HARMONIC MAPS BETWEEN GRAPHS

by

ELIZABETH SAMANTHA ELDER

Submitted in Partial Fulfillment of the Requirements
for the Degree of
Master of Science

Advisor: Dr. E. Jerome Benveniste

Department of Mathematics
Case Western Reserve University
May 2002

Contents

Acknowledgments

I would like to thank my advisor, Jerome Benveniste for sharing his insight into mathematics with me, for skilled guidance and patient explanations. I would also like to thank Mark Meckes for helpful discussions and help with typesetting.

Harmonic Maps Between Graphs

Abstract

by

ELIZABETH SAMANTHA ELDER

In the first chapter we present historical background. We present definitions and existence results for harmonic maps between Riemannian manifolds and harmonic maps from Riemannian manifolds into nonpositively curved metric spaces. We also present a classical Poincaré inequality. In the second chapter we define harmonicity for a certain class of maps between arbitrary finite graphs. We give several examples of harmonic maps and prove an existence result. We also prove a Poincaré inequality for this class of maps between graphs.

Chapter 1

Background and Motivation

In this section we discuss the definitions and results about harmonic maps in the context of maps between Riemannian manifolds and maps into singular spaces as motivation for the approach taken in section 2 for maps between graphs.

For the discussion of the differential equations approach to harmonic maps between Riemannian manifolds, we essentially follow [4]. Let $f : M \to N$ be a smooth map between Riemannian manifolds M and N where $dim(M) = m$ and $dim(N) = n$. The differential df of f can be viewed as a section of the bundle $T^*M \otimes f^{-1}TN$, and there is a connection ∇ on $T^*M \otimes f^{-1}TN$ induced by the connections on T^*M and $f^{-1}TN$. One then defines the operator

$$\tau(f) = trace(\nabla df).$$

A *harmonic* map $h : M \to N$ is a map with

$$\tau(h) = 0.$$

In local coordinates, this equation has the form:

$$\sum_{\alpha,\beta,j,k} \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma} \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} f^i \right) + \gamma^{\alpha\beta} \Gamma^i_{jk} \frac{\partial}{\partial x^\alpha} f^j \frac{\partial}{\partial x^\beta} f^k = 0$$

for all i, where $(\gamma_{\alpha\beta})$ is the metric tensor on M with respect to local coordinates $(x^1, ..., x^n)$, $\gamma^{\alpha\beta} = (\gamma_{\alpha\beta})^{-1}$, $\gamma = \det(\gamma_{\alpha\beta})$, and Γ^i_{jk} are the Christoffel symbols on N.

There is the following existence theorem of Eells and Sampson [1] with improvements by Hartman [3] for harmonic maps between Riemannian manifolds:

Theorem 1 (Eells-Sampson-Hartman). *Let M and N be compact Riemannian manifolds of dimensions m and n respectively, and suppose N is nonpositively curved. Then any continuous map $f : M \to N$ is homotopic to a harmonic map.*

There is another way of defining what it means for $f : M \to N$ to be a harmonic map between Riemannian manifolds M and N, and that approach is the one which suggests how one may proceed in the case of maps into singular spaces, when defining harmonicity via differential equations is no longer an option. Here we follow [6].

Let M and N be Riemannian manifolds of dimensions m and n respectively, and let γ be the Riemannian inner product on M and g the Riemannian inner product on N. Then for a smooth map $f : M \to N$, the *energy density* of f is given by:

$$e(f) = Trace_{\gamma}(f^{*}g) = \sum_{\alpha,\beta,i,j} \gamma^{\alpha\beta}(x) g_{ij}(f(x)) \frac{\partial f^{i}}{\partial x^{\alpha}} \frac{\partial f^{j}}{\partial x^{\beta}}$$

with notation as above. The energy functional is then:

$$E(f) = \int_{M} e(f) d\mu_{\gamma}$$

where $d\mu_{\gamma}$ is the Riemannian measure on M.

The following theorem ties the energy functional to harmonic maps (see [6], theorem 3.5).

Theorem 2. *Let M be a compact Riemannian manifold and N a complete Riemannian manifold with non-positive sectional curvatures. If $f : M \to N$ is a smooth harmonic map, then $E(f) \leq E(g)$ for any $g : M \to N$ homotopic (rel. ∂M) to f.*

This suggests a way to define what it means for a map from a Riemannian manifold into a general complete metric space to be harmonic; namely, define a similar energy functional and define harmonic maps to be those which minimize the energy functional. This approach is described in much more detail in [7].

Let (Ω^n, g) be a Riemannian domain, i.e. an n-dimensional open subset of a Riemannian manifold whose closure is compact, and (X, d) a complete metric space. Let $f \in L^p(\Omega, X)$, i.e., $\int_\Omega d^p(f(x), Q)d\mu_g(x) < \infty$ for some $Q \in X$. Then define the energy density of f at a point $x \in \Omega$ to be:

$$e_\epsilon(x) = (n+p)\int_{B(x,\epsilon)} \frac{d^p(f(x), f(y))}{\epsilon^p}\frac{dy}{\epsilon^n}$$

for $\epsilon > 0$. One next defines, for $\varphi \in C_c(\Omega)$, the energy functional associated with f:

$$E_\epsilon(\varphi) = \int_\Omega \varphi e_\epsilon(x)dx.$$

Finally, the energy of f is:

$$E(f) = \sup\{\limsup_{\epsilon \longrightarrow 0} E_\epsilon(\varphi) : 0 \leq \varphi \leq 1, \varphi \in C_c(\Omega)\}.$$

Although it is possible to define harmonicity for general $L^p(\Omega, X)$ maps, we will define it here only for continuous maps with finite energy. A continuous map $h : \Omega \to X$ is harmonic if it minimizes energy in its homotopy class.

In order to state an existence result for harmonic maps of this type, we need to introduce the concept of non-positive curvature for a metric space.

Definition 3. *A complete metric space (X, d) is said to be non-positively curved (NPC) if the following conditions hold:*

1. *(X, d) is a length space.*
2. *For any three points x, y, z in X and geodesics γ_{xy}, γ_{yz}, and γ_{zx} connecting them, the following property holds: For $\lambda \in (0, 1)$, let y_λ be the point on γ_{yz} such that if d_{yz} is the distance from y to z, then $d(y, y_\lambda) = \lambda d_{yz}$ and $d(z, y_\lambda) = (1-\lambda)d_{yz}$. On the Euclidean triangle of side lengths d_{xy}, d_{yz}, d_{zx} and opposite vertices $\overline{z}, \overline{x}, \overline{y}$ there is a point*

$$\overline{y_\lambda} = (1-\lambda)\overline{y} + \lambda\overline{z}.$$

The NPC condition is that $d(x, y\lambda)$ is smaller than the corresponding Euclidean distance $\|\overline{x} - \overline{y_\lambda}\|_2$. The actual bound is

$$d^2(x, y_\lambda) \leq (1-\lambda)d^2(x, y) + \lambda d^2(x, z) - \lambda(1-\lambda)d^2(y, z).$$

Remark 4. *The terminology is due to the fact that a manifold with non-positive sectional curvatures possesses the second property in the definition of NPC for metric spaces above. (See [5], section 2.7, "Comparison Theorems for Triangles".)*

We then have the following existence result (see [7], section 2, "Harmonic Maps Into Non-Positively Curved Metric Spaces"):

Theorem 5. *If (Ω^n, g) is a Riemannian domain and (X, d) is a complete metric space with NPC, then in every homotopy class of maps from Ω to X, there is a harmonic map.*

As the last proposition in section 2 is a Poincaré inequality, we now present a classical Poincaré inequality (see [2], Chapter 7, "Sobolev Spaces").

Theorem 6. *Let Ω be a bounded domain in $\mathbb{R}^n$. For $f \in W_0^{1,2}(\Omega)$, the following inequality holds:*

$$\|f\|_2 \leq \left(\frac{1}{\omega_n}|\Omega|\right)^{\frac{1}{n}} \|Df\|2$$

where $\omega_n = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ *and $|\Omega|$ is the Lebesgue measure of Ω.*

Works cited

[1] Eells, Jr., J., and J. H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math. 86* (1964), 109–160.

[2] Gilbarg, D. and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order, 2e.* Springer-Verlag, Berlin, 1983.

[3] Hartman, P. On homotopic harmonic maps. *Canad. J. Math. 19* (1967), 673–687.

[4] Jost, J. *Nonlinear Methods in Riemannian and Kählerian Geometry.* Birkhäuser Verlag, Basel, 1988.

[5] Klingenberg, W. *Riemannian Geometry.* Walter de Gruyter & Co., Berlin, 1982.

[6] Schoen, R., and S. T. Yau. Analytic aspects of the harmonic map problem. In *Lectures on Harmonic Maps.* International Press, Cambridge, MA, 1997, ch. IX.

[7] Schoen, R., and S. T. Yau. Sobolev spaces and harmonic maps for metric space targets. In *Lectures on Harmonic Maps.* International Press, Cambridge, MA, 1997, ch. X.

Chapter 2

Maps Between Graphs

In this section we extend the concept of the energy functional to maps between graphs.

We first assign a metric d to a graph by associating each edge with the unit interval and measuring distance between two points on the same edge with the standard Euclidean metric. For two points on different edges, the distance between the points will be the length of the shortest path connecting them, lengths of paths measured by adding the length contained in each edge.

We also assign a probability measure to each graph. If $G' \subseteq G$ where G is our graph, then

$$\mu(G') = \frac{\displaystyle\sum_{e \in E} m(G' \cap e)}{\displaystyle\sum_{e \in E} m(e)}$$

where E is the set of edges in G and m is Lebesgue measure on each edge.

Definition 7. *A vertex-preserving map (VPM) f between graphs G_1 and G_2 is a continuous map $f : G_1 \to G_2$ such that if v is a vertex in G_1, then $f(v)$ is a vertex in G_2.*

We now define an energy density and an energy functional for VPMs, similar to that defined in section 1.

Definition 8. *The local energy density $e_x(f)$ of a VPM $f : G_1 \to G_2$ at a point $x \in G_1$, where x is not a vertex is:*

$$e_x(f) = \lim_{\epsilon \to 0} \frac{d(f(x_\epsilon), f(x))}{\epsilon}$$

where x_ϵ is a point in G_1 with $d(x_\epsilon, x) = \epsilon$, when this limit exists. The energy density is defined to be 0 for $x \in G_1$ a vertex.

Definition 9. *The energy of a VPM* $f : G_1 \to G_2$ *is defined by*

$$E(f) = \int_{x \in G_1} (e_x(f))^2 d\mu(x).$$

when $e_x(f) \in L^2(G_1)$.

Denote by $H^1(G_1, G_2)$ the space of VPMs $f : G_1 \to G_2$ such that f is absolutely continuous on each edge of G_1 when viewed as a map from $[0, 1]$ to $\mathbb{R}$, and that $e_x(f)$ is a continuous function of $x \in G_1$ when restricted to the interior of each edge of G_1 with $e_x(f) \in L^2(G_1)$. (It is proved in the course of the proof of lemma (12) that any VPM can be considered as a map from $[0, 1]$ to $\mathbb{R}$ when restricted to any edge of G_1.)

Note that our decision that the energy density at a vertex be 0 does not affect the value of the energy of f as the set of vertices in G_1 has measure 0.

In analogy with section 2, $f \in H^1(G_1, G_2)$ will be said to be *harmonic* if $E(f) = \inf\{E(g) : g : G_1 \to G_2 \text{is in the same homotopy class as} f\}$.

Proposition 10. *A map* $f \in H^1(G_1, G_2)$ *has* $E(f) = 0$ *iff* f *is constant on connected components of* G_1.

Proof: This is a simple consequence of the definitions:

$$E(f) = \int_{x \in G_1} (e_x(f))^2 d\mu(x) = 0$$

implies that $e_x(f) = 0$ a.e., as $(e_x(f))^2 > 0$. By definition of $H^1(G_1, G_2)$, $e_x(f)$ is a continuous real-valued function when restricted to the interior of an edge, so we have if $e_x(f)$ is a.e. 0 on the interior of an edge, then $e_x(f) = 0$ on that edge. Recall that $e_x(f)$ was defined to be 0 on vertices, so we have that $e_x(f) \equiv 0$. Now, given $x \in G_1$ not a vertex, $e_x(f)$ is just $|f'(x)|$, when f is viewed as a function from $[0, 1]$ to $\mathbb{R}$. So we have that on the interior of each edge, $|f'(x)| = 0$, hence $f'(x) = 0$, hence f is constant on the interior of each edge in G_1. But since f is a continuous map, any two adjacent edges must be taken by f to the same point, hence f is constant on connected components of G_1. $\square$

From this point on, we will assume that all of our graphs are connected.

Examples:

1. *If* $f \in H^1(G_1, G_2)$ *and* G_2 *is a tree, then* f *is harmonic iff* f *is constant.*

Proof: "If" is trivial, because it is obvious from the definition that a constant map f has $E(f) = 0$ and that $E(f) \geq 0$. To show that the only harmonic $H^1(G_1, G_2)$ maps into trees are constant, it suffices to note that a tree is contractible, hence all continuous maps into trees are homotopic to a constant map. As this homotopy class contains some $f \in H^1(G_1, G_2)$ with $E(f) = 0$, the harmonic maps are exactly those with 0 energy, and the proposition shows that those are exactly the constant maps. □

2. *Characterization of harmonic maps between polygons.*

We will first need

Lemma 11. *When considering $f \in H^1(N, M)$ where N is an n-gon and M is an m-gon, it suffices to consider f with $deg(f) = 0$ or $deg(f) = 1$.*

Proof: Let LM be an lm-gon. Then we will see that LM is an l-fold covering of M. Let v_N be a particular vertex in N, let $f : N \to M$ be a map of degree l with $f(v_N) = v_M$, and choose a vertex v_{LM} in LM. Define a map $k : LM \to M$ in the following way: First define $k(v_{LM}) = v_M$. Then, moving clockwise around LM and M, map the first edge in LM into the first edge in M via the identity on the unit interval, the second edge in LM into the second edge in M in the same way, etc. Continue in this way until the mth edge of LM has been mapped into the last edge of M. Then map the $m + 1$st edge of LM into the first edge of M, the $m + 2$nd edge of LM into the second edge of M, etc. In this way we get a surjective map $k : LM \to M$, which is in fact a covering map. Thus a lifting $\tilde{f} : N \to LM$ of f exists iff

$$f_*(\pi_1(N, v_N)) \subseteq k_*(\pi_1(LM, v_{LM})).$$

By the construction of k, we can see that $k_*(\pi_1(LM, v_{LM}))$ is the subgroup of $\pi_1(M, v_M)$ generated by a loop of degree l. Likewise, as f is a degree l map, $f_*(\pi_1(N, v_N))$ is also the subgroup of $\pi_1(M, v_M)$ generated by a loop of degree l. So the above inclusion holds (with equality), and thus $\tilde{f}$ exists. Note that $\tilde{f}$ is a VPM by construction (the only preimages of vertices in M are vertices in LM). In fact, $E(f) = E(\tilde{f})$. This is just because LM is locally isometric to M under k, and so $e_x(f) = e_x(\tilde{f})$ $\forall x \in G_1$. Also, if $f, g \in H^1(N, M)$ are homotopic with $deg(f) = deg(g) = l$, then we may assume that $f(v_N) = g(v_N) = v_M$ (possibly needing to compose one or both maps by a rotation of M which would not affect computation of energy). Then we can just lift the homotopy as we lifted f above, and we get that the liftings $\tilde{f}$ and $\tilde{g}$ are homotopic. So considering the liftings of maps to LM instead of considering the maps themselves preserves energy computations and homotopy classes. Also, instead of considering maps of negative degree, one can consider the map of positive degree which is just the original map composed with a reflection of M through a bisector containing v_M. This reflection preserves homotopy classes and does not affect the computation of energy as it is an isometry. □

Now, note that if G_1 consists of one edge and is being mapped into G_2 which also consists of one edge, and the two vertices of G_1 are not both mapped to the same vertex in G_2, then the minimal energy is achieved if the mapping is just the identity on the unit interval:
If $f : G_1 \to G_2$ is such a map, then f can be thought of as a map from the unit interval to itself, with $f(0) = 0$ and $f(1) = 1$. In this case, we have:

$$\begin{aligned} E(f) &= \int_0^1 |f'(x)|^2 dx \\ &\geq \left(\int_0^1 |f'(x)| dx \right)^2 \\ &\geq \left(\int_0^1 f'(x) dx \right)^2 \\ &= (f(1) - f(0))^2 = 1 \end{aligned}$$

where the first inequality follows because on a probability space, $\| \cdot \|_{L^2} \geq \| \cdot \|_{L^1}$, and the use of the fundamental theorem of calculus is justified because f is absolutely continuous on G_1 when viewed as a map from $[0, 1]$ to itself. In fact, in this inequality we have equality iff $f'(x)$ is constant, which in this situation can happen iff f is actually the identity map, and so in this situation the identity is the unique harmonic map.

Another important thing to generally note about harmonic maps is that two edges in G_1 being mapped isometrically into two edges in G_2 will add less to the total energy than one edge in G_1 being mapped into two edges in G_2 and a second edge in G_1 being mapped to a single point. This is because in the second case, for a harmonic map, $e_x(f) = 2$ on one edge and $e_x(f) = 0$ on the other edge, so the contribution to $E(f)$ will be 4 times the weight of one edge in G_1. But in the first case, $e_x(f) = 1$ on two edges, and so the contribution to $E(f)$ will be 2 times the weight of one edge in G_1. So in our quest for harmonic maps, we will be looking for maps which are local isometries when possible, and if that is not possible, then we look for the maps with as little stretching as possible. Of course, as we consider only VPMs, one edge in G_1 cannot be mapped into less than an entire edge in G_2 unless the whole edge is mapped to a single point.

We can now characterize harmonic maps $f : N \to M$ where N is an n-gon and M is an m-gon. Applying lemma (11), we may consider degree one and degree zero maps only. We will do this by considering three separate cases:

(a) $n > m$.

As remarked above, the harmonic maps in the degree one homotopy class will be the ones which are isometric when possible. If $n > m$, then we can map m edges of N isometrically into M, and map the remaining $n - m$ edges of N into single vertices. Any VPM with these properties is clearly harmonic in this homotopy class, because each edge of N which is mapped non-trivially into M is making the minimal contribution to the energy by being mapped isometrically, and we cannot have fewer than m edges mapped into M in this way or the homotopy class would become the trivial class. For such maps, we have that $e_x(f) = 1$ on m of N's edges and $e_x(f) = 0$ on the remaining $n - m$ of N's edges, and so $E(f) = \frac{m}{n}$. Note that harmonic maps are not at all unique, because it doesn't matter which of the edges of N are mapped non-trivially into M and which are mapped trivially into M; all that matters is the number.

In this as in the other two cases, for the trivial homotopy class; i.e., maps of degree 0, the same argument as in example (1) shows that the harmonic maps are exactly the constant maps.

(b) $n = m$.

In this case the homotopy class of degree 1 maps is just the identity homotopy class. By the remark above, the harmonic maps will be those which are isometries, and as those maps have $e_x(f) = 1 \ \forall x \in N$, hence $E(f) = 1$ for the harmonic maps in this homotopy class. Note again that harmonic maps are not unique, as any rotation of N which is a VPM will be harmonic.

(c) $n < m$.

In this case, what we would like to do is to map each edge of N into $\frac{m}{n}$ edges of M, because that way no edge in N is being stretched any more than any other. An argument exactly analogous to the one showing that isometry is best when possible shows that it is best to have all edges being stretched the same amount.

However, $\frac{m}{n}$ is not in general an integer, and if it is not, then the map we get in this way would not be a VPM. So let $k = \left\lfloor \frac{m}{n} \right\rfloor$, the greatest integer less than or equal to $\frac{m}{n}$. Then we want to stretch some number of edges of N over k edges each of M, and the rest of the edges of N over $k+1$ edges each of M, so that the map we get is injective. If p is the number of edges in N to be stretched over k edges each in M, then we have that p satisfies

$$pk + (n-p)(k+1) = m$$

$$\Leftrightarrow p = n(k+1) - m$$

For a harmonic map f which behaves the way described above, we will have that $e_x(f) = k$ on p of the edges of N and $e_x(f) = k+1$ on $n-p$ edges of N. This gives us:

$$E(f) = \frac{pk^2 + (n-p)(k+1)^2}{n}$$

and substituting the value above for p yields:

$$E(f) = \frac{2mk + m - nk^2 - nk}{n}.$$

Note that the formula above developed for the case $n < m$ actually is valid in all three cases. □

A natural question which arises is that of existence: Is there a harmonic map in every homotopy class of maps in $H^1(G_1, G_2)$ between two general graphs G_1 and G_2? In fact, we will see that the answer is yes. To prove this, we will first need:

Lemma 12. *Let $f \in H^1(G_1, G_2)$ be such that for any edge $e \in G_1$ and $x_1, x_2 \in e$, if $f(x_1) = f(x_2)$, then for every x between x_1 and x_2, $f(x) = f(x_1) = f(x_2)$. (Call an f with this property semi-injective.) Then*

$$\int_e (e_x(f))^2 d\mu = \mu(e) \int_0^1 (\varphi'(x))^2 dx \tag{2.1}$$

for some $\varphi : [0,1] \to [0,m]$ which is continuous, non-decreasing, and $\varphi(0) = 0$ and $\varphi(1) = m$, where m is the length of the path $f|_e$ in G_2. Further, φ is homotopic to the function $\psi(x) = mx$ (rel. endpoints), and this ψ gives rise to a map $\tilde{f}$ which is homotopic to f and such that $e_x(\tilde{f}) = m$ for all $x \in e$.

Proof: Construct φ as follows:

Associate e in the usual way with $[0,1]$. Then the endpoint 0 is mapped to a vertex v_0 in G_2. If all of $[0,1]$ is not mapped by f to v_0, then there is an ϵ such that $f(\epsilon) \neq v_0$ and such that $f(\epsilon)$ is on an edge e_0 emanating from v_0. But since f is semi-injective, there must be a t_1 such that $f(t_1) = v_1$, the vertex adjacent to v_0 on the other side of e_0 and $f([0,t_1]) = e_0$; if not, then because f is continuous and vertex-preserving, we would have to have $f(x) = v_0$ for some $x > \epsilon$. But that contradicts the semi-injectivity of f. So $f([0,t_1]) = e_0$, and e_0 is isometric to $[0,1]$. Let $i_0 : e_0 \to [0,1]$ be the isometry of e_0 and $[0,1]$ such that $i_0(v_0) = 0$. Define

$$\varphi|_{[0,t_1]}(x) = i_0 \circ f(x).$$

Because $e_x(f) = \lim_{\epsilon \to 0} \frac{d(f(x_\epsilon), f(x))}{\epsilon} = |\varphi'(x)|$ on $[0,t_1]$, we may calculate the contribution of $[0,t_1]$ to $E(f)$ by computing $\mu(e) \int_0^{t_1} (\varphi'(x))^2 dx$.

Now, if $[t_1,1]$ is not all mapped by f into v_1, then there there is some ϵ such that $f(t_1+\epsilon) \neq v_1$ and such that $f(t_1+\epsilon)$ is on an edge e_1 (not the same as e_0) emanating from v_1. Since f is semi-injective, there is a t_2 such that $f(t_2) = v_2$, the vertex on the other side of e_1, and $f([t_1,t_2]) = e_1$ by exactly the same reasoning as above. Let $i_1 : e_1 \to [1,2]$ be the isometry with $i_1(v_1) = 1$. Define

$$\varphi|_{[t_1,t_2]}(x) = i_1 \circ f(x).$$

Note that this agrees with the earlier definition of φ at t_1, and that $e_x(f) = |\varphi'(x)|$ for all $x \in [t_1,t_2]$.

Continue this process to define φ on all of $[0,1]$. Then $im(\varphi) = [0,m]$ where m is the length of the path $f|_e : [0,1] \to G_2$. At each $x \in e = [0,1]$, we have that $e_x(f) = |\varphi'(x)|$, and so we have shown (2.1). Note that φ is continuous, non-decreasing, and $\varphi(0) = 0$ and $\varphi(1) = m$ by construction.

Next we show that $\varphi : [0,1] \to [0,m]$ is homotopic to $\psi : [0,1] \to [0,m]$ (rel. endpoints), where $\psi(x) = mx$. Let $h(s,t) = tms + (1-t)\varphi(s)$. Then h is a continuous function of s and t, $h(s,0) = \varphi(s)$, $h(s,1) = ms = \psi(s)$, $h(0,t) = 0$, and $h(1,t) = m$. So h is a homotopy (rel. endpoints) of φ and ψ.

To get $\tilde{f}$ from ψ, we essentially reverse the process by which φ was constructed from f. Note that $\psi([0,\frac{1}{m}]) = [0,1]$. Define

$$\tilde{f}|_{[0,\frac{1}{m}]}(x) = i_0^{-1} \circ \psi(x)$$

where i_0 is the isometry of e_0 and $[0,1]$ such that $i_0(v_0) = 0$ as before. Next note that

$\psi([\frac{1}{m},\frac{2}{m}]) = [1,2]$. Define

$$\tilde{f}|_{[\frac{1}{m},\frac{2}{m}]}(x) = i_1^{-1} \circ \psi(x)$$

where i_1 is the isometry of e_1 and $[1,2]$ such that $i_1(v_1) = 1$ as above. Continue this process to define $\tilde{f}$ on all of $[0,1] = e$. Now, $\tilde{f}|_e$ is homotopic to $f|_e$ (rel. endpoints). This is because the homotopy h between φ and ψ can be pulled back to a homotopy $\overline{h}$ of $f|_e$ and $\tilde{f}|e$ via the same process which pulled ψ back to $\tilde{f}$:
Note that for each fixed $t \in [0,1]$, $h(\cdot,t) : [0,1] \to [0,m]$ is a continuous, non-decreasing function. Thus there is an $s_1 \in [0,1]$ such that $h(s_1,t) = 1$ and such that $h([0,s_1],t) = [0,1]$. For this fixed t, define

$$\overline{h}|_{[0,s_1]}(s,t) = i_0^{-1} \circ h(s,t).$$

Next we can find an $s_2 > s_1$ such that $h(s_2,t) = 2$ and such that $h([s_1,s_2],t) = [1,2]$. For t, define

$$\overline{h}|_{[s_1,s_2]}(s,t) = i_1^{-1} \circ h(s,t).$$

Continue this pulling back process until $\overline{h}(s,t)$ is defined on all of $[0,1]$ for this fixed t. Since t was arbitrary, this defines a map for each t, so $\overline{h}(s,t)$ is defined on $[0,1]^2$, and $\overline{h}$ is a homotopy of $f|_e$ and $\tilde{f}|_e$ relative to the endpoints of e by construction. Hence we may define $\tilde{f}$ on G_1 by defining $\tilde{f}(x) = f(x)$ on $G_1 \setminus e$ and $\tilde{f}$ is then homotopic to f. On the edge e, $e_x(\tilde{f}) = |\psi'(x)| = m$. $\square$

Proposition 13. *If G_1 and G_2 are arbitrary (finite) graphs, then in every homotopy class of $H^1(G_1,G_2)$ maps, there is at least one harmonic map.*

Proof: Let $f \in H^1(G_1,G_2)$ be given, and let $\{\mathfrak{F}_i\}_{i=1}^n$ be classes of $H^1(G_1,G_2)$ maps such that f_i is homotopic to f for all $f_i \in \mathfrak{F}_i$ and for all i, and that any two maps in the same class $\mathfrak{F}_i$ agree on the vertices of G_1. Note that the set of these classes of maps is finite, because there are only finitely many different ways to send the vertices of G_1 to the vertices of G_2. We will show that in each $\mathfrak{F}_i$, there is a VPM $\tilde{f}_i$ of minimal energy. Once we have shown this, we can obtain a harmonic map $\tilde{f}$ in the homotopy class of f by choosing $\tilde{f}$ to be the $\tilde{f}_k$ with $E(\tilde{f}_k) = \min\{E(\tilde{f}_i) : 1 \leq i \leq n\}$

So it remains to show that within each $\mathfrak{F}_i$, there is a VPM of minimal energy. This can be done in the following way:
Choose a representative $f_i \in \mathfrak{F}_i$. Restricting our attention to two adjacent vertices in G_1, we examine the image under f_i in G_2 of the edge e which joins them. Look for pairs of points $x_1, x_2 \in e$ s.t. $f_i(x_1) = f_i(x_2)$ and s.t. the restriction of f_i to the segment e' of e between x_1 and x_2 is homotopic to (but not equal to) the constant map on e'. For any such pair, get

a new map from f_i by replacing $f_i|_{e'}$ with the constant map. This new map is still in $\mathcal{F}_i$, and it has smaller energy than f_i. Continue this until there are no more such pairs. What we have at this point is a semi-injective map f_i^* as described in the statement of lemma 12. We can thus apply the lemma and get a new map $f_i^\dagger$ homotopic to f_i^* such that $e_x(f_i^\dagger) = m$ on e, where m is the length of the path f_i^* in G_2. The result of this construction yields the minimal possible contribution of the edge e to the total energy of a map in $\mathcal{F}_i$. One can perform this procedure on every edge of G_1 as it is a finite graph, and the result will be a map in $\mathcal{F}_i$, which has minimal energy in that class of maps. $\square$

Remark 14. *As noted in the example of harmonic maps between polygons, harmonic maps are not unique within homotopy classes.*

We next prove a Poincaré inequality for maps in $H^1(G_1, G_2)$.

Proposition 15. *The following inequality holds for all maps $f \in H^1(G_1, G_2)$, where C is a constant depending only on the number of vertices in G_1:*

$$|f|_2 \leq CE(f).$$

Here, $|f|_2 = \min\left\{\int_{x\in G_1} d^2(f(x), y)d\mu(x) : y \in G_2\right\}$.

Proof: Suppose not. Then there is an n and a G_1 with n vertices such that for all constants C, there is an $f \in H^1(G_1, G_2)$ such that $|f|_2 > CE(f)$. Let m be the number of edges contained in $f(G_1)$. Then $E(f) \geq \dfrac{2mk + m - nk^2 - nk}{n}$, where $k = \left\lfloor \dfrac{m}{n} \right\rfloor$. This estimate is obtained with exactly the same reasoning as the polygon example. Also, we can trivially estimate $|f|_2 < m^2$. This gives us the following string of inequalities:

$$m^2 > |f|_2 > CE(f) > C\frac{2mk + m - nk^2 - nk}{n}.$$

We can absorb the n in the denominator of the RHS into C as n is a fixed integer such that the statement holds. Now, $2mk + m - nk^2 - nk \geq 2m(\frac{m}{n} - 1) + m - n(\frac{m}{n})^2 - n(\frac{m}{n}) = \frac{m^2}{n} - 2m$, so we get that $\dfrac{nm}{m - 2n} > C$. So there is some finite n such that this holds for a fixed arbitrary m for *any* constant. This is clearly not possible, and we have a contradiction. $\square$

Note that this proof gives no indication of the way in which C depends on n. The following example indicates that the dependence is probably fairly simple.

Example:

This example is important because it is the case in which $|f|_2$ is biggest and $E(f)$ is still fairly small.

Suppose that G_1 has n edges (hence at least n vertices) and that the image of G_1 in G_2 is a line segment made up of m edges, where m is even. Suppose that f sends $\left\lfloor \frac{n-1}{2} \right\rfloor$ edges to one of the endpoints of the line segment, all but one of the remaining edges to the other endpoint of the line segment, and the one remaining edge is stretched to span the m edges of the segment. Then $E(f) = \dfrac{m^2}{n}$. We get that $|f|_2 = \dfrac{\frac{m^2}{4}(n-1) + \frac{m}{4}}{n}$ because the fixed point $y \in G_2$ which minimizes $\displaystyle\int_{x \in G_1} d^2(f(x), y) d\mu(x)$ is the point exactly in the center of the segment. In this case then, we would need C to be at least $\dfrac{|f|_2}{E(f)} = \dfrac{n-1}{4} + \dfrac{1}{4m} \leq \dfrac{n}{4} - \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{n}{4}$. So here the n dependence of C is very simple; it is linear in n.

Bibliography

EELLS, JR., J., AND J. H. SAMPSON. Harmonic mappings of Riemannian manifolds. *Amer. J. Math. 86* (1964), 109–160.

GILBARG, D. AND N. S. TRUDINGER. *Elliptic Partial Differential Equations of Second Order, 2e.* Springer-Verlag, Berlin, 1983.

HARTMAN, P. On homotopic harmonic maps. *Canad. J. Math. 19* (1967), 673–687.

JOST, J. *Nonlinear Methods in Riemannian and Kählerian Geometry.* Birkhäuser Verlag, Basel, 1988.

KLINGENBERG, W. *Riemannian Geometry.* Walter de Gruyter & Co., Berlin, 1982.

SCHOEN, R., AND S. T. YAU. Analytic aspects of the harmonic map problem. In *Lectures on Harmonic Maps.* International Press, Cambridge, MA, 1997, ch. IX.

SCHOEN, R., AND S. T. YAU. Sobolev spaces and harmonic maps for metric space targets. In *Lectures on Harmonic Maps.* International Press, Cambridge, MA, 1997, ch. X.