $g < \varepsilon$ would be missing from $V_{\mathcal{T}}(\varepsilon)$.

REMARK 6.10 (Comparison with the Steiner formula). In the 'trivial' case when the spray consists only of finitely many scaled copies of the generators (i.e., when the scaling measure $\eta$ is supported on a finite set), and the generators are convex, the geometric zeta function will have no poles in $\mathbb{C}$. Therefore, the tube formula becomes a sum over only the numbers $0, 1, \ldots, d-1$ (recall from (6.25) that $a_{qd} = 0$, so the $d^{\text{th}}$ summand vanishes), for which the residues simplify greatly as in (6.33). In this case, $\zeta_\eta(i) = \rho_1^i + \cdots + \rho_N^i$, so each residue from (6.33) becomes a finite sum

$$\begin{aligned} g_q^s \zeta_\eta(i) \kappa_{qi}(\varepsilon) &= \rho_1^i \kappa_{qi} \varepsilon^{d-i} + \cdots + \rho_N^i \kappa_{qi} \varepsilon^{d-i} \\ &= \kappa_{qi}(r_{w_1} G_q) \varepsilon^{d-i} + \cdots + \kappa_{qi}(r_{w_N} G_q) \varepsilon^{d-i} \end{aligned}$$

where $N$ is the number of scaled copies of the generator $G_q$, and $r_{w_n}$, $n = 1, \ldots, N$ is the corresponding scaling factor. Thus, for each $q$ and each $n$, we obtain a Steiner-like polynomial for the volume of the inner $\varepsilon$-neighbourhood of the scaled basic shape $r_{w_n} G_q$. Recall that for a self-similar tiling, every tile $R_m$ is congruent to $r_w G_q$ for some $q$ and some $w \in \mathcal{W}$.

REMARK 6.11. In the case when $\zeta_\eta$ is not only languid but also strongly languid, then by Rem. 5.6, we may choose $W = \mathbb{C}$ and the error term vanishes, i.e. $\mathcal{R}(\varepsilon) \equiv 0$. Indeed, each individual error term obtained in the proof of Thm. 6.8 vanishes identically in that case. This is just as in [**La-vF4**, Thm. 8.1].

In particular, a self-similar tiling will always be strongly languid. This is explained in detail in [**La-vF4**, §6.4] and follows from the fact that a (normalized) self-similar fractal string $\eta$ is formally indistinguishable from the scaling measure of a self-similar tiling, as described in Rem. 3.8. Hence Thm. 6.8 above may be strengthened for self-similar tilings to yield Thm. 6.13 just below. (See also Rem. 3.8 above.) See Appendix A for the definitions of languid and strongly languid.

REMARK 6.12 (Reality principle). The nonreal complex dimensions appear in complex conjugate pairs and produce terms with coefficients which are also complex conjugates, in the general tube formula for fractal sprays. This ensures that formulas (6.11) and (6.37) are real-valued.

## 6.2. The self-similar case

The following corollary of Thm. 6.8 provides a higher-dimensional counterpart of the tube formula obtained for self-similar strings in [**La-vF4**, §8.4]. It should be noted that Thm. 6.13 applies to a slightly smaller class of test functions than Thm. 6.8. Indeed, the support of the test functions must be bounded away from 0 by $\mu_d(C) g_Q / r_J$, where $C = [F]$ is the hull of the attractor (as in §2), $g_Q$ is the smallest generating inradius (as in (3.8)), and $r_J$ is the smallest scaling ratio of

$\Phi$ (as in (2.1)). This technicality is discussed further in [**La-vF4**, Def. 5.3 and Thm. 5.27, §6.4, and Thm. 8.1].

THEOREM 6.13 (Tube formula for self-similar tilings). *Suppose a self-similar tiling* $\mathcal{T} = (\{\Phi_j\}_{j=1}^J, \{G_q\}_{q=1}^Q)$, *has generating inradii* $g_q = \rho(G_q)$ *and zeta function* $\zeta_\mathcal{T}$. *Also suppose that each generator is Steiner-like, as in Def. 4.1. Then the d-dimensional volume of the inner tubular neighbourhood of* $\mathcal{T}$ *is given by the following distributional explicit formula:*

$$V_\mathcal{T}(\varepsilon) = \sum_{\omega \in \mathcal{D}_\mathcal{T}} \operatorname{res}\left(\zeta_\mathcal{T}(\varepsilon, s); \omega\right), \tag{6.37}$$

*where* $\mathcal{D}_\mathcal{T} = \mathcal{D}_\mathcal{T}(\mathbb{C}) = \mathcal{D}_\mathfrak{s}(\mathbb{C}) \cup \{0, 1, \dots, d-1\}$.

PROOF. Note that in this (self-similar) case, one has $\zeta_\eta(s) = \zeta_\mathfrak{s}(s)$ and $\mathcal{D}_\eta(\mathbb{C}) = \mathcal{D}_\mathfrak{s}(\mathbb{C})$, with $\eta = \eta_\mathfrak{s}$ as in (3.9), in the notation of §3.2. The proof follows [**La-vF4**, §6.4]. According to Thm. 3.5, the scaling zeta function of a self-similar tiling has the form

$$\zeta_\mathfrak{s}(s) = \frac{1}{1 - \sum_{j=1}^J r_j^s}.$$

Let $r_J$ be the smallest scaling ratio. Then from

$$|\zeta_\mathfrak{s}(s)| \ll \left(\frac{1}{r_J}\right)^{-|\sigma|} \quad \text{as } \sigma = \operatorname{Re}(s) \to -\infty,$$

we deduce that $\zeta_\mathcal{T}$ is strongly languid, in the sense of Def. A.3. Hence, we can apply the extension of Thm. 6.8 mentioned in Rem. 6.11. This argument follows from the analogous ideas regarding fractal strings, which may be found in [**La-vF4**, Ch. 2–3]. The relevance of this reference is discussed in Rem. 3.7–3.8. □

COROLLARY 6.14 (Measurability and the lattice/nonlattice dichotomy). *A self-similar tiling is Minkowski measurable if and only if it is nonlattice.*

PROOF. We define a self-similar tiling $\mathcal{T}$ to be Minkowski measurable iff

$$0 < \lim_{\varepsilon \to 0^+} V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)} < \infty, \tag{6.38}$$

i.e., if the limit in (6.38) exists *and* takes a value in $(0, \infty)$. A tiling has infinitely many complex dimensions with real part $D$ iff it is lattice type, as mentioned in Rem. 3.8. Furthermore, all the poles with real part $D$ are simple in that case. A glance at (6.40) then shows that $V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)}$ is a sum containing infinitely many purely oscillatory terms $c_\omega \varepsilon^{\mathbf{i}n\mathbf{p}}$, $n \in \mathbb{Z}$, where $\mathbf{p}$ is some fixed period. Thus, the limit (6.38) cannot exist; see also [**La-vF4**, §8.4.2]. Conversely, the tiling is nonlattice iff $D$ is the only complex dimension with real part $D$. In this case, $D$ is simple and no term in the sum $V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)}$ is purely oscillatory; thus the tiling $\mathcal{T}$ is measurable. See also [**La-vF4**, §8.4.4]. □

The following corollary of Thm. 6.13 will be used in §7. Since this corollary pertains to tilings with a single generator $G$, we suppress dependence on $q$ for convenience. From (6.35)–(6.36), it is clear that an analogous result holds in the case of multiple generators.

COROLLARY 6.15. *If, in addition to the hypotheses of Thm. 6.13, $\mathcal{T}$ is a self-similar tiling with one generator and $\zeta_{\mathcal{T}}(s)$ has only simple poles, then*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \sum_{i=0}^{d} \operatorname{res}\left(\zeta_{\mathfrak{s}}(s);\omega\right) g^{\omega} \kappa_i(\varepsilon) \tfrac{\varepsilon^{d-\omega}}{\omega - i} + \sum_{i=0}^{d-1} g^i \kappa_i(\varepsilon) \zeta_{\mathfrak{s}}(i) \varepsilon^{d-i}. \tag{6.39}$$

It is not an error that the first sum extends to $d$ in (6.39), while the second stops at $d-1$; see Rem. 6.9. Note that in Cor. 6.15, $\mathcal{D}_{\mathfrak{s}}$ does not contain any integer $i = 0, 1, \ldots, d-1$, because this would imply that $\zeta_{\mathcal{T}}$ has a pole of multiplicity at least 2. In general, at most one integer can possibly be a pole of $\zeta_{\mathfrak{s}}$; see Rem. 3.7.

REMARK 6.16. For self-similar tilings satisfying the hypotheses of Cor. 6.15, it is clear that the general form of the tube formula is

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} c_{\omega}(\varepsilon) \varepsilon^{d-\omega}, \tag{6.40}$$

where for each fixed $\omega \in \mathcal{D}_{\mathfrak{s}}$,

$$c_{\omega}(\varepsilon) := \operatorname{res}\left(\zeta_{\mathfrak{s}}(s);\omega\right) \sum_{i=0}^{d} \frac{g^{\omega} \kappa_i(\varepsilon)}{\omega - i}, \tag{6.41}$$

and for each fixed $\omega \in \{0, 1, \ldots, d-1\}$,

$$c_{\omega}(\varepsilon) := \zeta_{\mathfrak{s}}(\omega) g^{\omega} \kappa_{\omega}(\varepsilon). \tag{6.42}$$

If the curvature matrix $\boldsymbol{\kappa}$ is constant on $(0, g)$, as is frequently the case, then each $c_{\omega}$ will also be independent of $\varepsilon$. The analogous statement to (6.40) will also hold in the case of multiple generators, as long as all complex dimensions are simple poles of $\zeta_{\mathcal{T}}$ (in this case, the analogue of (6.41) would also contain a sum over $q = 1, \ldots, Q$). In fact, this remark holds more generally, as alluded to in the Introduction. By (6.35), the tube formula for fractal sprays has essentially the same form as (6.40).

REMARK 6.17. The oscillatory nature of the geometry of $\mathcal{T}$ is apparent in (6.40). In particular, the existence of the limit in (6.38) can be determined by examining (6.40) and $\mathcal{D}_{\mathcal{T}}$.

REMARK 6.18. In the literature regarding the 1-dimensional case, the terms "gaps" and "multiple gaps" have been used where we have used "generators". See [**La-vF4**] and [**Fra**].

### 6.3. Recovering the tube formula for fractal strings

In this section, we discuss a result which is true for general (i.e., not necessarily self-similar) fractal strings and which can be recovered from Thm. 6.8. Suppose $\mathcal{L} = \{\ell_n\}_{n=1}^{\infty}$ is a fractal string with associated measure $\eta = \sum_{n=1}^{\infty} \delta_{1/\ell_n}$, as in (3.2), and geometric zeta function $\zeta_{\mathcal{L}}$, as in (3.1). However, write $\mathcal{L}$ as $L = \{L_n\}_{n=1}^{\infty}$ to emphasize the fact that we are thinking of it as a spray instead of as a string. The spray $L$ has a single 1-dimensional generator $G = (0, 1)$.[3] In keeping with the

[3]Even a self-similar string with multiple generators can be thought of as a fractal spray on one generator (albeit with a different scaling measure) in this fashion, as all open intervals are homothetic to each other. See also Example 7.1 and Rem. 7.5 of Example 7.4.

current perspective, we use inradii $\rho_n = \frac{1}{2}\ell_n$ instead of lengths. Thus

$$\zeta_{\mathcal{L}}(s) = g^s \zeta_{\mathfrak{s}}(s) = \left(\tfrac{1}{2}\right)^s \sum_{n=1}^{\infty} \rho_n^s. \tag{6.43}$$

By inspection, we find the generator tube formula (for $d = 1$)

$$\gamma_G(x,\varepsilon) = \sum_{i=0}^{d-1} \kappa_i(\varepsilon)(x)^i \varepsilon^{d-i} = \kappa_0(x)^0 = 2\varepsilon. \tag{6.44}$$

and

$$\mu_d(xG) = -\kappa_1(\varepsilon)x^1 = 2x, \tag{6.45}$$

so that we have

$$\boldsymbol{\kappa} = [2,\ -2] \tag{6.46}$$

$$\mathcal{E}(\varepsilon, s) = \left[\tfrac{\varepsilon^{1-s}}{s}, \tfrac{\varepsilon^{1-s}}{s-1}\right] = \left[\tfrac{1}{s}, \tfrac{1}{s-1}\right] \varepsilon^{1-s}. \tag{6.47}$$

Since the generator of a string (i.e., an open interval) is always convex, the terms $\kappa_i = \kappa_i(\varepsilon)$ will be constants (in particular, independent of $\varepsilon$).

Using (6.43), one obtains[4]

$$\zeta_L(\varepsilon, s) = \left(\tfrac{1}{2}\right)^s \zeta_{\mathfrak{s}}(s) \left(\frac{2}{s} + \frac{-2}{s-1}\right) \varepsilon^{1-s} = \zeta_{\mathfrak{s}}(s) \frac{(2\varepsilon)^{1-s}}{s(1-s)} \tag{6.48}$$

so the volume (6.11) becomes

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}(W) \cup \{0\}} \operatorname{res}\left(\zeta_{\mathfrak{s}}(s) \frac{(2\varepsilon)^{1-s}}{s(1-s)}; \omega\right) + \mathcal{R}(\varepsilon) \tag{6.49}$$

and we have exactly recovered (5.8), the tube formula for fractal strings discussed in Thm. 5.7. Indeed, from (6.48), (6.13) and (6.12) we see that even the error term $\mathcal{R}(\varepsilon)$ is the same (and satisfies the same estimate) as in (5.9)–(5.10). In addition, we gain a geometric interpretation of the terms appearing in (6.49), in view of (6.44)–(6.48). This will be discussed further in [**LaPe2**].

[4]In (6.48), we use the symbol $\zeta_L$ to denote the tiling zeta function of the string, as in Def. 6.5, in contrast to $\zeta_{\mathcal{L}}$, the geometric zeta function of a string defined in (3.1).

CHAPTER 7

# Tube Formula Examples

Each of the examples chosen in this section is a *lattice* self-similar tiling, in the sense of Rem. 3.8: the scaling ratios $r_j$ are all integral powers of some number $r \in (0,1)$. (See Remark 7.4 for a discussion of how one may construct nonlattice examples.) As will be verified, all examples in this section have Steiner-like generators in the sense of Def. 4.1. This allows us to find the tube formula for each tile from the tube formula for each generator $G$. Indeed, $\gamma_G(x,\varepsilon) = V_{(1/xg)G}(\varepsilon)$, as defined in (4.1). Recall that $\gamma_G(x,\varepsilon)$ gives the tube formula for a tile which is congruent to $G$, but has inradius $1/x$. In each example, $\gamma_G(1,\varepsilon)$ or $\gamma_G(1/g,\varepsilon) = V_G(\varepsilon)$ is computed 'by hand' so that the homogeneity property (4.3) can be used to obtain $\gamma_G(x,\varepsilon)$. Then, when $\gamma_G(x,\varepsilon)$ is integrated with respect to $d\eta_{\mathfrak{g}}(x)$, each point $x = 1/\rho$ in the support of $\eta_{\mathfrak{g}}$ will contribute

$$\gamma_G(x,\varepsilon) = V_{(\rho/g)G}(\varepsilon) = V_{r_1^{e_1}\dots r_J^{e_J}G}(\varepsilon) = V_{\Phi_w(G)}(\varepsilon) = V_{R_n}(\varepsilon) \tag{7.1}$$

to the integral, corresponding to the tile $R_n = \Phi_w(G_q)$.

Moreover, the scaling zeta function $\zeta_{\mathfrak{s}}$ of each example has only simple poles, with a single line of complex dimensions distributed periodically on the line $\operatorname{Re} s = D$. Thus, the tube formula may be substantially simplified via Cor. 6.15.

Remark 7.1. Much as in the case of fractal strings where $d = 1$ (see [**La-vF4**, §8.4.2]), it follows from Thm. 6.13 that for a lattice self-similar tiling $\mathcal{T}$,[1] each line of complex dimensions $\beta + \mathbb{i}n\mathbf{p}$ gives rise to a function which consists of a multiplicatively periodic function times $\varepsilon^{d-\beta}$. Here, $\beta$ is some real constant and $\mathbf{p} = 2\pi/\log r^{-1}$ is the oscillatory period of $\mathcal{T}$; recall that for lattice fractals, the scaling ratios are all integral powers of some number $r \in (0,1)$. Since all the examples considered in this section are of this type, and also have a single line of simple complex dimensions of the form $D + \mathbb{i}n\mathbf{p}$, we have

$$V_{\mathcal{T}}(\varepsilon) = h\left(\log_{r^{-1}}(\varepsilon^{-1})\right)\varepsilon^{d-\beta} + P(\varepsilon), \tag{7.2}$$

where $h$ is an additively periodic function of period 1, and $P$ is a polynomial in $\varepsilon$. For example, the periodic function appearing in the tube formula (7.14) for the Koch tiling $\mathcal{K}$ (in Example 7.2 below) has the following Fourier expansion:

$$h(u) = \tfrac{g}{\log 3} \sum_{n\in\mathbb{Z}} g^{\mathbb{i}n\mathbf{p}} \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right) e^{2\pi\mathbb{i}nu}, \tag{7.3}$$

where $g = \sqrt{3}/18$, $D = \log_3 4$, $r = 1/\sqrt{3}$, and $\mathbf{p} = 4\pi/\log 3$.

Remark 7.2. For some purposes, it might also be helpful to truncate the tube formula by using a suitable screen (and restricting to the corresponding visible

[1] Here, $\mathcal{T}$ is assumed to have generators which all have piecewise constant curvature coefficients $\kappa_{qi}$, as in Rem. 6.16.

complex dimensions) and applying a special case of the tube formula with error term (6.11). This is needed, for example, to give a detailed proof of Cor. 6.14. The interested reader may wish to consult [**La-vF4**, Thm. 5.31 and Thm. 8.36], but this will not be discussed further in this paper.

## 7.1. The Cantor tiling

First, we compute the tube formula for the Cantor tiling $\mathcal{C}$ (called the Cantor string in [**La-vF4**, §1.1.2 and §2.3.1]) using these techniques. The Cantor tiling $\mathcal{C}$ is constructed via the self-similar system

$$\Phi_1(x) = \tfrac{x}{3}, \qquad \Phi_2(x) = \tfrac{x+2}{3}.$$

The associated self-similar set is, of course, the classical ternary Cantor set. Thus $d = 1$ and we have one scaling ratio $r = \frac{1}{3}$, and one generator $G = \left(\frac{1}{3}, \frac{2}{3}\right)$ which has generating inradius $g = \frac{1}{6}$. The corresponding self-similar string has inradii $\rho_k = gr^k$ with multiplicity $2^k$ for $k = 0, 1, 2, \ldots$, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s}}, \tag{7.4}$$

with complex dimensions

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} \,.\, n \in \mathbb{Z}\} \qquad \text{for } D = \log_3 2,\ \mathbf{p} = \tfrac{2\pi}{\log 3}. \tag{7.5}$$

Now an application of (6.49) from the previous section gives the following tube formula for $\mathcal{C}$:

$$\begin{aligned} V_{\mathcal{C}}(\varepsilon) &= \sum_{n \in \mathbb{Z}} \operatorname{res}\left(\tfrac{3^{-s}}{1-2\cdot 3^{-s}}; D + \mathbb{i}n\mathbf{p}\right)\left(\tfrac{(2\varepsilon)^{1-s}}{(D+\mathbb{i}n\mathbf{p})(1-D-\mathbb{i}n\mathbf{p})}\right) + 2\varepsilon\zeta_{\mathfrak{s}}(0) \\ &= \frac{1}{2\log 3} \sum_{n \in \mathbb{Z}} \frac{(2\varepsilon)^{1-D-\mathbb{i}n\mathbf{p}}}{(D + \mathbb{i}n\mathbf{p})(1 - D - \mathbb{i}n\mathbf{p})} - 2\varepsilon, \end{aligned} \tag{7.6}$$

exactly as obtained for the Cantor string in [**La-vF4**, §1.1.2].

Alternatively, this may be written as a series in $\left(\frac{\varepsilon}{g}\right)$ as

$$V_{\mathcal{C}}(\varepsilon) = \frac{1}{3\log 3} \sum_{n \in \mathbb{Z}} \left(\frac{1}{D + \mathbb{i}n\mathbf{p}} - \frac{1}{D - 1 + \mathbb{i}n\mathbf{p}}\right)\left(\frac{\varepsilon}{g}\right)^{1-D-\mathbb{i}n\mathbf{p}} - 2\varepsilon, \tag{7.7}$$

with $g = \frac{1}{6}$, $D = \log_3 2$, and $\mathbf{p} = 2\pi/\log 3$. It is this form of the tube formula which is closer in appearance to the following examples. Note that by (7.5), we have $3^{1-D} = \frac{3}{2}$ and $3^{-\mathbb{i}n\mathbf{p}} = 1$ for all $n \in \mathbb{Z}$.

## 7.2. The Koch tiling

The standard Koch tiling $\mathcal{K}$ (see Fig. 1) is constructed via the self-similar system

$$\Phi_1(z) := \xi\overline{z} \quad \text{and} \quad \Phi_2(z) := (1 - \xi)(\overline{z} - 1) + 1. \tag{7.8}$$

with $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$ and $z \in \mathbb{C}$. Thus we have one scaling ratio $r = |\xi| = 1/\sqrt{3}$, and one generator $G$ which is an equilateral triangle of side length $\frac{1}{3}$. Then the height of $G$ is $\frac{\sqrt{3}}{6}$ and the generating inradius is $g = \frac{\sqrt{3}}{18}$; see Fig. 2. The self-similar set defined by $\{\Phi_1, \Phi_2\}$ is, of course, the classical von Koch curve.

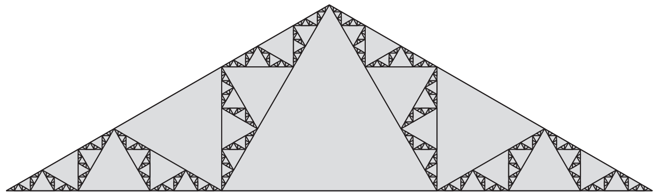

FIGURE 1. The Koch tiling $\mathcal{K}$.

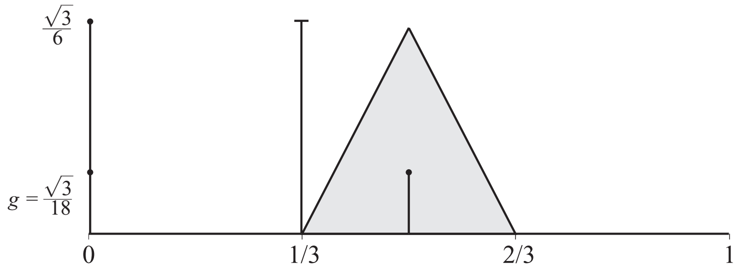


FIGURE 2. The generator for the Koch tiling.

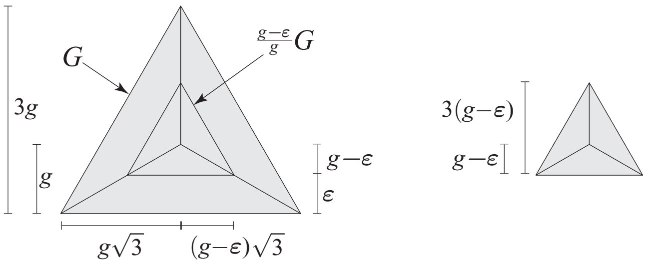


FIGURE 3. The volume $V_G(\varepsilon)$ of the generator of the Koch tiling.

This tiling has inradii $\rho_k = gr^k$ with multiplicity $2^k$ for $k = 0, 1, 2, \dots$, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s/2}}, \tag{7.9}$$

with complex dimensions

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_3 4,\ \mathbf{p} = \tfrac{4\pi}{\log 3}. \tag{7.10}$$

Now we find the tube formula for the generator. By inspection of Fig. 2 and Fig. 3, we would like to find $\gamma_G(x, \varepsilon)$ so that $\gamma_G(1/g, \varepsilon)$ gives

$$V_G(\varepsilon) = \mathrm{vol}_2(G) - \mathrm{vol}_2\left(\tfrac{g-\varepsilon}{g} G\right) = 3^{3/2}\,(2g - \varepsilon)\,\varepsilon \quad \text{ for } \varepsilon < g. \tag{7.11}$$

The reasoning for (7.11) is as follows: $G$ has inradius $g$, so subtract the volume of a smaller, scaled copy which has inradius $g - \varepsilon$, as in Fig. 3. Then the tube formula for a scaled copy of $G$ with $\rho(G) = x$ is simply obtained by replacing $g$ with $x$:

$$\gamma_G(x,\varepsilon) = \kappa_0(\varepsilon)x^0 + \kappa_1(\varepsilon)x^1 = 3^{3/2}\left(-\varepsilon^2 + 2\varepsilon x\right), \tag{7.12}$$

$$\mu_2(xG) = \kappa_2(\varepsilon)x^2 = 3^{3/2}x^2. \tag{7.13}$$

For a given tile, $x$ is fixed as $\varepsilon \to 0^+$ (since $x$ represents the inradius of the tile) and it is clear that the expressions (7.12) and (7.13) coincide when $\varepsilon = x$. Thus we have

$$\begin{aligned} g^s\zeta_{\mathfrak{s}}(s) &= \frac{g^s}{1 - 2\cdot 3^{-s/2}} \\ \boldsymbol{\kappa}(\varepsilon) &= [\kappa_0,\kappa_1,\kappa_2] = 3^{3/2}\,[-1,2,-1], \\ \mathcal{E}(\varepsilon,s) &= \left[\frac{1}{s},\frac{1}{s-1},\frac{1}{s-2}\right]\varepsilon^{2-s}. \end{aligned}$$

Now applying (6.39), the tube formula for the Koch tiling $\mathcal{K}$ is

$$\begin{aligned} V_{\mathcal{K}}(\varepsilon) &= 3^{3/2}g^2 \sum_{\omega\in\mathcal{D}_{\mathfrak{s}}} \operatorname{res}\left(\frac{1}{1-2\cdot 3^{-s/2}};\omega\right)\left(-\tfrac{1}{\omega} + \tfrac{2}{\omega-1} - \tfrac{1}{\omega-2}\right)\left(\tfrac{\varepsilon}{g}\right)^{2-\omega} \\ &\qquad + \tfrac{g}{2}\zeta_{\mathfrak{s}}(0)\operatorname{res}\left(-\tfrac{1}{s};0\right)\left(\tfrac{\varepsilon}{g}\right)^{2-0} + \tfrac{g}{2}\zeta_{\mathfrak{s}}(1)\operatorname{res}\left(\tfrac{2}{s-1};1\right)\left(\tfrac{\varepsilon}{g}\right)^{2-1} \\ &= \frac{g}{\log 3}\sum_{n\in\mathbb{Z}}\left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right)\left(\tfrac{\varepsilon}{g}\right)^{2-D-\mathbb{i}n\mathbf{p}} \\ &\qquad + 3^{3/2}\varepsilon^2 + \tfrac{1}{1-2\cdot 3^{-1/2}}\varepsilon, \end{aligned} \tag{7.14}$$

where $D = \log_3 4$ and $\mathbf{p} = \frac{4\pi}{\log 3}$ as before. The last equality in (7.14) comes by observing that $g = \frac{\sqrt{3}}{18} = \frac{3^{1/2}}{2\cdot 3^2} = \frac{1}{2}3^{-3/2}$, so we have $3^{3/2}g^2 = \frac{g}{2}$, and then $\varepsilon^2/2g = 3^{3/2}\varepsilon^2$.

Remark 7.3. In [**LaPe1**], a tube formula was obtained for the actual inner $\varepsilon$-neighbourhood of the Koch curve (rather than of the tiling associated with it) and the possible complex dimensions of this curve were inferred to be

$$\mathcal{D}_{\mathcal{K}^\star} = \{D + \mathbb{i}n\mathbf{p} : n\in\mathbb{Z}\} \cup \{0 + \mathbb{i}n\mathbf{p} : n\in\mathbb{Z}\},$$

where $D = \log_3 4$ and $\mathbf{p} = \frac{2\pi}{\log 3}$. It is pleasing to see that $\mathcal{D}_{\mathcal{K}}$ is a subset of this. We note that we did not define a zeta function for the Koch curve prior to the present paper. In both [**La-vF1**, §10.3] and [**LaPe1**], one reasoned by analogy with the tube formula (5.8) to deduce the possible complex dimensions. In the present work, however, we define both the scaling and tiling zeta functions, and then define the complex dimensions as the poles of the scaling zeta function.

Remark 7.4 (Nonlattice Koch tilings). By replacing $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$ in (7.8) with any other complex number satisfying

$$|\xi|^2 + |1-\xi|^2 < 1,$$

we can easily construct family of examples of nonlattice self-similar tilings. The computation of the tube formula parallels that of the lattice case, almost identically. The lattice Koch tilings correspond to those $\xi \in B(\frac{1}{2},\frac{1}{2})$ (the ball of radius $\frac{1}{2}$ centered at $\frac{1}{2}\in\mathbb{C}$) for which $\log_r|\xi|$ and $\log_r|1-\xi|$ are both positive integers, for some fixed $0 < r < 1$. Any other choice of $\xi \in B(\frac{1}{2},\frac{1}{2})$ will produce a nonlattice tiling. See [**Pe1**] for further discussion (and illustrations) of nonlattice Koch tilings.

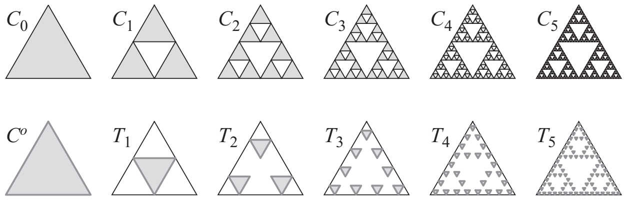


FIGURE 4. The Sierpinski gasket tiling.

## 7.3. The Sierpinski Gasket tiling

The Sierpinski gasket tiling $\mathcal{SG}$ (see Fig. 4) is constructed via the self-similar system

$$\Phi_1(z) := \tfrac{1}{2}z, \quad \Phi_2(z) := \tfrac{1}{2}z + \tfrac{1}{2}, \quad \Phi_3(z) := \tfrac{1}{2}z + \tfrac{1+\mathbb{i}\sqrt{3}}{4}.$$

Thus we have one scaling ratio $r = 1/2$, and one generator $G$ which is an equilateral triangle with generating inradius $g = \frac{1}{4\sqrt{3}}$.

This tiling has inradii $\rho_k = gr^k$ with multiplicity $3^k$ for $k = 0, 1, 2, \ldots$, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 3 \cdot 2^{-s}}, \tag{7.15}$$

with complex dimensions

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_2 3,\ \mathbf{p} = \tfrac{2\pi}{\log 2}. \tag{7.16}$$

Except for $\zeta_{\mathfrak{s}}(s)$, the calculation for the tube formula for the Sierpinski tiling $\mathcal{SG}$ is just like that for the Koch tiling, so we omit the details and give the result:

$$\begin{aligned} V_{\mathcal{SG}}(\varepsilon) &= 3^{3/2} \sum_{\omega \in \mathcal{D}_{\mathcal{SG}}} \operatorname{res}\left( \frac{g^s}{1 - 3 \cdot 2^{-s}} \left( -\tfrac{1}{s} + \tfrac{2}{s-1} - \tfrac{1}{s-2} \right) \varepsilon^{2-s}; \omega \right) \\ &= \tfrac{\sqrt{3}}{16 \log 2} \sum_{n \in \mathbb{Z}} \left( -\tfrac{1}{D + \mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}} \right) \left( \tfrac{\varepsilon}{g} \right)^{2-D-\mathbb{i}n\mathbf{p}} \\ &\qquad + \tfrac{3^{3/2}}{2} \varepsilon^2 - 3\varepsilon. \end{aligned} \tag{7.17}$$

In fact, a similar formula can be obtained for higher-dimensional analogues of the Sierpinski gasket, where the generator is a simplex instead of a triangle. The computations for the Sierpinski carpet, and its higher-dimensional analogue (the Menger sponge) are also extremely similar. In each case, the primary complication is to obtain the tube formula for the generator.

## 7.4. The Pentagasket tiling

The Pentagasket tiling $\mathcal{P}$ (see Fig. 5) is constructed via the self-similar system defined by the five maps

$$\Phi_j(x) = \tfrac{3-\sqrt{5}}{2}x + p_j, \qquad j = 1, \ldots, 5,$$

with common scaling ratio $\phi^{-2}$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio:

$$r = \tfrac{3-\sqrt{5}}{2} = \left( \tfrac{\sqrt{5}-1}{2} \right)^2 = \phi^{-2},$$

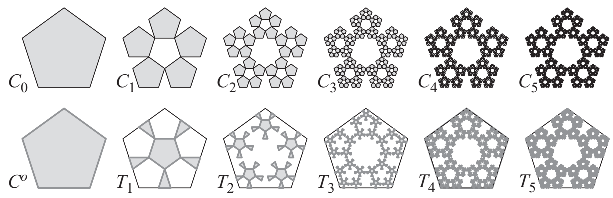


FIGURE 5. The Pentagasket tiling.

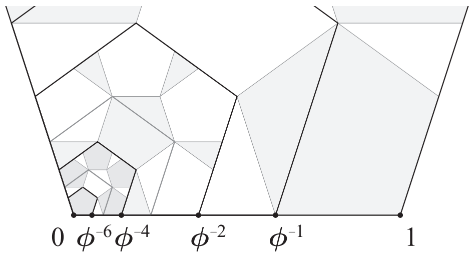


FIGURE 6. The pentagasket and the golden ratio $\phi$.

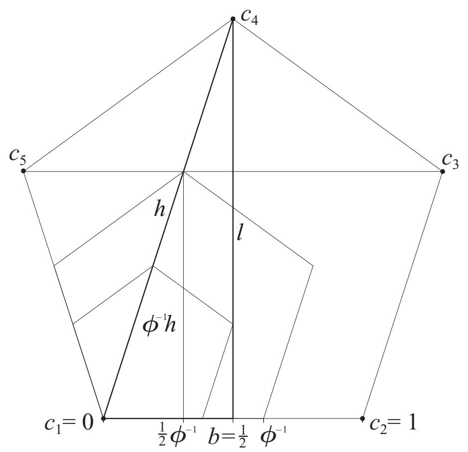


FIGURE 7. The vertices of the pentagasket.

and the points $\frac{p_j}{1-r} = c_j$ form the vertices of a regular pentagon of side length 1. See Fig. 6 and Fig. 7. Then $\Phi_j(c_j) = rc_j + c_j(1-r) = c_j$ shows that $c_j$ is the fixed point of $\Phi_j$, for $j = 1, \ldots, 5$.

The Pentagasket is the first example of multiple generators $G_q$. In fact, in the notation of (2.7), we have $T_1 = G_1 \cup \cdots \cup G_6$ where $G_1$ is a regular pentagon and $G_2, \ldots, G_6$ are congruent isosceles triangles (see Fig. 8). There are two distinct generating inradii:

$$g_1 = \frac{\phi^2}{2} \tan \frac{3}{10}\pi, \text{ and}$$
$$g_2 = \cdots = g_6 = \frac{\phi^3}{2} \tan \frac{\pi}{5}. \tag{7.18}$$

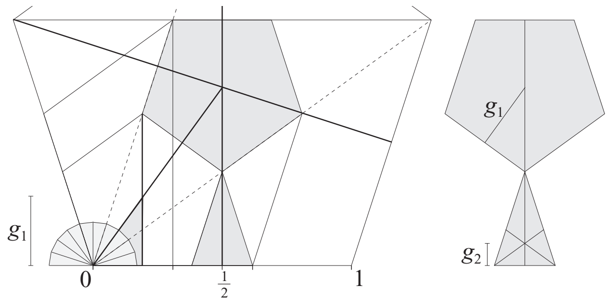


FIGURE 8. The generator of the pentagasket tiling.

We omit the exercise of finding volumes for the pentagonal and triangular generators and simply give the results:

$$V_p(\rho) = 5\cot\left(\tfrac{3}{10}\pi\right)\rho^2 = \alpha_1\rho^2, \tag{7.19}$$

where $\alpha_1 := 5\cot\frac{3}{10}\pi$, and

$$V_t(\rho) = \frac{\cot\frac{\pi}{5}}{\left(1-\tan^2\frac{\pi}{5}\right)}\rho^2 = \alpha_2\rho^2 \tag{7.20}$$

where $\alpha_2 := (\cot\frac{\pi}{5})/\left(1-\tan^2\frac{\pi}{5}\right)$.

The pentagasket tiling has inradii $\rho_k = g_q r^k$, for $q = 1, 2$, with multiplicity $5^k$ for $k = 0, 1, 2, \ldots$, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 5\cdot r^{-s}}, \tag{7.21}$$

with complex dimensions

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_{1/r} 5,\ \mathbf{p} = \tfrac{2\pi}{\log r^{-1}}. \tag{7.22}$$

As in previous examples, for $\varepsilon < g$ we have

$$V_{G_q}(\varepsilon) = \mu_2(G_q) - \mu_2((1-\varepsilon)G_q) = \alpha_q(2\varepsilon - \varepsilon^2). \tag{7.23}$$

We find the generator tube formulas

$$\gamma_q(x,\varepsilon) = \kappa_{q0}(\varepsilon)x^0 + \kappa_{q1}(\varepsilon)x^1 = \alpha_q\left(-\varepsilon^2 + 2\varepsilon x\right),$$
$$\mu_{q2}(G) = \kappa_{q2}(\varepsilon)x^2 = \alpha_q x^2.$$

REMARK 7.5. Since $G_2, \ldots, G_6$ are congruent, we can avoid writing a $6\times 3$ matrix, and instead write a $2\times 3$ matrix, by multiplying the contribution from one triangle by 5.

Thus we have the matrices

$$\mathfrak{g}(s)\zeta_{\mathfrak{s}}(s) = [g_1^s \ \ 5g_2^s]\zeta_{\mathfrak{s}}(s)$$
$$\boldsymbol{\kappa}(\varepsilon) = \begin{bmatrix} -\alpha_1 & 2\alpha_1 & -\alpha_1 \\ -\alpha_2 & 2\alpha_2 & -\alpha_2 \end{bmatrix},$$
$$\mathcal{E}(\varepsilon,s) = \left[\frac{1}{s}, \frac{1}{s-1}, \frac{1}{s-2}\right]\varepsilon^{2-s}.$$

The geometric zeta function of the tiling is

$$\zeta_{\mathcal{P}}(\varepsilon, s) = \sum_{q=1}^{6} \frac{\alpha_q g_q^s}{1 - 5 \cdot r^{-s}} \left(-\tfrac{1}{s} + \tfrac{2}{s-1} - \tfrac{1}{s-2}\right) \varepsilon^{2-s}.$$

Hence the tube formula for the Pentagasket tiling $\mathcal{P}$ is

$$\begin{aligned} V_{\mathcal{P}}(\varepsilon) &= \sum_{q=1}^{6} \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \operatorname{res}\left(\frac{\alpha_q g_q^2}{1 - 5 \cdot r^{-s}}; \omega\right) \left(-\tfrac{1}{\omega} + \tfrac{2}{\omega-1} - \tfrac{1}{\omega-2}\right) \left(\tfrac{\varepsilon}{g_q}\right)^{2-\omega} \\ &\quad + \sum_{q=1}^{6} \alpha_q g_q^2 \left[\zeta_{\mathfrak{s}}(0) \operatorname{res}\left(-\tfrac{1}{s}; 0\right) \left(\tfrac{\varepsilon}{g_q}\right)^2 + \zeta_{\mathfrak{s}}(1) \operatorname{res}\left(\tfrac{2}{s-1}; 1\right) \left(\tfrac{\varepsilon}{g_q}\right)\right] \\ &= \frac{\alpha_1}{\log r^{-1}} \sum_{n \in \mathbb{Z}} g_1^2 \left(-\tfrac{1}{D + \mathbb{i} n \mathbf{p}} + \tfrac{2}{D - 1 + \mathbb{i} n \mathbf{p}} - \tfrac{1}{D - 2 + \mathbb{i} n \mathbf{p}}\right) \left(\tfrac{\varepsilon}{g_1}\right)^{2 - D - \mathbb{i} n \mathbf{p}} \\ &\quad + \frac{5\alpha_2}{\log r^{-1}} \sum_{n \in \mathbb{Z}} g_2^2 \left(-\tfrac{1}{D + \mathbb{i} n \mathbf{p}} + \tfrac{2}{D - 1 + \mathbb{i} n \mathbf{p}} - \tfrac{1}{D - 2 + \mathbb{i} n \mathbf{p}}\right) \left(\tfrac{\varepsilon}{g_2}\right)^{2 - D - \mathbb{i} n \mathbf{p}} \\ &\quad + \left[\left(\tfrac{\alpha_1}{4} + \tfrac{5\alpha_2}{4}\right) \varepsilon^2 + \tfrac{(2\alpha_1 g_q + 10\alpha_1 g_q r) r}{r-5} \varepsilon\right], \end{aligned} \tag{7.24}$$

with $r = \phi^{-2}$, $\alpha_1 = 5 \cot \frac{3}{10}\pi$, $\alpha_2 = (\cot \frac{\pi}{5}) / \left(1 - \tan^2 \frac{\pi}{5}\right)$, $g_1$ and $g_2$ given by (7.18), $D = \log_{1/r} 5$ and $\mathbf{p} = \frac{2\pi}{\log r^{-1}}$ as before.

CHAPTER 8

# Concluding Remarks

## 8.1. Observations

**8.1.1.** In [**La-vF4**], a new definition of fractality is proposed; it states that the presence of complex dimensions characterize an object as being fractal. More specifically, [**La-vF4**] states that a fractal is an object with nonreal complex dimensions that have a positive real part. In this sense, the present work confirms the fractal nature of all the examples discussed in §7, and more generally, of all self-similar tilings considered in this paper.

**8.1.2.** Thm. 6.8 and its corollaries for self-similar tilings (especially Thm. 6.13 and Cor. 6.15) provide a fractal analogue of the classical Steiner formula and a higher-dimensional analogue of the tube formula for fractal strings (5.8) obtained in [**La-vF4**]. The present work can be considered as a further step towards a higher-dimensional theory of fractals, especially in the self-similar case. A first step was taken in [**La-vF1**, §10.2 and §10.3]. A second step (in the same spirit, but with more precise results) was taken in our earlier paper [**LaPe1**] as discussed in Rem. 7.3.

In [**LaPe1**], the emphasis was on obtaining an inner tube formula for the self-similar set itself (in that case, the Koch curve), rather than for the associated self-similar tiling or system. Moreover, because the geometry of the $\varepsilon$-neighbourhoods and hence the resulting computations were very complicated, the coefficients of the tube formula could not be explicitly calculated. More precisely, they could only be expressed in terms of the Fourier coefficients of a certain periodic function. Here, in contrast to [**LaPe1**], the scaling and tiling zeta functions are defined and used to obtain the explicit tube formula for a self-similar tiling. One may then obtain the complex dimensions directly from these zeta functions.

**8.1.3.** The geometry of the generators of a self-similar tiling is much more complicated in dimensions greater than 1. However, it also sheds new light on the 1-dimensional tube formula. In particular, as was mentioned in §6.3, it reveals the geometric interpretation of the second sum in (5.3) as a form of Steiner's formula, and how this sum may be naturally included in the first sum, as in (6.33) and (6.49).

**8.1.4.** Despite the fact that our tube formulas are for the $\varepsilon$-neighbourhoods of self-similar *tilings* rather than of self-similar *sets*, they give us valuable information about self-similar geometries (and their associated dynamical systems). Indeed, given a self-similar set in $\mathbb{R}^d$, we can define its complex dimensions as those of the self-similar tiling canonically associated to it as in [**Pe1**]. This is accomplished by turning to the self-similar system which defines the given self-similar set, and focusing on the dynamical system induced by the self-similar system. For example,

§7.2 shows how the complex dimensions of the Koch curve really depend on the self-similar system $\Phi$.

**8.1.5.** By using the inner $\varepsilon$-neighbourhoods of the generators, we believe that the curvature coefficients $c_\omega$ appearing in the tube formula

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} c_\omega \varepsilon^{d-\omega} \tag{8.1}$$

are intrinsic to the self-similar tiling. This should be the case, provided the curvature matrix $\boldsymbol{\kappa}$ is also intrinsic (i.e., does not depend on the embedding of $\mathcal{T}$ in the ambient space $\mathbb{R}^d$). Hermann Weyl also gave a tube formula for smooth submanifolds of $\mathbb{R}^d$ in [**We**], expressed as a polynomial in $\varepsilon$ with coefficients defined in terms of curvatures that are intrinsic to the submanifold. See [**BeGo**, §6.6–6.9] and [**Gr**]. This was further extended by Herbert Federer (in [**Fed**]) who unified Weyl's tube formula for manifolds with Steiner's formula for convex sets. In particular, he accomplished this task by introducing the notion of "sets of positive reach" and appropriate corresponding curvature measures. We plan to examine this issue in [**LaPe2**] and later work [**LaPe3**] connected to possible 'generalized curvature measures' associated to each complex and integral dimension.

**8.1.6.** Many classical fractal curves are attractors of more than one self-similar system. For example, the Koch curve discussed in §7.2 is also the attractor of a system of 4 mappings, each with scaling ratio $r = \frac{1}{3}$. In this particular example, changes in the scaling zeta function produce a different set of complex dimensions. In fact, we obtain a subset of the original complex dimensions: $\{\log_3 4 + \mathbb{i} n \mathbf{p} : n \in \mathbb{Z}, \mathbf{p} = 4\pi/\log 3\}$. This has a natural geometric interpretation which is to be discussed in later work. In particular, it would be desirable to determine precisely which characteristics remain invariant between different tilings which are so related.

## 8.2. Acknowledgements

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APPENDIX A

# Languid and Strongly Languid

The following definitions are excerpted from [**La-vF4**, §5.3]. The technical details described here are used in the proof of Thm. 6.8, especially in Appendix B and Appendix C.

Definition A.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuous function. Then the *screen* is $S = \{f(t) + \mathbb{i}t : t \in \mathbb{R}\}$, the graph of a function with the axes interchanged. We let

$$\inf S := \inf_t f(t) = \inf\{\operatorname{Re} s : s \in S\}, \text{ and} \tag{A.1}$$

$$\sup S := \sup_t f(t) = \sup\{\operatorname{Re} s : s \in S\}. \tag{A.2}$$

The screen is thus a vertical contour in $\mathbb{C}$. The region to the right of the screen is the set $W$, called the *window*:

$$W := \{z \in \mathbb{C} : \operatorname{Re} z \geq f(\operatorname{Im} z)\}. \tag{A.3}$$

Definition A.2. The generalized fractal string $\eta$ (as in Def. 5.1) is said to be *languid* if its associated zeta function $\zeta_\eta$ satisfies certain growth conditions.[1] Specifically, let $\{T_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ such that $T_{-n} < 0 < T_n$ for $n \geq 1$, and

$$\lim_{n\to\infty} T_n = \infty, \lim_{n\to\infty} T_{-n} = -\infty, \text{ and } \lim_{n\to\infty} \frac{T_n}{|T_{-n}|} = 1. \tag{A.4}$$

For $\eta$ to be languid, there must exist real constants $M, c > 0$ and a sequence $\{T_n\}$ as described in (A.4), such that

**L1** For all $n \in \mathbb{Z}$ and all $\sigma \geq f(T_n)$,

$$|\zeta_\eta(\sigma + \mathbb{i}T_n)| \leq c \cdot (|T_n| + 1)^M, \text{ and} \tag{A.5}$$

**L2** For all $t \in \mathbb{R}$, $|t| \geq 1$,

$$|\zeta_\eta(f(t) + \mathbb{i}t)| \leq c \cdot |t|^M. \tag{A.6}$$

In this case, $\eta$ is said to be *languid of order $M$*.

Definition A.3. The generalized fractal string $\eta$ is said to be *strongly languid* if it satisfies **L1** and the condition **L2'**, which is clearly stronger than **L2**:

**L2'** There exists a sequence of screens $S_m(t) = f_m(t) + \mathbb{i}t$ for $m \geq 1$, $t \in \mathbb{R}$, with $\sup S_m \to -\infty$ as $m \to \infty$, and with a uniform Lipschitz bound, for which there exist constants $a, c > 0$ such that

$$|\zeta_\eta(f(t) + \mathbb{i}t)| \leq c \cdot a^{|f_m(t)|}(|t| + 1)^M, \tag{A.7}$$

for all $t \in \mathbb{R}$ and $m \geq 1$.

[1] We take $\zeta_\eta$ to be meromorphically continued to an open neighbourhood of $W$, as in Def. 5.3.

APPENDIX B

# The Definition and Properties of $\zeta_{\mathcal{T}}$

In this section, we confirm some basic properties of the geometric zeta function of a tiling $\zeta_{\mathcal{T}}$. However, we first require some facts about Mellin transformation. If $\varphi \in \mathbb{D} = C_c^\infty(0,\infty)$, it is elementary to check that for every $s \in \mathbb{C}$, the Mellin transform $\widetilde{\varphi}(s)$ is given by the well-defined integral (5.6) and satisfies $|\widetilde{\varphi}(s)| \leq \widetilde{|\varphi|}(\operatorname{Re} s) < \infty$. We will need additional estimates in what follows. We will also use the forthcoming fact that $\widetilde{\varphi}(s)$ is an entire function. This remains true when $\varphi$ is replaced by $h\varphi$, for any bounded function $h$.

Lemma B.1. *Suppose that $S \subseteq \mathbb{C}$ is horizontally bounded, so that $\inf_S \operatorname{Re} s$ and $\sup_S \operatorname{Re} s$ are finite. Let $K$ be a compact interval containing the support of $\varphi \in C_c^\infty(0,\infty)$. Then there is a constant $c_K > 0$ depending only on $K$ such that*

$$\sup_{s\in S} |\widetilde{\varphi}(s)| \leq c_K \|\varphi\|_\infty. \tag{B.1}$$

*This is the case, in particular, if $S$ is a screen as in Def. A.1. In* (B.1), *$\|\varphi\|_\infty$ is the supremum norm of $\varphi$.*

Proof. Let $K$ be a compact interval containing the support of $\varphi$. Since

$$|x^{s-1}| = x^{\operatorname{Re} s-1} \leq \begin{cases} x^{\sup S-1}, & x \geq 1 \\ x^{\inf S-1}, & 0 < x < 1, \end{cases} \tag{B.2}$$

one can define a bound

$$b_K := \sup_{x\in K} \max\{x^{\sup S-1}, x^{\inf S-1}\}.$$

Note that $b_K$ is finite because the function $x \mapsto \max\{x^{\sup S-1}, x^{\inf S-1}\}$ is continuous on the compact set $K$, and hence is bounded. Then we use (B.2) to bound $\widetilde{\varphi}$ as follows:

$$\begin{aligned} |\widetilde{\varphi}(s)| &\leq \int_0^\infty |x^{s-1}| \cdot |\varphi(x)|\, dx \\ &= \int_K x^{\operatorname{Re} s-1} |\varphi|(x)\, dx = \widetilde{|\varphi|}(\operatorname{Re} s) \qquad \text{(B.3)} \\ &\leq b_K \|\varphi\|_\infty \cdot \mathit{length}(K). \end{aligned}$$

□

Corollary B.2. *If $h$ is a bounded measurable function on $(0,\infty)$, and $\varphi$ and $K$ are as in Lemma B.1, then*

$$\sup_{s\in S} |\widetilde{h\varphi}(s)| \leq c_K \|h\|_\infty \|\varphi\|_\infty, \tag{B.4}$$

*where $c_K > 0$ depends only on $K$. In particular, $\widetilde{\kappa_{qi}\varphi}$ is always uniformly bounded on the screen $S$.*

PROOF. The argument is identical to that of Lemma B.1. For each $i = 0, \dots, d$, $\kappa_{qi}(\varepsilon)$ is bounded for $\varepsilon \leq g$ and constant for $\varepsilon > g$ (see Def. 4.1). Thus $\kappa_{qi}$ is globally bounded and the corollary applies. □

REMARK B.3. The exact counterpart of Lemma B.1 and Cor. B.2 holds if $\widetilde{\varphi}(s)$ or $\widetilde{h\varphi}(s)$ is replaced by a translate $\widetilde{\varphi}(s-s_0)$ or $\widetilde{h\varphi}(s-s_0)$, for any $s_0 \in \mathbb{C}$. Therefore, under the same assumptions as in Cor. B.2, we have

$$\sup_{s\in S} |\widetilde{h\varphi}(s-s_0)| \leq c_{K,s_0} \|h\|_\infty \|\varphi\|_\infty, \tag{B.5}$$

where $c_{K,s_0} := b_{K,s_0} \cdot length(K)$, and

$$b_{K,s_0} := \sup_{x\in K} \max\{x^{\sup S - \operatorname{Re} s_0 - 1}, x^{\inf S - \operatorname{Re} s_0 - 1}\} < \infty. \tag{B.6}$$

In particular, for any compact interval $K$ containing the support of $\varphi$, and for each fixed integer $k \geq 0$,

$$\sup_{s\in S} |\widetilde{\kappa_{qi}\varphi}(s-d+k+1)| \leq c_{K,k} \|\varphi\|_\infty, \tag{B.7}$$

where $c_{K,k}$ is a finite and positive constant, independent of $q$ and $i$.

LEMMA B.4. *Let $(X, \mu)$ be a measure space. Define an integral transform by $F(s) = \int_X f(x,s)\, d\mu(x)$ where*

$$|f(x,s)| \leq G(x), \quad \text{for } \mu\text{-a.e. } x \in X,$$

*for some $G \in L^1(X, \mu)$, and for all $s$ in some neighbourhood of $s_0 \in \mathbb{C}$. If the function $s \mapsto f(x,s)$ is holomorphic for $\mu$-a.e. $x \in X$, then $F(s)$ is well-defined and holomorphic at $s_0$.*

The proof is a well-known application of Lebesgue's Dominated Convergence Theorem. We use Lemma B.4 and Cor. B.2 to obtain the following corollary, which is used to prove Thm. 6.8 and Thm. B.8.

COROLLARY B.5. *For $\varphi \in C_c^\infty(0,\infty)$, $\widetilde{\varphi}(s)$ is entire. Further, if $h(x)$ is a bounded measurable function, then $\widetilde{h\varphi}(s)$ is also entire. In particular, $\widetilde{\kappa_{qi}\varphi}(s)$ is entire for all $q = 1, \dots, Q$ and $i = 0, \dots, d$.*

PROOF. Fix $s_0 \in \mathbb{C}$. If $s$ is in a compact neighbourhood of $s_0$, then $\operatorname{Re} s$ is bounded, say by $\alpha \in \mathbb{R}$. Then for almost every $x > 0$,

$$\left|x^{s-1} h(x)\varphi(x)\right| \leq x^{\alpha-1} \|h\|_\infty \|\varphi\|_\infty \chi_\varphi, \tag{B.8}$$

where $\chi_\varphi$ is the characteristic function of the compact support of $\varphi$. Upon application of Lemma B.4, one deduces that $\varphi$ is holomorphic at $s_0$. □

Note that this does not combine with Lemma B.1 (or Cor. B.2) to imply that $\widetilde{\varphi}$ (or $\widetilde{h\varphi}$) is constant; indeed, Liouville's Theorem does not apply because $s$ is restricted to $S$ in these two propositions.

DEFINITION B.6. For $T(\varepsilon, s)$ to be a *weakly meromorphic distribution-valued function* on $W$, there must exist

(i) a discrete set $\mathcal{P}_T \subseteq \mathbb{C}$, and
(ii) for each $\omega \in \mathcal{P}_T$, an integer $n_\omega < \infty$,

such that $\Psi(s) = \langle T(\varepsilon, s), \varphi(\varepsilon)\rangle$ is a meromorphic function of $s \in W$, and each pole $\omega$ of $\Psi$ lies in $\mathcal{P}_T$ and has multiplicity at most $n_\omega$.

To say that the distribution-valued function $T : W \to \mathbb{D}'$ given by $s \mapsto T(\varepsilon, s)$ is *(strongly) meromorphic* means that, as a $\mathbb{D}'$-valued function, it is truly a meromorphic function, in the sense of the proof of Lemma B.7. Recall that we are working with the space of distributions $\mathbb{D}'$, defined as the dual of the space of test functions $\mathbb{D} = C_c^\infty(0, \infty)$.

LEMMA B.7. *If $T$ is a weakly meromorphic distribution-valued function, then it is a meromorphic distribution-valued function.*

PROOF. For $\omega \notin \mathcal{P}_T$, note that as $s \to \omega$,

$$\frac{T(\varepsilon, s) - T(\varepsilon, \omega)}{s - \omega} \tag{B.9}$$

converges to a distribution (call it $T'(\varepsilon, \omega)$) in $\mathbb{D}'$, by the Uniform Boundedness Principle for a topological vector space such as $\mathbb{D}$; see [**Rud**, Thm. 2.5 and Thm. 2.8]. Hence, the $\mathbb{D}'$-valued function $T$ is holomorphic at $\omega$.

For $\omega \in \mathcal{P}_T$, apply the same argument to

$$\lim_{s\to\omega} \frac{1}{(n_\omega - 1)!} \left(\frac{d}{ds}\right)^{n_\omega - 1} \Big((s-\omega)^{n_\omega} T(\varepsilon, s)\Big), \tag{B.10}$$

which must therefore define a distribution, i.e., exist as an element of $\mathbb{D}'$. Thus $T$ is truly a meromorphic function with values in $\mathbb{D}'$, and with poles contained in $\mathcal{P}_T$. □

THEOREM B.8. *Under the hypothesis of Thm. 6.8 or Thm. 6.13, the geometric zeta function of a fractal spray or tiling*

$$\begin{aligned} \zeta_{\mathcal{T}}(\varepsilon, s) &= \sum_{i=0}^{d} \sum_{q=1}^{Q} g_q^s \zeta_\eta(s) \frac{\varepsilon^{d-s}}{s-i} \kappa_{qi}(\varepsilon) \\ &= \langle \mathfrak{g}(s)\zeta_\eta(s), \mathcal{E}(\varepsilon, s)\rangle_{\boldsymbol{\kappa}(\varepsilon)} \end{aligned} \tag{B.11}$$

*is a (strongly) distribution-valued meromorphic function on $W$, with poles contained in $\mathcal{D}_{\mathcal{T}}$.*

PROOF. Let $\mathcal{P}_T = \mathcal{D}_{\mathcal{T}}$ and note that

$$\begin{aligned} \langle \zeta_{\mathcal{T}}(\varepsilon, s), \varphi(\varepsilon)\rangle_{\boldsymbol{\kappa}} &= \sum_{q,i} \int_0^\infty g_q^s \zeta_\eta(s) \frac{\varepsilon^{d-s}}{s-i} \kappa_{qi}(\varepsilon)\varphi(\varepsilon)\, d\varepsilon \\ &= \sum_{q,i} g_q^s \zeta_\eta(s) \frac{\widetilde{\kappa_{qi}\varphi}(d-s+1)}{s-i}. \end{aligned} \tag{B.12}$$

By Cor. B.5, this is a finite sum of meromorphic functions and hence meromorphic on $W$, for any test function $\varphi$. Applying Lemma B.7, one sees that $\zeta_{\mathcal{T}}$ is a meromorphic function with values in $\mathbb{D}'$. □

REMARK B.9. Note that for each $\varphi \in \mathbb{D}$, the poles of the $\mathbb{C}$-valued function

$$s \mapsto \langle \zeta_{\mathcal{T}}(\varepsilon, s), \varphi(\varepsilon)\rangle \tag{B.13}$$

are contained in $\mathcal{D}_{\mathcal{T}}$. Further, if $m_\omega$ is the multiplicity of $\omega \in \mathcal{D}_{\mathcal{T}}$ as a pole of $\zeta_\eta(s)$, then the multiplicity of $\omega$ as a pole of (B.13) is bounded by $m_\omega + 1$.

Corollary B.10. *The residue of $\zeta_{\mathcal{T}}$ at a pole $\omega \in \mathcal{D}_{\mathcal{T}}$ is a well-defined distribution.*

Proof. This follows immediately from the second part of the proof of Lemma B.7, with $\mathcal{P}_T = \mathcal{D}_{\mathcal{T}}$. $\square$

Corollary B.11. *The sum of residues appearing in Thm. 6.8 and Thm. 6.13 is distributionally convergent, and is thus a well-defined distribution.*

Proof. In view of the proof of Thm. B.8, this comes by applying the Uniform Boundedness Principle to an appropriate sequence of partial sums, in a manner similar to the proof of Lemma B.7. Again, see [**La-vF4**, Rem. 5.21]. $\square$

APPENDIX C

# The Error Term and Its Estimate

As promised, we give a proof of the expression for the error term and its estimate, as stated in Thm. 6.8. First, we require a definition.

Definition C.1 (Primitives of distributions). Let $T_\eta$ be a distribution defined by a measure as

$$\langle T_\eta, \varphi \rangle := \int \varphi \, d\eta.$$

Then the $k^{\text{th}}$ *primitive* (or $k^{\text{th}}$ antiderivative) of $T_\eta$ is defined by

$$\langle T_\eta^{[k]}, \varphi \rangle := (-1)^k \langle T_\eta, \varphi^{[k]} \rangle, \tag{C.1}$$

where $\varphi^{[k]}$ is the $k^{\text{th}}$ primitive of $\varphi \in C_c^\infty(0,\infty)$ that vanishes at $\infty$ together with all its derivatives. For $k \geq 1$, for example,

$$\langle T_\eta^{[k]}, \varphi \rangle = \int_0^\infty \int_y^\infty \frac{(x-y)^{k-1}}{(k-1)!} \varphi(x) \, dx \, d\eta(y). \tag{C.2}$$

Theorem C.2. *The Mellin transform of the $k^{th}$ primitive of a test function is given by $\widetilde{\varphi^{[k]}}(s) = \tilde{\varphi}(s+k)\psi_k(s)$, where $\psi_k$ is the meromorphic function*

$$\psi_k(s) := \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!(s+j)}. \tag{C.3}$$

Proof. By direct computation,

$$\begin{aligned}
\widetilde{\varphi^{[k]}}(s) &= \int_0^\infty \varepsilon^{s-1} \int_\varepsilon^\infty \frac{(x-\varepsilon)^{k-1}}{(k-1)!} \varphi(x) \, dx \, d\varepsilon \\
&= \frac{1}{(k-1)!} \int_0^\infty \int_\varepsilon^\infty \sum_{j=0}^{k-1} \binom{k-1}{j} x^{k-1-j} (-\varepsilon)^j \varepsilon^{s-1} \varphi(x) \, dx \, d\varepsilon \\
&= \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!} \int_0^\infty \int_\varepsilon^\infty x^{k-1-j} \varepsilon^{s+j-1} \varphi(x) \, dx \, d\varepsilon \\
&= \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!} \int_0^\infty x^{k-1-j} \varphi(x) \int_0^x \varepsilon^{s+j-1} \, d\varepsilon \, dx \\
&= \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!(s+j)} \int_0^\infty x^{s+k-1} \varphi(x) \, dx \qquad \text{(C.4)} \\
&= \tilde{\varphi}(s+k)\psi_k(s).
\end{aligned}$$

Again, the formula (C.3) for $\psi_k$ is valid for $\operatorname{Re} s > k$ by (C.4), but then extends to being valid for $s \in \mathbb{C}$ by meromorphic continuation. □

COROLLARY C.3. *We also have* $\left|\widetilde{\kappa_{qi}\varphi^{[k]}}(s)\right| \leq |\widetilde{\kappa_{qi}\varphi}(s+k)\psi_k(s)|$.

PROOF. Repeat the proof of Thm. C.2 with $\kappa_{qi}$ in the integrand, or follow (C.17) below, with $a = 1$. □

REMARK C.4. For $s \in S$, we also have

$$|\psi_k(s)| \leq \frac{c_\psi}{|t|^k}, \quad \text{for } t = \operatorname{Im} s \text{ and } c_\psi > 0. \tag{C.5}$$

We are now in a position to provide the proofs previously promised.

THEOREM C.5. *As stated in* (6.11) *of Thm. 6.8, the error term is given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_S \zeta_{\mathcal{T}}(\varepsilon, s)\, ds, \tag{C.6}$$

*and is a well-defined distribution.*

PROOF. Applying (5.6) to (6.31) for $i = 0, \ldots, d$ gives[1]

$$\langle \mathcal{R}, \varphi \rangle_{qi} = \frac{1}{2\pi \mathbb{i}} \int_S g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} \varphi(\varepsilon)\, d\varepsilon\, ds. \tag{C.7}$$

To see that this gives a well-defined distribution $\mathcal{R}$, we apply the descent method, as described in [**La-vF4**, Rem. 5.20]. The first step is to show that $\left\langle \mathcal{R}^{[k]}, \varphi \right\rangle_{qi}$ is a well-defined distribution for sufficiently large $k$; specifically, for any integer $k > M$, where $M$ is as in Def. A.2. Note that we can break the integral along the screen $S$ into two pieces and work with each separately:

$$\left\langle \mathcal{R}^{[k]}, \varphi \right\rangle_{qi} = \frac{(-1)^k}{2\pi \mathbb{i}} \int_{|\operatorname{Im} s|>1} g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} \varphi^{[k]}(\varepsilon)\, d\varepsilon\, ds \tag{C.8}$$

$$+ \frac{(-1)^k}{2\pi \mathbb{i}} \int_{|\operatorname{Im} s|\leq 1} g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} \varphi^{[k]}(\varepsilon)\, d\varepsilon\, ds. \tag{C.9}$$

Here and throughout the rest of this appendix, it is understood that such integrals (as in (C.8)–(C.9)) are for $s \in S$. Since the screen avoids the integers $0, \ldots, d$ by assumption, the quantity $|s - i|$ is bounded away from 0. Recall from the proof of Cor. B.2 that $\kappa_{qi}$ is bounded on the support of $\varphi$ by some constant $c_{qi} > 0$. Since the screen avoids the poles of $\zeta_{\mathfrak{s}}$ by hypothesis, $\zeta_{\mathfrak{s}}(s)$ is continuous on the compact set $\{s \in S : |\operatorname{Im} s| \leq 1\}$. Therefore, it is clear that (C.9) is a well-defined integral.

[1]In the proof of Thm. 6.8, the quantity (C.7) was denoted by $\langle \mathcal{R}_{qi}, \varphi \rangle$, so that $\mathcal{R}$ could easily be written (formally) as a function in (6.32). Since we work with test functions, this quantity is instead denoted by $\langle \mathcal{R}, \varphi \rangle_{qi}$ throughout this proof.

We focus now on (C.8):

$$\begin{aligned}
&\left| \frac{1}{2\pi \mathbb{i}} \int_{|\operatorname{Im} s|>1} g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} \varphi^{[k]}(\varepsilon)\, d\varepsilon\, ds \right| \\
&\quad \leq \frac{1}{2\pi} \int_{\operatorname{Im} s>1} \left| \frac{g_q^s \zeta_{\mathfrak{s}}(s)}{s-i} \right| \cdot \left| \widetilde{\kappa_{qi}\varphi^{[k]}}(s-d+1) \right| ds \\
&\quad \leq c_1 \int_1^\infty |t|^{M-1} \cdot |\widetilde{\kappa_{qi}\varphi}(s-d+k+1)| \cdot |\psi_k(s-d+1)|\, dt \\
&\quad \leq c_1 \int_1^\infty |t|^{M-1} \cdot c_K c_{qi} \|\varphi\|_\infty \cdot \frac{c_\psi}{|t|^k}\, dt, \\
&\quad = C\|\varphi\|_\infty \int_1^\infty |t|^{M-1-k}\, dt, \qquad \text{(C.10)}
\end{aligned}$$

which is clearly convergent for $k > M$. The second inequality in (C.10) comes by (A.6) and Cor. C.3. Also, recall that for $s \in S$, the real part of $s$ is given by a function $f$ which is Lipschitz, and hence is almost every differentiable and has bounded derivatives on the support of $\varphi$. The third comes by Cor. B.2, or rather, inequality (B.7) of Rem. B.3, along with Rem. C.4. This establishes the validity of $\left\langle \mathcal{R}^{[k]}, \varphi \right\rangle_{qi}$ and thus shows that $\mathcal{R}^{[k]}$ defines a linear functional on $\mathbb{D}$.

To check that the action of $\mathcal{R}^{[k]}$ is continuous on $\mathbb{D}$, let $\varphi_n \to 0$ in $\mathbb{D}$, i.e., suppose $K$ is a compact set which contains the support of every $\varphi_n$, and $\|\varphi_n\|_\infty \to 0$. Then

$$\left| \langle \mathcal{R}^{[k]}, \varphi_n \rangle \right| \leq C \cdot |\widetilde{\varphi}_n(s-d+k+1)| \leq c_K \|\varphi_n\|_\infty \xrightarrow{n\to\infty} 0, \qquad \text{(C.11)}$$

by following (C.10) and then applying Lemma B.1, along with its extensions as stated in Rem. B.3. Thus, $\mathcal{R}^{[k]}$ is a well-defined distribution. If we differentiate it distributionally $k$ times, we obtain $\mathcal{R}$. This shows that $\mathcal{R}$ is a well-defined distribution and concludes the proof. □

Before finally checking the error estimate, we define what is meant by the expression $T(x) = O(x^\alpha)$ as $x \to \infty$, when $T$ is a distribution.

Definition C.6. When $\mathcal{R}(x) = O(x^\alpha)$ as $x \to \infty$ (as in (5.5)), we say as in [**La-vF4**, §5.4.2] that $\mathcal{R}$ is of *asymptotic order at most* $x^\alpha$ as $x \to \infty$. To understand this expression, first define

$$\varphi_a(x) := \tfrac{1}{a}\varphi\left(\tfrac{x}{a}\right), \qquad \text{(C.12)}$$

for $a > 0$ and for any test function $\varphi$. Then "$\mathcal{R}(x) = O(x^\alpha)$ as $x \to \infty$" means that

$$\langle \mathcal{R}, \varphi_a \rangle = O(a^\alpha), \qquad \text{as } a \to \infty,$$

for every test function $\varphi$. The implied constant may depend on $\varphi$. Similarly, "$\mathcal{R}(x) = O(x^\alpha)$ as $x \to 0^+$" (as in (6.13)) is defined to mean that

$$\langle \mathcal{R}, \varphi_a \rangle = O(a^\alpha), \qquad \text{as } a \to 0^+,$$

for every test function $\varphi$.

Theorem C.7 (Error estimate). *As stated in Thm. 6.8, the error term* $\mathcal{R}(\varepsilon)$ *in* (C.6) *is estimated by*

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{d-\sup S}), \qquad \text{as } \varepsilon \to 0^+. \qquad \text{(C.13)}$$

Proof. As in the proof of Thm. C.5, we use the descent method and begin by splitting the integral into two pieces. Since $\langle \mathcal{R}^{[k]}, \varphi_a \rangle = (-1)^k \langle \mathcal{R}, (\varphi_a)^{[k]} \rangle$, we work with

$$\left\langle \mathcal{R}, (\varphi_a)^{[k]} \right\rangle_{qi} = \frac{1}{2\pi \mathbb{i}} \int_{|\operatorname{Im} s|>1} g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} (\varphi_a)^{[k]}(\varepsilon)\, d\varepsilon\, ds \tag{C.14}$$

$$+ \frac{1}{2\pi \mathbb{i}} \int_{|\operatorname{Im} s|\le 1} g_q^s \zeta_{\mathfrak{s}}(s) \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} (\varphi_a)^{[k]}(\varepsilon)\, d\varepsilon\, ds. \tag{C.15}$$

The $k^{\text{th}}$ primitive of $\varphi_a$ is given by

$$\begin{aligned}(\varphi_a)^{[k]}(\varepsilon) &= \int_\varepsilon^\infty \frac{(u-\varepsilon)^{k-1}}{(k-1)!} \tfrac{1}{a} \varphi\left(\tfrac{u}{a}\right)\, du \\ &= \int_{\varepsilon/a}^\infty \frac{(au-\varepsilon)^{k-1}}{(k-1)!} \varphi(u)\, du.\end{aligned} \tag{C.16}$$

By following the same calculations as in Thm. C.2, one observes that

$$\begin{aligned}
&\left| \int_0^\infty \kappa_{qi}(\varepsilon) \frac{\varepsilon^{d-s}}{s-i} \int_{\varepsilon/a}^\infty \frac{(au-\varepsilon)^{k-1}}{(k-1)!} \varphi(u)\, du\, d\varepsilon \right| \\
&= \left| \int_0^\infty \int_0^{au} \frac{\kappa_{qi}(\varepsilon)}{s-i} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!} (au)^{k-1-j} \varepsilon^{d-s+j} \varphi(u)\, d\varepsilon\, du \right| \\
&\le \frac{1}{|s-i|} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!} \int_0^\infty \left| (au)^{k-1-j} \varphi(u) \right| \int_0^{au} \left| \kappa_{qi}(\varepsilon) \varepsilon^{d-s+j}\, d\varepsilon \right|\, du \\
&\le \frac{c_{qi}}{|s-i|} \psi_k(d - \operatorname{Re} s + 1) \int_0^\infty (au)^{k-1-j} (au)^{d-\operatorname{Re} s+j+1} |\varphi(u)|\, du \\
&= a^{d-\operatorname{Re} s+k} \frac{c_{qi}}{|s-i|} \psi_k(d - \operatorname{Re} s + 1) \widetilde{|\varphi|}(d - \operatorname{Re} s + k).
\end{aligned} \tag{C.17}$$

Using (C.5) for $\psi_k$ and (B.3) for $\widetilde{|\varphi|}$ (see Rem. B.3), we bound (C.14) by

$$\frac{c_{qi}}{2\pi} \int_{|\operatorname{Im} s|>1} a^{d-\operatorname{Re} s+k} \cdot \frac{|g_q^s \zeta_{\mathfrak{s}}(s)|}{|s-i|} \cdot \frac{c_\psi}{|t|^k} \cdot c_K \|\varphi\|_\infty\, ds \tag{C.18}$$

$$\le a^{d-\sup S+k} \left( C \int_1^\infty |t|^{M-1-k}\, dt \right), \tag{C.19}$$

for any $0 < a < 1$, as in (C.10). Since the integral in (C.19) clearly converges for $k > M$, we have established the estimate for $\mathcal{R}^{[k]}$, along the part of the integral where $|\operatorname{Im} s| > 1$. Recall that all our contour integrals are taken along the screen $S$. The proof for (C.15), where $|\operatorname{Im} s| > 1$, readily follows from the corresponding argument in the proof of Thm. C.5. Thus we have established that

$$\left| \langle \mathcal{R}^{[k]}(\varepsilon), \varphi_a(\varepsilon) \rangle \right| \le a^{d-\sup S+k} c_k, \quad \text{for all } 0 < a < 1. \tag{C.20}$$

In (C.20)–(C.22), the constants $c_k$ are allowed to depend on the test function $\varphi$.[2]

[2]Note that $c_{k-1}$ does not correspond to $c_k$ when $k$ is replaced by $k-1$; rather, $c_{k-1}$ depends on the support of $\varphi'$. The notation is just used to indicate the analogous roles the constants $c_k$ play.

By iterating the following calculation:

$$\begin{aligned}\left|\langle \mathcal{R}^{[k-1]}(\varepsilon), \varphi_a(\varepsilon)\rangle\right| &= \left|\langle \mathcal{R}^{[k]}(\varepsilon), \left(\tfrac{1}{a}\varphi\left(\tfrac{\varepsilon}{a}\right)\right)'\rangle\right| \\ &= \left|\tfrac{1}{a}\langle \mathcal{R}^{[k]}(\varepsilon), (\varphi')_a(\varepsilon)\rangle\right| \\ &\leq a^{d-\sup S+k-1} c_{k-1}, \end{aligned} \tag{C.21}$$

one sees that

$$|\langle \mathcal{R}(\varepsilon), \varphi_a(\varepsilon)\rangle| \leq a^{d-\sup S} c_0, \quad \text{for all } 0 < a < 1. \tag{C.22}$$

By Def. C.6, this implies that $\mathcal{R}(\varepsilon) = O(\varepsilon^{d-\sup S})$ as $\varepsilon \to 0^+$. □

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# Symbol Glossary

$(\mathcal{C}(\mathbb{R}^d), \delta)$ the space of nonempty compact subsets of $\mathbb{R}^d$, under Hausdorff metric, 5
$A_\varepsilon$ (inner) $\varepsilon$-neighbourhood of the set $A$, 7
$A_j$ orthogonal matrix (giving the rotational part of $\Phi_j$), 5
$B$ basic shape (generator) of a spray, 19
$B^k$ unit ball in $\mathbb{R}^k$, 2
$C_i$ $i^{\text{th}}$ curvature measure, 13
$D$ similarity dimension, 9
$S$ screen, 15, 41
$T'$ distributional derivative of $T \in \mathbb{D}'$, 45
$T^{[k]}$ $k^{\text{th}}$ primitive of the distribution $T \in \mathbb{D}'$, 47
$T_\eta$ distribution defined by integration with respect to a measure $\eta$, 47
$V_A$ tube formula for the set $A$, 1
$V_{\mathcal{L}}$ tube formula for a string $\mathcal{L}$, 2
$V_G$ tube formula for a generator, 11
$V_\eta$ tube formula for a (generalized) string $\eta$, 16
$V_{\mathcal{T}}$ tube formula for a tiling $\mathcal{T}$, 1
$V_q$ the part of $V_{\mathcal{T}}$ corresponding to $G_q$, 22
$W$ window, 15, 41
$[A]$ convex hull of the set $A$, 5
$\mathcal{D}(W)$ visible complex dimensions, 15
$\mathcal{D}_\eta$ complex dimensions, poles of $\zeta_\eta$, 15
$\mathcal{D}_{\mathcal{K}^\star}$ complex dimensions of the Koch *curve*, 34
$\mathcal{D}_{\mathfrak{s}}$ set of complex dimensions, 9
$\mathcal{L}$ an (ordinary) fractal string, 7
$\mathcal{P}_T$ set of realizable poles of the meromorphic distribution-valued function $T$, 44
$\mathcal{R}$ distributional error term, 16
$\mathcal{W}_k$ set of words of length $k$, 5
$F$ attractor of $\Phi$; a self-similar set, 5
$\mathbb{D}$ the set of test functions $C_c^\infty(0,\infty)$, 43
$\mathbb{D}'$ the class of distributions defined as the dual of $\mathbb{D}$, 45
$\mathbb{R}^d$ $d$-dimensional Euclidean space, 5
$\overline{A}$ closure of the set $A$, 6
$\overline{z}$ complex conjugate of $x \in \mathbb{C}$, 32
$\mathbb{K}^d$ the space of nonempty convex, compact subsets of $\mathbb{R}^d$, 13
$\boldsymbol{\kappa}$ 'curvature matrix', 20
$\partial$ boundary operator, 1
$\ell_n$ a length of fractal string, an element of $\mathcal{L}$, 7
$\delta_{1/r}(x)$ Dirac mass at $1/r$, 8
$G_q$ the generators, the connected components of $T_1$, 6
$g_q$ inradius of generator $G_q$, 8
$\mathfrak{g}$ vector of generating inradii, 20
$\varphi$ a test function for distributions, 15
$\varphi^{(n)}$ $n^{\text{th}}$ derivative of $\varphi$, 15
$\varphi^{[n]}$ $n^{\text{th}}$ primitive of $\varphi$, 47
$\varphi_{qi}$ truncated action of $\kappa_{qi}$ on $\varphi$, 22
$\widetilde{\varphi}$ Mellin transform of $\varphi$, 16
$\widetilde{\varphi}_{qi}$ Mellin transform of $\varphi_{qi}$, 23
$\eta_{\mathcal{L}}$ measure associated to $\mathcal{L}$, 7
$\eta_{\mathfrak{g}}$ geometric measure, 8
$\eta_{\mathfrak{g}q}$ $q^{\text{th}}$ geometric measure, 8
$\eta_{\mathfrak{s}}$ scaling measure, 8
$\kappa_i$ $i^{\text{th}}$ 'inner curvature measure', 11
$\mu_i$ $i^{\text{th}}$ invariant/intrinsic measure, 13
$\rho(A)$ inradius of the set $A$, 8
$\sigma$ real part of $s \in \mathbb{C}$, 27
$\gamma_G$ adaptive generator tube formula, 11
$\xi$ a complex parameter, 32
$\psi_k$ a certain meromorphic function relating $\widetilde{\varphi^{[k]}}(s)$ to $\tilde{\varphi}(s+k)$, 47
$\zeta_{\mathcal{L}}$ geometric zeta function of an ordinary fractal string $\mathcal{L}$, 7
$\zeta_{\mathcal{T}}$ zeta function of a tiling $\mathcal{T}$, 27 or of a fractal spray, 20, 21
$\zeta_L$ zeta function of a string viewed as a 1-dimensional tiling, 29
$\zeta_\eta$ (scaling) zeta function of a fractal spray, 19
$\zeta_{\mathfrak{s}}$ scaling zeta function, 8
$C$ convex hull of the attractor $F$, 5
$C_k$ $k^{\text{th}}$ iterate of $C$ under $\Phi$, 5
$i$ integer dimension, 2
$\mathbb{i}$ the imaginary number $\sqrt{-1}$, 3
$\langle \cdot, \cdot \rangle_{\boldsymbol{\kappa}}$ bilinear form given by the matrix $\boldsymbol{\kappa}$, 20

$\langle \eta, \varphi \rangle$ distributional action of the measure $\eta$ on the test function $\varphi$, 14
$\mathbf{p}$ oscillatory period, 31
$\phi$ the golden ratio $(1+\sqrt{5})/2$, 36
$\mathcal{C}$ Cantor tiling (or string), 32
$\mathcal{E}$ vector of boundary terms, 20
$\mathcal{K}$ Koch tiling, 32
$\mathcal{P}$ Pentagasket tiling, 35
$\mathcal{R}^{[k]}$ $k^{\text{th}}$ primitive of the distributional error term $\mathcal{R}$, 48
$\mathcal{SG}$ Sierpinski Gasket tiling, 35
$\Phi$ self-similar system, 5
$\Phi_j$ contractive similarity mapping; element of the self-similar system $\Phi$, 5
$\Phi_w$ composition of the mappings $\Phi_j$, 5
$R_n$ a tile of a self-similar tiling, 6
$\mathcal{R}$ the set of tiles of a self-similar tiling, 6
$T$ relative interior of the convex hull $C$, 5
$T_k$ $k^{\text{th}}$ iterate of $T$ under $\Phi$, 6
$\mathcal{T}$ a self-similar tiling, 6
$\text{vol}_k$ $k$-dimensional Lebesgue measure, 7
$\{f(i,\varepsilon)\}$ a summand which is not always present, 27
$a_{qi}$ the constant $\widetilde{\kappa_{qi}\varphi}(d-i+1)$, 23
$c_K$ constant depending on $K \in \mathbb{K}$, 43
$c_\omega$ a constant depending on $\omega \in \mathcal{D}_{\mathcal{T}}$, 28
$c_{K,s_0}$ constant depending on $K$ and $s_0 \in \mathbb{C}$, 44
$d$ ambient Euclidean dimension, 2
$e_j$ integer exponent giving the multiplicity of $\Phi_j$, 8
$j = 1, \dots, J$ indices for the self-similar system $\Phi$, 5
$q = 1, \dots, Q$ indices of the generators, 6
$r_j$ scaling ratio (of $\Phi_j$), 5
$s$ a complex variable, 8
$w$ word on the alphabet $\{1, 2, \dots, J\}$, 5
**L1** horizontal growth condition for *languid*, 41
**L2** vertical growth condition for *languid*, 41
**L2′** vertical growth condition for *strongly languid*, 41

# Subject Index