CANONICAL SELF-SIMILAR TILINGS BY IFS.

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Abstract. An iterated function system consisting of contractive similarity mappings has a unique attractor $F \subseteq \mathbb{R}^d$ which is invariant under the action of the system, as was shown by Hutchinson [Hut]. This paper shows how the action of the function system naturally produces a tiling $\mathcal{T}$ of the convex hull of the attractor. These tiles form a collection of sets whose geometry is typically much simpler than that of F, yet retains key information about both F and Φ. In particular, the tiles encode all the scaling data of Φ. We give the construction, along with some examples and applications. The tiling $\mathcal{T}$ is the foundation for the higher-dimensional extension of the theory of *complex dimensions* which was developed in [La-vF1] for the case $d = 1$.

1. Introduction

This paper presents the construction of a self-similar tiling which is canonically associated to a given self-similar system Φ, as in Def. 1. The term "self-similar tiling" is used here in a sense quite different from the one often encountered in the literature. In particular, the region being tiled is the complement of the self-similar set F within its convex hull, rather than all of $\mathbb{R}^d$. Moreover, the tiles themselves are neither self-similar nor are they all of the same size; in fact, the tiles may even be simple polyhedra. However, the name "self-similar tiling" is appropriate because we will have a tiling of the convex hull: the union of the closures of the tiles is the entire convex hull, and the interiors of the tiles intersect neither each other, nor the attractor F. While the tiles themselves are not self-similar, the overall structure of the tiling is.[1] .

The construction of the tiling begins with definition of the generators, a collection of open sets obtained from the convex hull of F. The rest of the tiles will be seen to be images of these generators under the action of the original self-similar system. Thus, the tiling $\mathcal{T}$ essentially arises as a spray on the generators, in the sense of [LaPo2] and [La-vF1]. The tiles thus obtained form a collection of sets whose geometry is typically much simpler than that of F, yet retains key information about both F and Φ. In particular, the tiles encode all the scaling data of Φ.

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[1]Technically, the tiling is *subselfsimilar* in that $\Phi(\mathcal{T}) \subseteq \mathcal{T}$.

Section §2 gives the tiling construction and illustrates the method with several familiar examples, including the Koch snowflake curve, Sierpinski gasket and pentagasket. Section §3 describes the basic properties of the tiling. Section §4 discusses potential applications, and gives a tube formula for the tiling.

It is shown in [LaPe2] that the tiles allow one to define a zeta function $\zeta_{\mathcal{T}}$ for Φ which is essentially a generating function for the geometry of F. This geometric zeta function, in turn, allows computation of an explicit tube formula for $\mathcal{T}$. Moreover, one may define the complex dimensions of $\mathcal{T}$ as the poles of $\zeta_{\mathcal{T}}$. The tube formula $V_{\mathcal{T}}(\varepsilon)$ of [LaPe2] is thus defined entirely in terms of the self-similar tiling constructed in this paper; see §4.

2. The Self-Similar Tiling

2.1. Basic terms.

Definition 1. A *self-similar system* is a family $\Phi := \{\Phi_j\}_{j=1}^J$ (with $J \geq 2$) of contraction similitudes

$$\Phi_j(x) := r_j A_j x + a_j, \quad j = 1, \ldots, J.$$

For $j = 1, \ldots, J$, we have $0 < r_j < 1, a_j \in \mathbb{R}^d$, and $A_j \in O(d)$, the orthogonal group of rigid rotations in d-dimensional Euclidean space $\mathbb{R}^d$. Thus, each Φ_j is the composition of an (affine) isometry and a homothety (scaling).

Remark 2. Note that different self-similar systems may give rise to the same self-similar set. In this paper, the emphasis is placed on the self-similar system and its corresponding dynamical system, rather than on the self-similar set.

Definition 3. The numbers r_j are referred to as the *scaling ratios* of Φ. For convenience, we may take the scaling ratios in nonincreasing order, i.e., reindex $\{\Phi_j\}$ so that

$$1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0. \tag{2.1}$$

Definition 4. A self-similar system is thus just a particular type of iterated function system (IFS). It is well known[2] that for such a family of maps, there is a unique and self-similar set F satisfying the fixed-point equation

$$F = \Phi(F) := \bigcup_{j=1}^{J} \Phi_j(F). \tag{2.2}$$

We call F the *attractor* of Φ, or the *self-similar set* associated with Φ. The *action of* Φ is the set map defined by (2.2). Thus, one says that F is invariant under the action of Φ.

Definition 5. We fix some notation for later use. Let

$$C := [F] \tag{2.3}$$

be the convex hull of the attractor F, that is, the intersection of all convex sets containing F. Since F is a compact set, it follows that C is also compact, by [Sch, Thm. 1.1.10]. Further, let

$$C^o := \text{int}(C) = C \sim \partial C. \tag{2.4}$$

[2]See [Hut], as described in [Fal] or [Kig], for example.

Remark 6. For this paper, it will suffice to work with the ambient dimension

$$d = \dim C, \tag{2.5}$$

restricting the maps Φ_j as appropriate. In (2.5), $\dim C$ is defined to be the usual topological dimension of the smallest affine space containing C. An appropriate change of coordinates allows one to think of this convention as using a minimal subspace $\mathbb{R}^d$; if F is a Cantor set in $\mathbb{R}^3$, we study it as if the ambient space were the line containing it, rather than $\mathbb{R}^3$. Note that this means C^o is open in the standard topology; and so we have $C^o \neq \varnothing$. This remark is intended to allay any fears about possibly needing to use relative interior instead of interior (see [KlRo] or [Sch]) and other unnecessary complications.

Definition 7. A self-similar system satisfies the *tileset condition* iff for $j \neq \ell$,

$$\operatorname{int} \Phi_j(C) \cap \operatorname{int} \Phi_\ell(C) = \varnothing. \tag{2.6}$$

It is shown in Cor. 23 that because $C = \overline{\operatorname{int} C}$, (2.6) implies that the images $\Phi_j(C)$ and $\Phi_\ell(C)$ can intersect only on their boundaries:

$$\Phi_j(C) \cap \Phi_\ell(C) \subseteq \partial\Phi_j(C) \cap \partial\Phi_\ell(C).$$

Here, $\partial A := \overline{A} \cap \overline{A^c}$, where A^c is the complement of A and $\overline{A}$ denotes the (topological) closure of A. To avoid trivialities, we also require

$$C^o \nsubseteq \Phi(C). \tag{2.7}$$

The *nontriviality condition* (2.7) disallows the case $C^o \sim \Phi(C) = \varnothing$, and hence guarantees the existence of the tiles in §2.2.

Remark 8. The tileset condition is a restriction on the overlap of the images of the mappings, comparable to the *open set condition* (OSC). The OSC requires a nonempty bounded open set U such that the sets $\Phi_j(U)$ are disjoint but $\Phi(U) \subseteq U$. See, e.g., [Fal, Chap. 9]. If one takes $U = \operatorname{int} C$, then it will be clear from Cor. 16 of §3 that the OSC follows from (2.6); see Rem. 17.

Definition 9. Denote the *words of length k* (of $\{1, 2, \ldots, J\}$) by

$$\begin{aligned} W_k = W_k^J &:= \{1, 2, \ldots, J\}^k \\ &= \{w = w_1 w_2 \ldots w_k \mid w_j \in \{1, 2, \ldots, J\}\}, \end{aligned} \tag{2.8}$$

and the set of all (finite) words by $W := \bigcup_k W_k$. Generally, the dependence of W_k^J on J is suppressed. For w as in (2.8), we use the standard IFS notation

$$\Phi_w(x) := \Phi_{w_k} \circ \ldots \circ \Phi_{w_2} \circ \Phi_{w_1}(x) \tag{2.9}$$

to describe compositions of maps from the self-similar system.

Definition 10. For a set $A \subseteq \mathbb{R}^d$, a *tiling* of A is a sequence of sets $\{A_n\}_{n=1}^\infty$ such that

(i) $A = \bigcup_{n=1}^N A_n$, and

(ii) $A_n \cap A_m = \partial A_n \cap \partial A_m$ for $n \neq m$.

We then say that the sets A_n *tile* A. Further, define a *tiling of A by open sets* to be a sequence of *open* sets $\{A_n\}$ satisfying $\overline{A} = \bigcup_{n=1}^N \overline{A_n}$, where $A_n \cap A_m = \varnothing$ for $n \neq m$. In general, N may be taken to be ∞. Fig. 1 shows an example of a tiling by open sets with $N = \infty$.

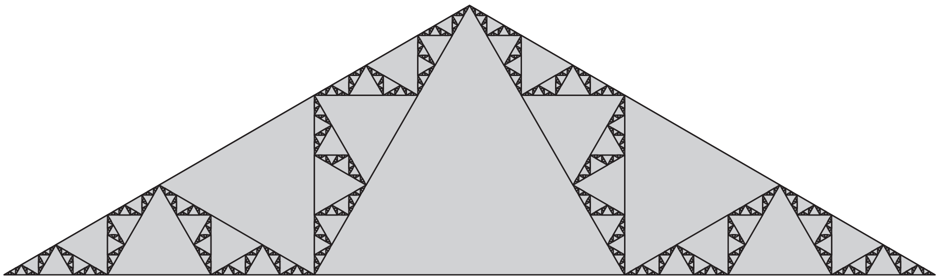

FIGURE 1. Tiling the complement of the Koch curve K. The equilateral triangles form an open tiling of the convex hull $C = [K]$, in the sense of Def. 10.

2.2. **The construction.** In this section, we construct a *self-similar tiling*, that is, a tiling which is constructed via the mappings of a self-similar system. Such a tiling will consequently have a self-similar structure, and is defined precisely in Def. 13 below. The reader is invited to look ahead at Figure 2, where the construction is illustrated step-by-step for the illuminative example of the Koch curve.

For the system $\{\Phi_j\}$ with attractor F, denote the hull of the attractor by

$$C_0 = C := [F]. \tag{2.10}$$

Denote the image of C under the action of Φ (in accordance with (2.2)) by

$$C_k := \Phi^k(C) = \bigcup_{w \in W_k} \Phi_w(C), \quad k = 1, 2, \dots . \tag{2.11}$$

Note that this is equivalent to the inductive definition

$$C_k := \Phi(C_{k-1}), \quad k = 1, 2, \dots . \tag{2.12}$$

Definition 11. The *tilesets* are the sets

$$T_k := \overline{C_{k-1} \sim C_k}, \quad k = 1, 2, \dots \tag{2.13}$$

Definition 12. The *generators* G_q of the aforementioned tiling $\mathcal{T}$ are the connected components of the open set

$$\operatorname{int}(C \sim \Phi(C)) = G_1 \sqcup G_2 \sqcup \dots \sqcup G_Q. \tag{2.14}$$

The symbol $\sqcup$ is used to indicate *disjoint* union.

As will be shown in Theorem 24, it follows from the tileset conditions (2.6)–(2.7) (and some other facts) that the tilesets and tiles are nonempty, and that each tileset is the closure of its interior. Also, Theorem 27 will justify the terminology "generators" by showing

$$T_k = \bigsqcup_{q=1}^{Q} \overline{\Phi^{k-1}(G_q)}, \tag{2.15}$$

that is, that any difference $C_{k-1} \sim C_k$ is (modulo some boundary points) the image of the generators under the action of Φ. The number Q of generators depends on the specific geometry of C and on the self-similar system Φ. It is conceivable that $Q = \infty$ for some systems Φ, but no such examples are known. This possibility will be investigated further in [LaPe3].

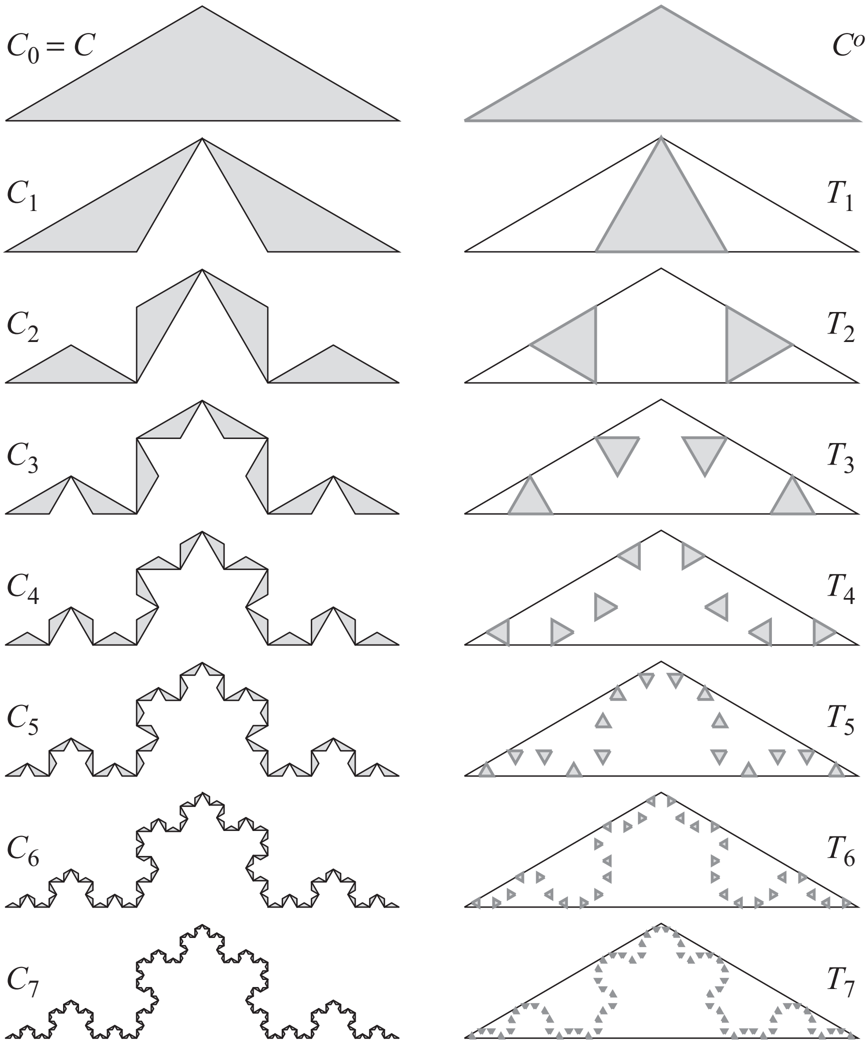

FIGURE 2. The left column shows images of the convex hull C under successive applications of Φ. The right column shows how the components of the T_k tile the complement; they are overlaid in Fig. 1. This tiling has one generator $G_1 = T_1$.

Definition 13. The *self-similar tiling* of F is

$$\mathcal{T} := \left(\{\Phi_j\}_{j=1}^{J}, \{G_q\}_{q=1}^{Q} \right). \tag{2.16}$$

We may also abuse the notation a little, and use $\mathcal{T}$ to denote the set of corresponding *tiles*:

$$\mathcal{T} = \{R_n\}_{n=1}^{\infty} = \{\Phi_w(G_q) \mid w \in W, q = 1, \ldots, Q\}, \tag{2.17}$$

where the sequence $\{R_n\}$ is an enumeration of the tiles. Clearly, each tile is nonempty and d-dimensional. Furthermore, Theorem 28 will confirm that (2.17) is an open tiling in the sense of Def. 10.

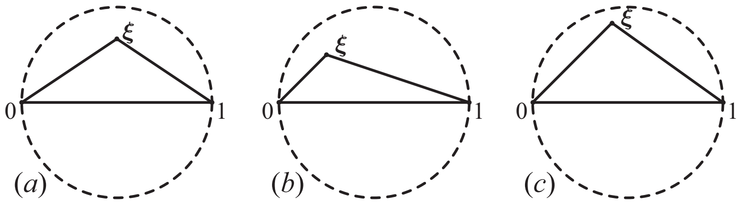

FIGURE 3. (a) The parameter ξ for the standard Koch curve, as depicted in Fig. 1 and Fig. 2. (b) ξ for a skinny Koch, as in Fig. 4. (c) ξ for a chunky Koch, as in Fig. 5.

2.3. **Examples.** All the examples discussed in this section have polyhedral generators, but this is not the general case. In fact, it is possible to have generators with boundary that is continuously differentiable, although it is not possible that they be twice continuously differentiable. This was observed to be true for the convex hull of an attractor in [StWa], and it immediately carries over to the generators as well. We will study this eventuality further in [LaPe3]. See also §3.1.

2.3.1. *The Koch curve.* Figure 1 shows the self-similar tiling of the Koch curve; the steps of the construction are illustrated in Figure 2. In this case, the tiling is $\mathcal{K} = \left(\{\Phi_j\}_{j=1}^2, \{G\}\right)$, and it is easiest to write down the similarities as maps $\Phi_j : \mathbb{C} \to \mathbb{C}$, with the natural identification of $\mathbb{C}$ and $\mathbb{R}^2$:

$$\Phi_1(z) := \xi\overline{z} \quad \text{and} \quad \Phi_2(z) := (1-\xi)(\overline{z}-1)+1 \tag{2.18}$$

for $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$. For this example, $r_1 = r_2 = 1/\sqrt{3}$ and the single generator G is the equilateral triangle of side length $\frac{1}{3}$ depicted as T_1 in Fig. 2. Here and henceforth, we reserve the symbol $\mathbb{i} = \sqrt{-1}$ for the imaginary number.

2.3.2. *The 1-parameter family of Koch curves.* There is an entire family of Koch curves generalizing the standard Koch curve. We use the same system as above:

$$\Phi_1(z) := \xi\overline{z} \quad \text{and} \quad \Phi_2(z) := (1-\xi)(\overline{z}-1)+1,$$

but now ξ may be any complex number satisfying

$$|\xi|^2 + |1-\xi|^2 < 1, \tag{2.19}$$

as shown in Figure 3. Geometric considerations show that (2.19) must be satisfied in order for the tileset condition (2.6) to be met.

For any member of this family, we have one isoceles triangle $G = G_1 = T_1$ for a generator. A key point of interest in this example is that, in the language of [La-vF1], curves from this family will generally be nonlattice, i.e., the logarithms of the scaling ratios will not be rationally dependent. Thus, one may use this example to construct tilings where the scaling ratios involved satisfy certain number-theoretic conditions, as studied in [La-vF1].

2.3.3. *The one-sided Koch curve.* Occasionally, one may wish to consider a set which is not self-similar, but which has a (piecewise) self-similar boundary. The Koch snowflake is an example of such a domain. When considering the area of the interior of the Koch curve, one is interested in tiling only the region on one side of the curve. This perspective is motivated by mimicking the calculation of the interior ε-neighbourhood of the snowflake, as opposed to a 2-sided ε-neighbourhood.

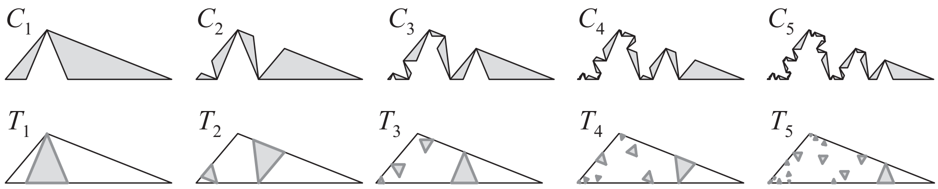

FIGURE 4. Self-similar tiling of nonstandard Koch curve (b).

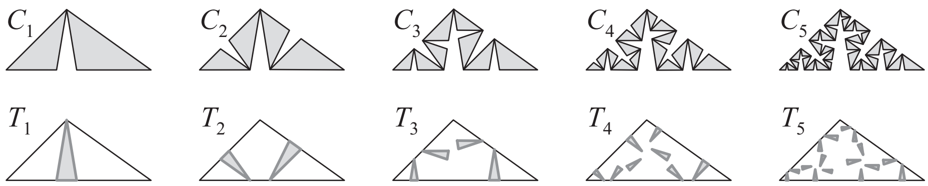

FIGURE 5. Self-similar tiling of nonstandard Koch curve (c).

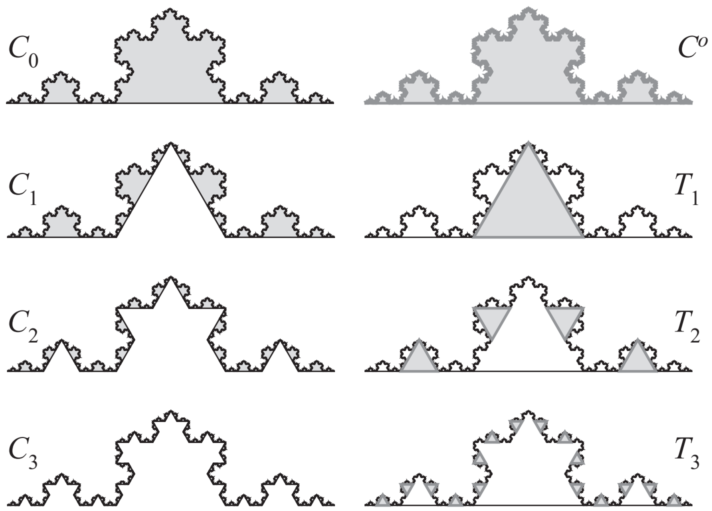

FIGURE 6. Self-similar tiling of the interior Koch curve.

For this approach, the previous IFS will not work; its alternating nature maps portions of the interior to the exterior and vice versa. We can, however, view the Koch curve as the self-similar set generated by an IFS consisting of 4 maps, each with scaling ratio $\frac{1}{3}$, in the obvious manner. Since we want each stage of the construction to generate only those triangles which lie inside the curve, it behooves us to take the intersection of the convex hull with the interior of the Koch curve, as seen in the shaded region of C_0 in Figure 6. The Koch curve may now be constructed using the 4-map IFS depicted in Figure 6 (note how $C_1 = \bigcup_{j=1}^{4} \Phi_j(C_0)$, etc).

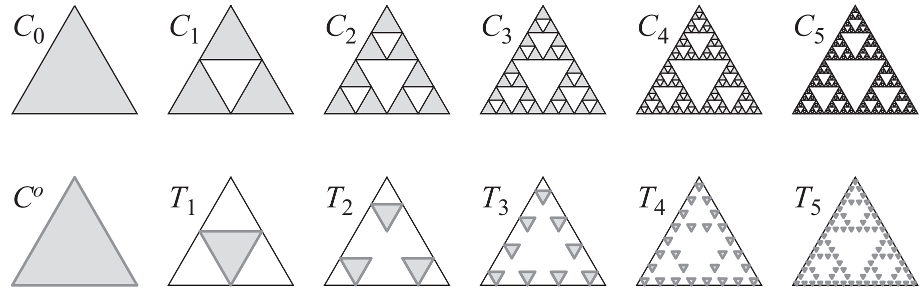

FIGURE 7. Self-similar tiling of the Sierpinski Gasket.

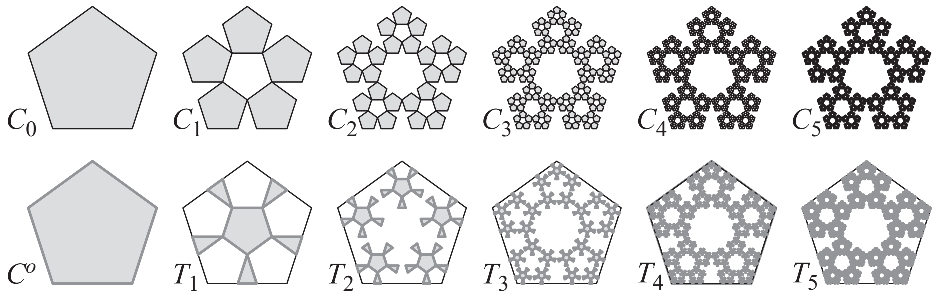

FIGURE 8. Self-similar tiling of the Pentagasket.

2.3.4. *The Sierpinski Gasket.* The Sierpinski gasket system consists of the three maps

$$\Phi_j(x) = \frac{1}{2}x + \frac{p_j}{2},$$

where the p_j are the vertices of an equilateral triangle; the standard example is $p_1 = 0, p_2 = 1,$ and $p_3 = (1 + \mathbb{i}\sqrt{3})/2$.

The convex hull of the gasket is the triangle with vertices p_1, p_2, p_3. The generator G is the 'middle fourth' of the hull (see T_1 in Figure 7).

2.3.5. *The Pentagasket.* The Pentagasket is constructed via five maps

$$\Phi_j(x) = \phi^{-2}x + p_j,$$

where the p_j form the vertices of a pentagon of side length 1, and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, so that the scaling ratio of each mapping is

$$r_j = \phi^{-2} = \tfrac{3-\sqrt{5}}{2}, \quad j = 1, \dots, 5.$$

The Pentagasket is the first example of multiple generators G_q. In fact, $T_1 = G_1 \cup \dots \cup G_6$ where G_1 is a pentagon and $G_2, \dots, G_6$ are triangles.

2.3.6. *The Sierpinski Carpet.* The Sierpinski carpet is constructed via eight maps

$$\Phi_j(x) = \tfrac{x}{3} + p_j,$$

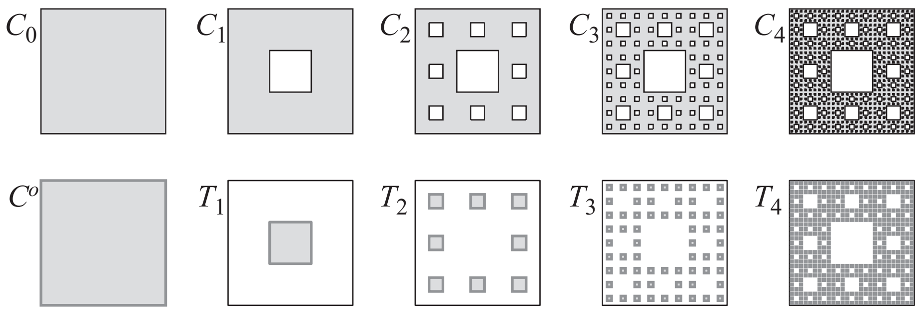

FIGURE 9. Self-similar tiling of the Sierpinski carpet.

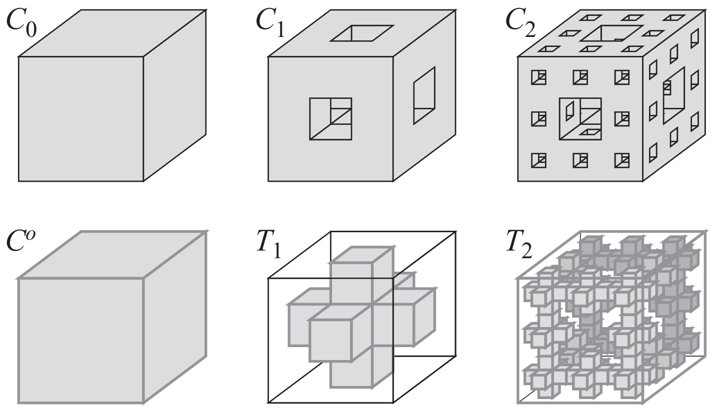

FIGURE 10. Self-similar tiling of the Menger sponge.

where $p_j = (a_j, b_j)$ for $a_j, b_j \in \{0, \frac{1}{3}, \frac{2}{3}\}$, excluding the single case $(1/3, 1/3)$. The Sierpinski carpet is an example which is not finitely ramified; indeed, it is not even post-critically finite (see [Kig]).

2.3.7. *The Menger Sponge.* The Menger sponge is constructed via twenty maps

$$\Phi_j(x) = \tfrac{x}{3} + p_j,$$

where $p_j = (a_j, b_j, c_j)$ for $a_j, b_j, c_j \in \{0, \frac{1}{3}, \frac{2}{3}\}$, except for the six cases when exactly two coordinate are $1/3$, and the single case when all three coordinates are $1/3$.

The Menger sponge system is the first example with an generator of dimension greater than 2, also the first example with a nonconvex generator.

3. Properties of the tiling

The results of this section indicate that a self-similar tiling may be constructed for any self-similar system satisfying the tileset condition of Def. 7. Throughout, we will use the fact that $\overline{A} = \operatorname{int} A \sqcup \partial A$, where we denote the closure of A by $\overline{A}$, the interior of A by $\operatorname{int} A$, and the boundary of A by $\partial A = \overline{A} \cap \overline{A^c}$, where $A^c = \mathbb{R}^d \sim A$. Recall that $\sqcup$ indicates *disjoint* union.

Theorem 14. *For each $k \in \mathbb{N}$, one has $C_{k+1} \subseteq C_k \subseteq C$.*

Proof. Any point $x \in C$ is a convex combination of points in F. Since similarity transformations preserve convexity, $\Phi_j(x)$ will be a convex combination of points in $\Phi_j(F) \subseteq F$. Hence $\Phi_j(C) \subseteq [F] = C$ for each j, so $\Phi(C) \subseteq C$. By iteration of this argument, we immediately have $\Phi^k(C) \subseteq C$ for any $k \in \mathbb{N}$. From (2.12), it is clear that

$$C_{k+1} = \Phi(C_k) = \Phi^{k+1}(C) = \Phi^k(\Phi(C)) \subseteq \Phi^k(C) = C_k, \tag{3.1}$$

where the inclusion follows by $\Phi(C) \subseteq C$, as established initially. □

Corollary 15. *The tileset condition is preserved under the action of Φ, i.e.,*

$$\operatorname{int} \Phi_j(C_k) \cap \operatorname{int} \Phi_\ell(C_k) = \varnothing, \quad \forall k \in \mathbb{N}. \tag{3.2}$$

Proof. From Theorem 14 we have $\operatorname{int} \Phi_j(C_k) \subseteq \operatorname{int} \Phi_j(C)$, and similarly for Φ_ℓ. The disjointness of $\operatorname{int} \Phi_j(C_k)$ and $\operatorname{int} \Phi_\ell(C_k)$ follows from the tileset condition (2.6). □

Corollary 16. *For $A \subseteq C_k$, we have $\Phi_w(A) \subseteq C_k$, for all $w \in W$. In particular, $F \subseteq C_k, \forall k$.*

Proof. By iteration of (3.1), it is immediate that $C_m \subseteq C_k$ for any $m \geq k$. Since $\Phi(A) \subseteq \Phi(C_k) = C_{k+1} \subseteq C_k$ by Theorem 14, the first conclusion follows. The special case follows by induction on k with basis case $A = F \subseteq C = C_0$. The inductive step is

$$F \subseteq C_k \quad \Longrightarrow \quad F = \Phi(F) \subseteq \Phi(C_k) = C_{k+1}.$$ □

Remark 17. With $k = 0$ and $A = \operatorname{int} C$, Cor. 16 shows that any system Φ satisfying the tileset condition (2.6) must also satisfy the open set condition; see Rem. 8.

Corollary 18. *The decreasing sequence of sets $\{C_k\}$ converges to F.*

Proof. Cor. 16 shows $F \subseteq C_k$ for every k, so it is clear that $F \subseteq \bigcap C_k$. For the reverse inclusion, suppose $x \notin F$, so that x must be some positive distance ε from F. Recall that r_1 is the largest scaling ratio of the maps $\{\Phi_j\}$, and that $0 < r_1 < 1$. For $w \in \mathcal{W}_k$, we have $diam(\Phi_w(C)) \leq r_1^k diam(C)$, which clearly tends to 0 as $k \to \infty$. Therefore, we can find k for which all points of $C_k = \Phi^k(C)$ lie within $\varepsilon/2$ of F. Thus x cannot lie in C_k and hence $x \notin \bigcap C_k$. □

Remark 19. Convergence also holds in the sense of Hausdorff metric, by a theorem of [Hut]; see also [Fal] or [Kig] for a nice discussion. Hutchinson showed that Φ is a contraction mapping on the metric space of compact subsets of $\mathbb{R}^d$, which is complete when endowed with the Hausdorff metric. An application of the contraction mapping principle then shows that Φ has a unique fixed point (as stated in Def. 4) and that any other point tends towards it under iteration of the action of Φ. This phenomenon is especially apparent in Figures 2, 7, 8, and 9.

Lemma 20. *The action of Φ commutes with set closure, i.e., $\Phi\left(\overline{A}\right) = \overline{\Phi(A)}$*

Proof. It is well known that closure commutes with finite unions, i.e., for any sets A, B, one has $\overline{A} \cup \overline{B} = \overline{(A \cup B)}$. See, e.g., [Mu, Chap. 2, §17]. Also, each Φ_j is a homeomorphism, and is thus a closed, continuous map. Therefore,

$$\Phi\left(\overline{A}\right) = \bigcup_{j=1}^{J} \Phi_j\left(\overline{A}\right) = \bigcup_{j=1}^{J} \overline{\Phi_j(A)} = \overline{\bigcup_{j=1}^{J} \Phi_j(A)} = \overline{\Phi(A)}.$$ □

Theorem 21. *If A is the closure of its interior, then so is $\Phi(A)$.*

Proof. Let $x \in \Phi(A)$ so that $x \in \Phi_j(A)$ for some $j = 1, \dots, J$. Because each map Φ_j is a homeomorphism, we know that $\Phi_j(A)$ is the closure of its interior, and hence that $x \in \overline{\operatorname{int} \Phi_j(A)} \subseteq \overline{\operatorname{int} \Phi(A)}$. The reverse inclusion $\overline{\operatorname{int} \Phi(A)} \subseteq \Phi(A)$ is immediate because A is closed by hypothesis (and thus $\Phi(A)$ is also closed). □

Corollary 22. *Each set C_k is the closure of its interior.*

Proof. The set $C = [F]$ is convex by definition, and compact by [Sch, Thm. 1.1.10]. Therefore, C is the closure of its interior by [Sch, Thm. 1.1.14]. The conclusion follows by iteration of Theorem 21. □

Corollary 23. *The tileset condition implies that images of the hull can only overlap on their boundaries:*

$$\Phi_j(C) \cap \Phi_\ell(C) \subseteq \partial\Phi_j(C) \cap \partial\Phi_\ell(C), \qquad \text{for } j \neq \ell. \tag{3.3}$$

Proof. Let $x \in \Phi_j(C) \cap \partial\Phi_\ell(C)$. Suppose, by way of contradiction, that $x \in \operatorname{int} \Phi_j(C)$. Then we can find an open neighbourhood U of x which is contained in $\operatorname{int} \Phi_j(C)$. Since $x \in \partial\Phi_\ell(C)$, there must be some $z \in U \cap \operatorname{int} \Phi_\ell(C)$, by Cor. 22. But then $z \in \operatorname{int} \Phi_j(C) \cap \operatorname{int} \Phi_\ell(C)$, in contradiction to the tileset condition. □

Theorem 24 (Nondegeneracy of tilesets)**.** *Each tileset is the closure of its interior.*

Proof. We need only show $T_k \subseteq \overline{\operatorname{int} T_k}$, since the reverse containment is clear by the closedness of T_k. Since $\overline{A} = \operatorname{int} A \sqcup \partial A$, take $x \in \operatorname{int}(C_{k-1} \sim C_k)$ to begin. Using Cor. 22, we have equality in the first step of the following derivation:

$$\begin{aligned} C_{k-1} \sim C_k &= \overline{\operatorname{int}(C_{k-1})} \sim \overline{C_k} \\ &\subseteq \overline{\operatorname{int}(C_{k-1}) \sim C_k} \\ &\subseteq \overline{\operatorname{int}(C_{k-1} \sim C_k)} \\ &\subseteq \overline{\operatorname{int}\left(\overline{C_{k-1} \sim C_k}\right)}. \end{aligned} \tag{3.4}$$

The containment (3.4) follows from

$$\operatorname{int}(C_{k-1}) \sim C_k = \operatorname{int}(\operatorname{int}(C_{k-1}) \sim C_k) \subseteq \operatorname{int}(C_{k-1} \sim C_k),$$

where one has the equality because the difference of an open and closed set is open, and the containment because $\operatorname{int}(C_{k-1}) \subseteq C_{k-1}$.

Now consider the case when $x \in \partial(C_{k-1} \sim C_k)$. Pick an open set U and find $z \in U \cap (C_{k-1} \sim C_k)$. Then $z \in \overline{\operatorname{int}\left(\overline{C_{k-1} \sim C_k}\right)}$ by the same argument as above. This means that x is a limit point of the closed set $\overline{\operatorname{int}\left(\overline{C_{k-1} \sim C_k}\right)}$, and hence must lie within it. □

The following corollary will be useful in the proof of Theorem 27.

Corollary 25. *For $j = 1, \dots, J$, $\overline{\Phi_j(C_{k-1}) \sim \Phi_j(C_k)}$ is the closure of its interior.*

Proof. Because each Φ_j is a homeomorphism, the set $\Phi_j(\overline{C_{k-1} \sim C_k})$ will be the closure of its interior by Theorem 24. However, we have

$$\Phi_j(\overline{C_{k-1} \sim C_k}) = \overline{\Phi_j(C_{k-1} \sim C_k)} = \overline{\Phi_j(C_{k-1}) \sim \Phi_j(C_k)}, \tag{3.5}$$

since Φ_j is closed and injective. □

We are now ready to prove the main result of this paper.

Theorem 26. *Each tileset is the image under Φ of its predecessor, i.e.,*

$$\Phi(T_k) = T_{k+1}, \quad \textit{for } k \in \mathbb{N}. \tag{3.6}$$

Proof. Using using Def. 11 and (3.5), we have the identities

$$\Phi(T_k) = \bigcup\nolimits_{j=1}^{J} \overline{\Phi_j(C_{k-1} \sim C_k)}, \text{ and} \tag{3.7}$$

$$T_{k+1} = \overline{C_k \sim C_{k+1}}. \tag{3.8}$$

($\subseteq$) To see that (3.7) is a subset of (3.8), pick $x \in \Phi(T_k)$, so that

$$x \in \overline{\Phi_j(C_{k-1} \sim C_k)} = \overline{\Phi_j(C_{k-1}) \sim \Phi_j(C_k)} \tag{3.9}$$

for some $j = 1, \ldots, J$. Since $\overline{A} = \operatorname{int} A \sqcup \partial A$, we proceed by cases. Here again, $\sqcup$ denotes the disjoint union.

(i) Let $x \in \operatorname{int}(\Phi_j(C_{k-1}) \sim \Phi_j(C_k))$. Then let $U \subseteq \Phi_j(C_{k-1}) \sim \Phi_j(C_k)$ be an open neighbourhood of x. Since $x \in U \subseteq \Phi_j(C_{k-1})$, we have $x \in \operatorname{int} \Phi_j(C_{k-1}) \subseteq C_k$.

By way of contradiction, suppose that $x \in C_{k+1}$. Then $x \in \Phi_\ell(C_k)$ for some ℓ. Note that $\ell \neq j$, since $x \notin \Phi_j(C_k)$ by initial choice of x. Inasmuch as Theorem 14 gives $x \in \Phi_\ell(C_{k-1})$, Cor. 23 implies

$$x \in \partial\Phi_j(C_{k-1}) \cap \partial\Phi_\ell(C_{k-1}), \tag{3.10}$$

contradicting the fact that $x \in \operatorname{int} \Phi_j(C_{k-1})$. So we may conclude that

$$x \in C_k \sim C_{k+1} \subseteq \overline{C_k \sim C_{k+1}}. \tag{3.11}$$

(ii) Now consider $x \in \partial(\Phi_j(C_{k-1}) \sim \Phi_j(C_k))$. Again, let U be an open neighbourhood of x. By Cor. 25, we can find $w \in U \cap \operatorname{int}(\Phi_j(C_{k-1}) \sim \Phi_j(C_k))$. By applying the arguments of part (i), we obtain $w \in \overline{C_k \sim C_{k+1}}$ and hence that x is a limit point of $\overline{C_k \sim C_{k+1}}$. Since this latter set is closed, we have shown that $x \in \overline{C_k \sim C_{k+1}}$ in case (ii), and completed the forward inclusion.

($\supseteq$) Now we need to show that (3.8) is a subset of (3.7). Since

$$x \in \overline{C_k \sim C_{k+1}} = \operatorname{int}(C_k \sim C_{k+1}) \sqcup \partial(C_k \sim C_{k+1}), \tag{3.12}$$

this will again require two parts.

(iii) Let $x \in \operatorname{int}(C_k \sim C_{k+1}) \subseteq \Phi(C_{k-1}) \sim \Phi(C_k)$. Then $x \in \Phi(C_{k-1})$ means that $x \in \Phi_j(C_{k-1})$ for some $j = 1, \ldots, J$. Furthermore, there must be some $y \in C_{k-1}$ with $\Phi_j(y) = x$. We know $y \notin C_k$, because otherwise

$$y \in C_k \quad \Longrightarrow \quad x = \Phi_j(y) \in \Phi_j(C_k) \subseteq C_{k+1}, \tag{3.13}$$

which contradicts the initial choice $x \notin C_{k+1}$. Thus $y \in C_{k-1} \sim C_k$, which implies

$$x = \Phi_j(y) \in \Phi_j(C_{k-1} \sim C_k) \subseteq \overline{\Phi_j(C_{k-1} \sim C_k)}. \tag{3.14}$$

(iv) Now consider $x \in \partial(C_k \sim C_{k+1})$, and again let U be an open neighbourhood of x. Then there is some $z \neq x$ with $z \in U \cap \operatorname{int}(C_k \sim C_{k+1})$. By applying the arguments of part (iii) to z, we see

$$z \in \operatorname{int}(C_k \sim C_{k+1}) \quad \Longrightarrow \quad z \in \overline{\Phi_j(C_{k-1} \sim C_k)}. \tag{3.15}$$

Therefore, we have shown that x is a limit point of $\overline{\Phi_j(C_{k-1} \sim C_k)}$, and is hence contained in it. This completes the proof of the equality (3.6). $\square$

Theorem 27. *The tilesets can be recovered as the closure of the images of the generators under the action of* Φ*, that is,*

$$T_k = \overline{\bigsqcup_{q=1}^{Q} \Phi^{k-1}(G_q)}. \tag{3.16}$$

Proof. First, observe that[3]

$$\overline{C \sim \Phi(C)} = \overline{\text{int}(C \sim \Phi(C))}, \tag{3.17}$$

as follows. If $x \in \overline{C \sim \Phi(C)}$, then any open neighbourhood U of x must intersect $\text{int}(C \sim \Phi(C))$, because $\overline{C \sim \Phi(C)}$ is the closure of its interior, by Theorem 24. Hence $x \in \overline{\text{int}(C \sim \Phi(C))}$. The reverse inclusion is clear. Using (3.17), we have

$$\bigcup_{q=1}^{Q} \overline{G_q} = \overline{\bigsqcup_{q=1}^{Q} G_q} = \overline{\text{int}(C \sim \Phi(C))} = \overline{C \sim \Phi(C)} = T_1. \tag{3.18}$$

Now take Φ^{k-1} of both sides, using Lemma 20 on the left and Theorem 26 on the right, to obtain the conclusion:

$$\overline{\bigsqcup_{q=1}^{Q} \Phi^{k-1}(G_q)} = \Phi^{k-1}\left(\overline{\bigsqcup_{q=1}^{Q} G_q}\right) = \Phi^{k-1}(T_1) = T_k. \tag{3.19}$$

The union $\bigsqcup_{q=1}^{Q} \Phi^{k-1}(G_q)$ is disjoint because each Φ_j is injective, $G_q \subseteq \text{int}\, C$, and the tileset condition (2.6) prohibits overlaps of interiors. ☐

Theorem 28. *The collection* $\mathcal{T} = \{\Phi_w(G_q)\}$ *is an open tiling of* C*, in the sense of Def. 10. In fact,* $\mathcal{T}$ *is an open tiling of* $C \sim F$*.*

Proof. (i) To see that $C = \bigcup \overline{R_n} = \bigcup \overline{\Phi_w(G_q)}$, it suffices to show $C = \bigcup T_k$, by Theorem 27. Pick $x \in C \sim F$. Since $F = \bigcap C_k$ by Cor. 18, this means we can find k such that $x \in C_{k-1}$ but $x \notin C_k$. Then

$$x \in C_{k-1} \sim C_k \subseteq \overline{C_{k-1} \sim C_k} = T_k. \tag{3.20}$$

The reverse inclusion is obvious from Theorem 14 and the definition of the tiles as subsets of the C_k, in (2.13).

(ii) To see that the tiles are disjoint, note first that the generators are disjoint by definition. Suppose R_n and R_m are both in the same tileset T_k. Then (3.19) shows that they are disjoint. Now suppose $R_n \subseteq T_k$ and $R_m \subseteq T_\ell$, where $k < \ell$. Then R_n is disjoint from C_k by definition of T_k, and it follows from Theorem 14 that R_n is disjoint from C_ℓ for all $\ell \geq k$. (See, e.g., Figure 2.)

It is also clear that $R_n \cap C_k = \varnothing$ implies that $R_n \cap F = \varnothing$, so no tiles intersect the attractor F. Thus, $\mathcal{T}$ is an open tiling of $C \sim F$. ☐

Corollary 29. *The tiling* $\mathcal{T}$ *is subselfsimilar in that* $\Phi(\mathcal{T}) = \mathcal{T} \sim \bigsqcup_q G_q$*.*

Figure 1 illustrates Theorems 27–28 for the Koch tiling $\mathcal{K}$.

[3] The equality (3.17) is not trivial because the right side has $C \sim \Phi(C)$, not $\overline{C \sim \Phi(C)}$.

3.1. **Properties of the generators.** What kinds of generators are possible? In general, this is a difficult question to answer; it is explored in detail in [LaPe3]. The generators inherit many geometric properties from the convex hull $C = [F]$ and may therefore have a finite or infinite number of nonregular boundary points. In fact, by an observation of [StWa], it is possible (even generic) for the boundary of a 2-dimensional generator to be a piecewise C^1 curve. However, it is impossible for it to be a piecewise C^2 curve.

4. Applications

The motivation behind the self-similar tiling was to find a means of extending the work of [La-vF1] to higher dimensions; this has been partially accomplished in [LaPe2]. The research monograph [La-vF1] is an investigation of the theory of fractal subsets of $\mathbb{R}$. The complement of a fractal within the interval containing it is called a *fractal string* and may be represented by a sequence of bounded open intervals of length l_j:

$$\mathcal{L} := \{l_j\}_{j=1}^{\infty}, \quad \text{with} \sum_{j=1}^{\infty} l_j < \infty. \tag{4.1}$$

Note that l_j is a length (i.e., a number) and not an interval. The authors are able to relate geometric and spectral properties of such objects through the use of zeta functions which encode this data. This information includes the fractal dimension and measurability of the fractal under consideration. One of the main results of [La-vF1] is an explicit formula for the volume $V_{\mathcal{L}}(\varepsilon)$ of the ε-neighbourhood of a fractal string, obtained by applying distributional methods to the geometric zeta function. In [LaPe2], we have obtained similar results for suitable fractal subsets of $\mathbb{R}^d$; specifically, for those with an associated self-similar tiling. The requisite zeta functions are defined in terms of the tiling as it is developed in this paper, and hinges upon the *inradius* of the generators of the tiling, a notion we now explain.

Definition 30. The *inner ε-neighbourhood* of a set A is

$$A_\varepsilon := \{x \in A \mid d(x, \partial A) < \varepsilon\}.$$

Definition 31. The (inner) *tube formula* of A, is

$$V_A(\varepsilon) := V(A_\varepsilon) = \mathrm{vol}_d(A_\varepsilon), \qquad d \in \mathbb{N}, \tag{4.2}$$

and it gives the d-dimensional Lebesgue measure of A_ε.

The tube formula $V_{\mathcal{T}}(\varepsilon)$ for a tiling is useful for studying the dimension and spectral asymptotics of $\mathcal{T}$ and the original self-similar set F; see [La-vF1] and [We], for example. To compute the tube formula for a tiling, note that the tiles R_n have disjoint interior (as shown in Theorem 28), so the formula will simply be a sum taken over the tiles:

$$V_{\mathcal{T}}(\varepsilon) = \sum V_{R_n}(\varepsilon). \tag{4.3}$$

This sum naturally splits into two parts; one with tiles which are entirely within ε of their own boundary, and one with larger tiles. This split is determined by the inradius.

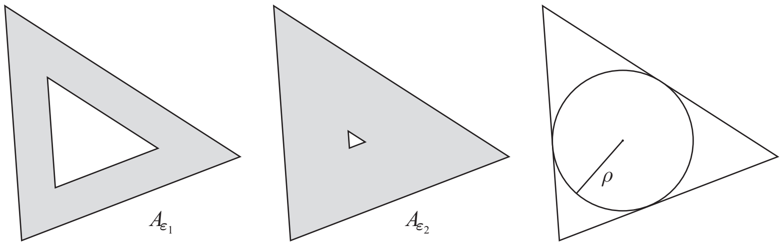

FIGURE 11. Here we see two inner ε-neighbourhoods of a triangle $A \subseteq \mathbb{R}^2$. As ε increases, $A_\varepsilon \to A$ (in the Hausdorff metric, for example). The inradius ρ is depicted at the far right.

Definition 32. The *inner radius* or *inradius* of a set A is

$$\rho = \rho_A = \rho(A) := \sup\{\varepsilon \mid V(A_\varepsilon) < V(A)\}. \tag{4.4}$$

Note that the supremum is taken over $\varepsilon > 0$, because $A_0 = \overline{A}$. The inradii replace the lengths l_j of the 1-dimensional theory.

Proposition 33. *In $\mathbb{R}^d$, the inradius is the furthest distance from a point of A to ∂A, or the radius of the largest ball contained in A, i.e.,*

$$\rho_A = \sup\{\varepsilon \mid V(A_\varepsilon) < V(A)\} \tag{4.5a}$$
$$= \sup\{d(x, \partial A) \mid x \in A\} \tag{4.5b}$$
$$= \sup\{\varepsilon \mid \exists x \text{ with } B(x, \varepsilon) \subseteq A\}. \tag{4.5c}$$

Proof. Let $A \subseteq \mathbb{R}^d$, and denote $m = \sup\{d(x, \partial A) \mid x \in A\}$. Then $V(A_\varepsilon) < V(A)$ implies there is a set of positive d-dimensional measure contained in the interior of A, which is further than ε from any point of ∂A. So $\varepsilon < \rho_A$ implies $\varepsilon < m$. Conversely, if $\varepsilon < m$, then there exists some nonempty open set $U \subseteq A$ for which $d(x, \partial A) > \varepsilon$, $\forall x \in U$. Such a set has positive d-dimensional measure. Thus $\varepsilon < m$ implies $\varepsilon < \rho_A$, whence $m = \rho_A$ and (4.5a) is equivalent to (4.5b).

Now let r be the radius of the largest circle which can be inscribed in A. For a point $x \in A$, we have $d(x, \partial A) \geq r$ if and only if a circle of radius r can be inscribed in A with center at x, i.e.

$$\varepsilon \leq m \iff \varepsilon \leq r.$$

This suffices to show the equivalence of (4.5b) and (4.5c). ☐

Definition 34. The *generating inradii* are the inradii of the generators G_q and denoted

$$g_q := \rho(G_q), \qquad q = 1, \dots, Q. \tag{4.6}$$

Equation (2.17) shows that under the action of Φ, the images of the G_q will be a sequence of tiles with inradii

$$\rho(\Phi_\omega(G_q)) = r_1^{e_1} \dots r_J^{e_J} g_q,$$

for some nonnegative integer exponents e_j, and $q = 1, \dots, Q$. It is precisely this well-behavedness of ρ under the action of Φ that makes it a useful concept.

Let us denote the inradius of the n^{th} tile by $\rho_n := \rho(R_n)$, and then collect all the inradii and index them in nonincreasing order of size. Now $\sigma < \tau$ implies $\rho_\sigma \geq \rho_\tau$, and we have

$$\rho_1 \geq \rho_2 \geq \dots .$$

We now return to the inner tube formula (4.3), beginning with the special case of a single generator. As in [LaPo1] and [La-vF1], this is

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\rho_n \geq \varepsilon/g} V_{R_n}(\varepsilon) + \sum_{\rho_n < \varepsilon/g} V(R_n),$$

where $g = \rho(G)$ is the inradius of the single generator, because $V_A(\varepsilon) = V(A)$ whenever $\varepsilon > \rho_A$. The inner tube formula of $\mathcal{R}_q = \{\Phi_w(G_q)\}$ is

$$V_{\mathcal{R}_q}(\varepsilon) = \sum_{\rho_q^w \geq \varepsilon/g_q} V_{R_q^w}(\varepsilon) + \sum_{\rho_q^w < \varepsilon/g_q} V(R_q^w),$$

Hence the inner tube formula of the tiling is

$$V_{\mathcal{T}}(\varepsilon) = \sum_{q=1}^{Q} \left(\sum_{\rho_q^w \geq \varepsilon/g_q} V_{R_q^w}(\varepsilon) + \sum_{\rho_q^w < \varepsilon/g_q} V(R_q^w) \right). \tag{4.7}$$

In [LaPe2], we compute an explicit formula for $V_{\mathcal{T}}(\varepsilon)$ analogous to [La-vF1, Thm. 6.1], using tools from geometric measure theory and convexity theory. We use the present construction in an essential manner in [LaPe2], along with the extended distributional explicit formula [La-vF1, Thm. 4.21], to obtain the following result.

Theorem 35. *The d-dimensional volume of the inner tubular neighbourhood of $\mathcal{T}$ is given by the following distributional explicit formula:*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon, s); \omega\right) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} c_\omega \varepsilon^{d-\omega}. \tag{4.8}$$

In this formula, $\zeta_{\mathcal{T}}$ is the geometric zeta function of the self-similar tiling whose residues define the constants c_ω, and $\zeta_{\mathcal{T}}$ is defined by a matrix product or bilinear form:

$$\zeta_{\mathcal{T}}(\varepsilon, s) := \langle \mathfrak{g}\zeta_{\mathfrak{s}}, \mathcal{E} \rangle_{\boldsymbol{\kappa}} = \left(\mathfrak{g}^\top \boldsymbol{\kappa} \mathcal{E}\right) \zeta_{\mathfrak{s}}, \tag{4.9}$$

where $\zeta_{\mathfrak{s}}(s) = \sum_w r_w^s$ is a zeta function encoding the combinatorics of the scaling ratios of Φ. The sum in (4.8) is taken over the set of complex dimensions

$$\mathcal{D}_{\mathcal{T}} := \{\text{poles of } \zeta_{\mathcal{T}}\} \cup \{0, 1, \dots, d-1\}.$$

In (4.9), $\mathfrak{g}$ is a vector which has each component of the form g^s (where g is a generating inradius). The matrix $\boldsymbol{\kappa}$ has a row for each generator, and the components of each row are the 0-dimensional through $(d-1)$-dimensional curvatures of that generator. Finally, $\mathcal{E}$ is a vector of 'boundary terms' $\frac{\varepsilon^{i-s}}{d-i}$, where i ranges through the integral dimensions $0, 1, \dots, d$ of the generators. Further discussion of these topics, however, is beyond the scope of the current paper; please see [LaPe2].

5. Concluding remarks

5.1. **Connection to classical results.**

As seen in the latter part of (4.8), the tube formula $V_{\mathcal{T}}(\varepsilon)$ of Theorem 35 is, in fact, a power series in ε, which is summed over the complex dimensions; i.e., $V_{\mathcal{T}}(\varepsilon)$ is a sum of terms $c_\omega \varepsilon^{d-\omega}$, one for each pole ω of $\zeta_{\mathcal{T}}$ and each integer $0, 1, \dots, d-1$. In the case of the integer terms, the coefficient is very closely related to the 0-dimensional through $(d-1)$-dimensional curvature measures. Thus, (4.8) is a fractal analogue or extension of the classical Steiner formula for a nonempty compact convex set A:

$$V_A(\varepsilon) = \sum_{i=0}^{d-1} c_i \varepsilon^{d-i}, \tag{5.1}$$

where the coefficients are given by $c_i := \mu_i(A)\mu_{d-i}(B^{d-i})$. The measure μ_i is the i-dimensional invariant measure (or intrinsic volume) and B^i is the unit ball in i dimensions. In [LaPe4], we hope to realize the other coefficients c_ω as some sort of suitably generalized curvature measures.

5.2. **Affine mappings.**

The construction presented in this paper remains true if the mappings are taken to be affine contractions, instead of similarities. Indeed, the key properties of similarity mappings that have been exploited to prove the theorems of §3 are as follows: similarity transformations are continuous, open, and closed mappings which preserve convexity.

However, I have not pursued the generalization to affine maps, as the tiling was developed as a tool for computing the tube formula associated with a system Φ. The strategy of [LaPe2] is to use tube formulas for the generators to obtain tube formulas for all the tiles. Under affine transformations, however, such an idea does not seem to work.

5.3. **The convex hull.**

One might also ask why the convex hull plays such a unique role in the construction of the tiling. There may exist other sets which are suitable for initiating the construction; however, here are some properties which seem to make the convex hull the natural choice:

(1) Any convex set is the closure of its interior, and hence so is any polyconvex set (as shown in the proof of Cor. 22).
(2) Theorem 14 holds for the convex hull, that is, $\Phi(C) \subseteq C$.
(3) The convex hull of F obviously contains F.

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