Math 205C - Topology Notes Erin Pearse

Definition 1. A **tangent vector** on M at p is a mapping $T_p : C^\infty(p) \to \mathbb{R}$ satisfying

i) $T_p\left(\alpha f + \beta g\right) = \alpha\left(T_p f\right) + \beta\left(T_p g\right)$, and

ii) $T_p\left(f \cdot g\right) = \left(T_p f\right) g\left(p\right) + f\left(p\right)\left(T_p g\right)$,

for all $\alpha, \beta \in \mathbb{R}$ and for all $f, g \in C^\infty(p)$, where $C^\infty(p)$ is the algebra of C^∞ functions whose domain of definition includes some open neighbourhood of p.

Definition 2. The **tangent space $T_p(M)$** of M at p is the collection of all tangent vectors T_p (as defined above) with vector space operations defined by $\left(T_p + S_p\right) f = T_p f + S_p f$ and $\left(\alpha T_p\right) f = \alpha\left(T_p f\right)$.

Definition 3. Let $F : M \to N$ be a C^∞ map of manifolds. Then for $\forall p \in M$, F induces a **pullback map F^***: $C^\infty\left(F(p)\right) \to C^\infty(p)$, which is defined by

$$F^*(f) = f \circ F.$$

Proposition 4. F^* is a homomorphism of algebras.

Definition 5. Let $F : M \to N$ be a C^∞ map of manifolds. Then the **differential of F** is the dual map **F_*: $T_p(M) \longrightarrow T_{F(p)}(N)$** induced by F^* and defined by

$$\left(F_*(X_p)\right)(f) = X_p\left(F^*(f)\right),\ \forall f \in F_*(X_p).$$

The vector $F_*(X_p) \in T_{F(p)}$ is a mapping $F_*(X_p) : C^\infty(F(p)) \to \mathbb{R}$.

Proposition 6.

i) F_* is a homomorphism of vector spaces.

ii) $F : M \to M$ is the identity $\implies$ F^*, F_* are the identity isomorphism.

iii) $H = G \circ F$ are C^∞ maps $\implies$ $H^* = F^* \circ G^*$ and $H_* = G_* \circ F_*$.

Remark. If (U, φ) is a coordinate neighbourhood on M, then the coordinate map φ induces an isomorphism

$$\varphi_* : T_p(M) \to T_{\varphi(p)}(\mathbb{R}^n),$$

at every point $p \in M$. Viewed in reverse,

$$\varphi_*^{-1} : T_a(R^n) \to T_p(M)$$

where $a = \varphi(p)$. We have a natural basis for $\mathbb{R}^n$ consisting of the vectors $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$, so we can extend this idea to determine a natural basis of the tangent space $(T_p(M))$ at each $p \in M$, by considering $\varphi_*^{-1} \circ \frac{\partial}{\partial x_i}$, as in the following definition.

Definition 7. The **coordinate frames $\{E_{1p}, \dots, E_{np}\}$** are the natural basis of $(T_p(M))$, defined as

$$E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right), \forall i = 1, \dots, n.$$

Proposition 8. $\dim T_p(M) = m = \dim M$.

Proof. M is locally diffeomorphic to $\mathbb{R}^n$, $\implies \dim M = n$. So $\mathbb{R}^n$ has a basis $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ of n vectors $\implies (T_p(M))$ has a basis $\{E_{1p}, \dots, E_{np}\}$ of n vectors. ☐

Remark. If $f \in C^\infty(U)$ is expressed as $\hat{f} = f \circ \varphi^{-1}$ in local coordinates relative to U, then $E_{ip} f = \left(\frac{\partial \hat{f}}{\partial x_i}\right)_{\varphi(p)}$ and, in fact, we can write this as

$$E_{ip} f = \left(\varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right)\right) f = \frac{\partial}{\partial x_i}\left(f \circ \varphi^{-1}\right)\Big|_{x=\varphi(p)}$$

Definition 9. A **C^r vector field** on M is a function assigning to every point $p \in M$ a vector $T_p \in T_p(M)$ whose components in the frames of any local coordinates $\{(U_p, \varphi_p)\}$ are functions of class C^r on U_p.

Proposition 10.

i) $\mathfrak{X}(M) = \{\text{all vector fields on M}\}$ forms a vector space.

ii) $\dim \mathfrak{X}(M) = \infty$

(i). For vector fields $X, Y \in \mathfrak{X}(M)$ and $\forall a, b \in \mathbb{R}$, $Z = aX + bY \in \mathfrak{X}(M)$. ☐

(ii). An element of $\mathfrak{X}(M)$ is a vector field T. In order to specify T completely, it is necessary to define T_p on a dense subset $\{p_i\}_{i=1}^\infty$. This requires at least a countable number of m-tuples (that is, tangent vectors T_p), so the basis of $\mathfrak{X}(M)$ is infinite. ☐

1. a) *Let $X, Y \in \mathfrak{X}(M)$ and show that $[X, Y] = XY - YX \in \mathfrak{X}(M)$.*

For $f \in C^\infty$, define $Z \in \mathfrak{X}(M)$ by

$$Z_p f = (XY - YX)_p f = X_p(Yf) - Y_p(Xf).$$

This is a well-defined linear map $C^\infty(p) \to \mathbb{R}$ because $\mathfrak{X}(M)$ is a $C^\infty(M)$-module. To see that Z_p satisfies the Leibniz rule, note that

$$\begin{aligned}
Z_p fg &= (XY - YX)_p fg \\
&= X_p (Yfg) - Y_p (Xfg) \\
&= X_p (fYg + gYf) - Y_p (fXg + gXf) \\
&= (X_p f)(Yg)_p + f(p) X_p (Yg) + (X_p g)(Yf)_p \\
&\qquad + g(p) X_p (Yf) - (Y_p f)(Xg)_p - f(p) Y_p (Xg) \\
&\qquad - (Y_p g)(Xf)_p - g(p)(Y_p Xf) \\
&= f(p)(XY - YX)_p g + g(p)(XY - YX)_p f \\
&= f(p) Z_p g + g(p) Z_p f
\end{aligned}$$

b) *Is XY a vector field in general? If not, provide a counterexample.*
No, XY is not a vector field in general. Counterexample:

2. a) *Define a Lie Algebra.*

A Lie Algebra $\mathfrak{L}$ is a vector space over $\mathbb{R}$ that is endowed with the additional structure of an operation $(X, Y) \to [X, Y] \in \mathfrak{L}$ satisfying

i) it is bilinear over $\mathbb{R}$:

$$\begin{aligned}
[\alpha_1 X_1 + \alpha_2 X_2, Y] &= \alpha_1 [X_1, Y] + \alpha_2 [X_2, Y] \\
[X, \alpha_1 Y_1 + \alpha_2 Y_2] &= \alpha_1 [X, Y_1] + \alpha_2 [X, Y_2]
\end{aligned}$$

ii) it is skew commutative:

$$[X, Y] = -[Y, X]$$

iii) it satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

b) *Prove that $\mathfrak{X}(M)$ with the Lie bracket operation forms a Lie Algebra.*

Let $\alpha, \beta \in \mathbb{R}$ and suppose X_1, X_2, Y are vector fields. Then

$$\begin{aligned}
[\alpha X_1 + \beta X_2, Y] f &= (\alpha X_1 + \beta X_2) Yf - Y(\alpha X_1 + \beta X_2) f \\
&= \alpha X_1 Y f + \beta X_2 Y f - \alpha Y X_1 f - \beta Y X_2 f \\
&= \alpha X_1 Y f - \alpha Y X_1 f + \beta X_2 Y f - \beta Y X_2 f \\
&= \alpha (X_1 Y - Y X_1) f + \beta (X_2 Y - Y X_2) f \\
&= \alpha [X_1, Y] f + \beta [X_2, Y] f
\end{aligned}$$

shows that $[X, Y]$ is linear in the first variable. Then $[Y, X] = YX - XY = -XY + YX = -(XY - YX) = -[X, Y]$ shows that $[X, Y]$ is skew-commutative.

Then

$$\begin{aligned}
[X,[Y,Z]]\,f &= X\left(Y\left(Zf\right)\right) - X\left(Z\left(Yf\right)\right) - Y\left(Z\left(Xf\right)\right) + Z\left(Y\left(Xf\right)\right)\\
[Y,[Z,X]]\,f &= Y\left(Z\left(Xf\right)\right) - Y\left(X\left(Zf\right)\right) - Z\left(X\left(Yf\right)\right) + X\left(Z\left(Yf\right)\right)\\
[Z,[X,Y]]\,f &= Z\left(X\left(Yf\right)\right) - Z\left(Y\left(Xf\right)\right) - X\left(Y\left(Zf\right)\right) + Y\left(X\left(Zf\right)\right)\\
&\implies [X,[Y,Z]]\,f + [Y,[Z,X]]\,f + [Z,[X,Y]]\,f = 0
\end{aligned}$$

3. a) $X, Y \in \mathfrak{X}(M)$ *and* $f, g \in C^\infty(M)$.
   Prove the identity. $[fX, gY] = fg\,[X,Y] + f\,(Xg)\,Y - g\,(Yf)\,X$.

   b) $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, $Y = z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ *in* $\mathbb{R}^3$.
      i) *Compute* $[X,Y]$.
      ii) *Compute* $[Y,Z]$.
      iii) *Compute* $[X,Z]$.