

A PRESENTATION FOR $Aut(F_n)$

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ABSTRACT. We study the action of the group $Aut(F_n)$ of automorphisms of a finitely generated free group on the degree 2 subcomplex of the spine of Auter space. Hatcher and Vogtmann showed that this subcomplex is simply connected, and we use the method described by K. S. Brown to deduce a new presentation of $Aut(F_n)$.

1. INTRODUCTION

In 1924 Nielsen produced the first finite presentation for the group $Aut(F_n)$ of automorphisms of a finitely-generated free group [6]. Other presentations have been given by B. Neumann [7] and J. McCool [5]. A very natural presentation for the index two subgroup $SAut(F_n)$ was given by Gersten in [3].

Nielsen, McCool and Gersten used infinite-order generators. Neumann used only finite-order generators of order at most n , but his relations are very complicated. P. Zucca showed that $Aut(F_n)$ can be generated by three involutions, two of which commute, but did not give a complete presentation [9].

In this paper we produce a new presentation for $Aut(F_n)$ which has several interesting features. The generators are involutions and the number of relations is fairly small. The form of the presentation for $n \geq 4$ depends only on the size of a signed symmetric subgroup.

The presentation is found by considering the action of $Aut(F_n)$ on a subcomplex of the *spine of Auter space*. This spine is a contractible simplicial complex on which $Aut(F_n)$ acts with finite stabilizers and finite quotient. A vertex of the spine corresponds to a basepointed graph Γ together with an isomorphism $F_n \rightarrow \pi_1(\Gamma)$. In [4] Hatcher and Vogtmann defined a sequence of nested invariant subcomplexes K_r of this spine, with the property that the r -th complex K_r is $(r-1)$ -connected. In particular, K_2 is simply-connected, and we use the method described by K. S. Brown in [2] to produce our finite presentation using the action of $Aut(F_n)$ on K_2 .

In order to describe the presentation, we fix generators a_1, \dots, a_n for the free group F_n and let W_n be the subgroup of $Aut(F_n)$ which permutes and inverts the generators. We let τ_i denote the element of

W_n which inverts a_i , and σ_{ij} the element which interchanges a_i and a_j :

$$\tau_i: \begin{cases} a_i \mapsto a_i^{-1} \\ a_j \mapsto a_j & j \neq i \end{cases} \quad \sigma_{ij}: \begin{cases} a_i \mapsto a_j \\ a_j \mapsto a_i \\ a_k \mapsto a_k & k \neq i, j. \end{cases}$$

There are many possible presentations of W_n . For instance, W_n is generated by τ_1 and by transpositions $s_i = \sigma_{i,i+1}$ for $1 \leq i \leq n-1$, subject to relations

$$\begin{aligned} s_i^2 &= 1 & 1 \leq i \leq n-1 \\ (s_i s_j)^2 &= 1 & j \neq i \pm 1 \\ (s_{i-1} s_i)^3 &= 1 & 2 \leq i \leq n-1 \\ \tau_1^2 &= 1 \\ (\tau_1 s_1)^4 &= 1 \\ (\tau_1 s_i)^2 &= 1 & 2 \leq i \leq n-1. \end{aligned}$$

Generators for $\text{Aut}(F_n)$ will consist of generators for W_n plus the following involution:

$$\eta: \begin{cases} a_1 \mapsto a_2^{-1} a_1 \\ a_2 \mapsto a_2^{-1} \\ a_k \mapsto a_k & k > 2. \end{cases}$$

The presentation we obtain is the following:

Theorem 1. *For $n \geq 4$, $\text{Aut}(F_n)$ is generated by W_n and η , subject to the following relations:*

- (1) $\eta^2 = 1$
- (2) $(\sigma_{12}\eta)^3 = 1$
- (3) $(\eta\tau_i)^2 = 1$ for $i > 2$
- (4) $(\eta\sigma_{ij})^2 = 1$ for $i, j > 2$
- (5) $((\eta\tau_1)^2\tau_2)^2 = 1$
- (6) $(\eta\sigma_{13}\tau_2\eta\sigma_{12})^4 = 1$
- (7) $\sigma_{12}\eta\sigma_{13}\tau_2\eta\sigma_{12}(\sigma_{23}\eta\sigma_{13}\tau_2\eta)^2 = 1$
- (8) $(\sigma_{14}\sigma_{23}\eta)^4 = 1$
- (9) relations in W_n .

The presentation we obtain for $n = 3$ differs only in that every relation involving indices greater than 3 is missing:

Corollary 1. *The group $\text{Aut}(F_3)$ is generated by W_3 and η , subject to the following relations:*

- (1) $\eta^2 = 1$
- (2) $(\sigma_{12}\eta)^3 = 1$

- (3) $(\eta\tau_3)^2 = 1$
- (4) $((\eta\tau_1)^2\tau_2)^2 = 1$
- (5) $(\eta\sigma_{13}\tau_2\eta\sigma_{12})^4 = 1$
- (6) $\sigma_{12}\eta\sigma_{13}\tau_2\eta\sigma_{12}(\sigma_{23}\eta\sigma_{13}\tau_2\eta)^2 = 1$
- (7) *relations in W_3 .*

For $n = 2$ we get:

Corollary 2. *The group $Aut(F_2)$ is generated by $\tau_1, \tau_2, \sigma_{12}$ and η , subject to the following relations:*

- (1) $\eta^2 = 1$
- (2) $(\sigma_{12}\eta)^3 = 1$
- (3) $((\eta\tau_1)^2\tau_2)^2 = 1$
- (4) $\sigma_{12}^2 = 1$
- (5) $\tau_1^2 = 1$
- (6) $(\tau_1\sigma_{12})^4 = 1$
- (7) $\tau_2 = \sigma_{12}\tau_1\sigma_{12}$.

2. BROWN'S THEOREM

To find our presentation, we use the method described by K. S. Brown in [2]. This method applies whenever a group G acts on a simply-connected CW-complex by permuting cells, but the description is simpler if the complex is simplicial and the action does not invert edges. Since this is the case for us, we describe this simpler version. We remark that a presentation of the fundamental group of a complex of groups, whether or not it arises from the action of a group on a complex, can be found in [1], Chapter III.C.

Let G be a group, and X a non-empty simply-connected simplicial complex on which G acts without inverting any edge. Let \mathcal{V}, \mathcal{E} and \mathcal{F} be sets of representatives of vertex-orbits, edge-orbits, and 2-simplex-orbits, respectively, under this action. The group G is generated by the stabilizers G_v of vertices in \mathcal{V} together with a generator for each edge $e \in \mathcal{E}$. There is a relation for each element of \mathcal{F} . Other relations come from loops in the 1-skeleton of the quotient X/G . In order to write down a presentation explicitly, we choose the sets \mathcal{V}, \mathcal{E} and \mathcal{F} quite carefully, as follows.

The 1-skeleton of the quotient X/G is a graph. Choose a maximal tree in this graph and lift it to a tree \mathcal{T} in X . The vertices of \mathcal{T} form \mathcal{V} , our set of vertex-orbit representatives for the action of G on X . Since the edges of \mathcal{T} are not a complete set of edge-orbit representatives, we complete the set \mathcal{E} by including for each missing orbit a choice of representative which is connected to \mathcal{T} . Finally, for the set \mathcal{F} , we

choose representatives for the 2-simplices so that they also have at least one vertex in \mathcal{T} .

We obtain a presentation for G as follows:

Generators. The group G is generated by the stabilizers G_v for $v \in \mathcal{V}$ together with a generator t_e for each $e \in \mathcal{E}$.

Relations. There are four types of relations: tree relations, edge relations, face relations and stabilizer relations. The *tree relations* are:

$$(1) \quad t_e = 1 \text{ if } e \in \mathcal{T}.$$

There are *edge relations* for each edge $e \in \mathcal{E}$ which identify the two different copies of G_e , the stabilizer of e , which can be found in the stabilizers of the endpoints of e . To make this explicit, we orient each edge $e \in \mathcal{E}$ so that the initial vertex $o(e)$ lies in \mathcal{T} , and let $i_e : G_e \rightarrow G_{o(e)}$ denote the inclusion map. There is also an inclusion $G_e \rightarrow G_{t(e)}$, where $t(e)$ is the terminal vertex. Note that when $t(e)$ is not in \mathcal{T} , $G_{t(e)}$ is not in our generating set. To encode the information of this inclusion map in terms of our generating set we must do the following. Since $t(e)$ is equivalent to some vertex $w(e)$ in \mathcal{T} , we choose $g_e \in G$ with $g_e w(e) = t(e)$ (if $t(e) \in \mathcal{T}$, we choose $g_e = 1$). Conjugation by g_e is an isomorphism from $G_{t(e)}$ to $G_{w(e)}$, so we set $c_e : G_e \rightarrow G_{w(e)}$ to be the inclusion $G_e \rightarrow G_{t(e)}$ followed by conjugation by g_e . Equating the two images of G_e gives us the edge relations, which are then:

$$(2) \quad \text{For } x \in G_e, \quad t_e i_e(x) t_e^{-1} = c_e(x).$$

There is a *face relation* for each 2-simplex $\Delta \in \mathcal{F}$. To describe this, we use the notation established in the previous paragraph.

We digress for a moment to consider an arbitrary oriented edge e' of X with $o(e') \in \mathcal{V}$. This edge is equivalent to some edge $e \in \mathcal{E}$. If the orientations on e' and e agree, then $e' = he$ for some $h \in G_{o(e')}$, and $t(e') = hg_e w(e)$. If the orientations do not agree, then $e' = hg_e^{-1}e$ for some $h \in G_{o(e')}$, and $t(e') = hg_e^{-1}o(e)$. The element h is unique modulo the stabilizer of e' .

Now let $e'_1 e'_2 e'_3$ be an oriented edge-path starting in \mathcal{T} and going around the boundary of Δ . Since e'_1 originates in \mathcal{T} , we can associate to it elements $h_1 \in G_{o(e'_1)}$ and $g_1 = h_1 g_{e'_1}^{\pm 1}$ as described above. Then e'_2 originates in $g_1 \mathcal{T}$, so $g_1^{-1} e'_2$ originates in \mathcal{T} , and we can find h_2 and $g_2 = h_2 g_{e'_2}^{\pm 1}$ for $g_1^{-1} e'_2$. Now e'_3 originates in $g_1 g_2 \mathcal{T}$ so we can find h_3 and $g_3 = h_3 g_{e'_3}^{\pm 1}$ associated to $g_2^{-1} g_1^{-1} e'_3$. Set $g_\Delta = g_1 g_2 g_3$, and note that g_Δ is in the stabilizer of the vertex $o(e'_1)$, so that the following is a relation among our generators:

$$(3) \quad \text{For each } \Delta \in \mathcal{F}, \quad h_1 t_{e'_1}^{\pm 1} h_2 t_{e'_2}^{\pm 1} h_3 t_{e'_3}^{\pm 1} = g_\Delta.$$

Here the sign on $t_{e'_i}$ is equal to the sign on g_{e_i} in the expression for g_i .

Finally, a *stabilizer relation* is a relation among the generators of a vertex stabilizer G_v .

Theorem 2. (Brown) *Let X be a simply-connected simplicial complex with a simplicial action by the group G which does not invert edges. Then G is generated by the stabilizers G_v ($v \in \mathcal{V}$) and symbols t_e ($e \in \mathcal{E}$) subject to all tree, edge, face and stabilizer relations as described above.*

3. COMPUTATIONS

3.1. The Degree 2 complex. We will apply Theorem 2 to a certain subcomplex of the spine of Auter space. The spine of Auter space is a contractible simplicial complex on which $Aut(F_n)$ acts with finite stabilizers and finite quotient. For full details on the construction of Auter space, we refer to [4].

A vertex in the spine of Auter space is a connected, basepointed graph Γ together with an isomorphism $g: \pi_1(\Gamma) \rightarrow F_n$, called a *marking*. (Note: often in the literature the marking goes in the other direction). We require all vertices of Γ to have valence at least three, and we also assume that Γ has no separating edges. One can describe the marking g by labeling certain edges of Γ as follows. Choose a maximal tree in Γ . The edges not in this maximal tree form a natural basis for the fundamental group $\pi_1(\Gamma)$. Orient each of these, and label them by their images in F_n . This description depends on the choice of maximal tree; for instance the labeled graphs in Figure 1 represent the same vertex.

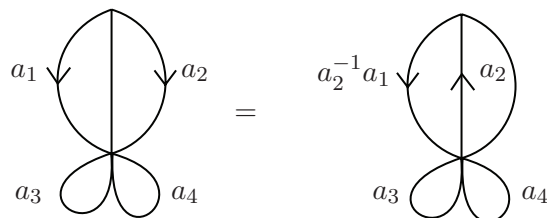


FIGURE 1. Two labeled graphs representing the same vertex in the spine of Auter space

Two marked graphs span an edge in the spine if one can be obtained from the other by collapsing a set of edges (this is called a *forest collapse*). A set of $k + 1$ vertices spans a k -simplex if each pair of vertices spans an edge.

The group $Aut(F_n)$ acts on Auter space on the left by $\alpha \cdot (g, \Gamma) = (\alpha \circ g, \Gamma)$. This is represented on a labeled graph by applying α to the edge-labels. Figure 2 shows the results of applying η to the graph from Figure 1. Note that this is the same marked graph, so that η fixes this vertex of the spine.

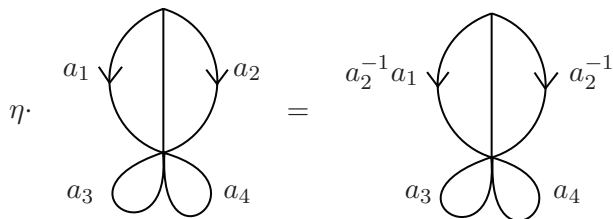


FIGURE 2. Action of η on a Nielsen graph.

The *degree* of a graph is defined to be $2n$ minus the valence of the basepoint. The only graph of degree 0 is a rose, and the only graph of degree 1 is the graph underlying the marked graph in Figure 1. There are five different graphs of degree 2.

A forest collapse cannot increase degree, so the vertices of degree at most i span a subcomplex K_i of the spine. Hatcher and Vogtmann proved that the subcomplexes K_i act like “skeleta” for the spine of Auter space:

Degree Theorem 1. [4] K_i is i -dimensional and $(i - 1)$ -connected.

In particular, the subcomplex K_2 spanned by graphs of degree at most 2 is a simply-connected 2-complex.

3.2. Quotient. The quotient of K_2 by the action of $Aut(F_n)$ was computed in [4]. For $n \geq 4$, this quotient has seven vertices, thirteen edges and seven triangles. Figure 3 shows a lift of these simplices to K_2 for $n = 4$. For $n > 4$, the picture is the same except one must add $n - 4$ loops at the basepoint. The darkest edges represent a choice of tree \mathcal{T} lifting a maximal tree in the 1-skeleton of $K_2/Aut(F_n)$, and the lighter solid edges represent the additional edges in \mathcal{E} . For $n = 3$, the picture is the same except that the backmost triangle is missing and every remaining graph has one fewer loop at the basepoint. For $n = 2$, there are only the three leftmost triangles and every remaining graph has two fewer loops at the basepoint.

3.3. Vertex stabilizers. We will use the labels in Figure 3 to refer to the vertices of \mathcal{T} , and we will denote by W_{n-i} the subgroup of W_n which permutes and inverts the last $n - i$ generators.

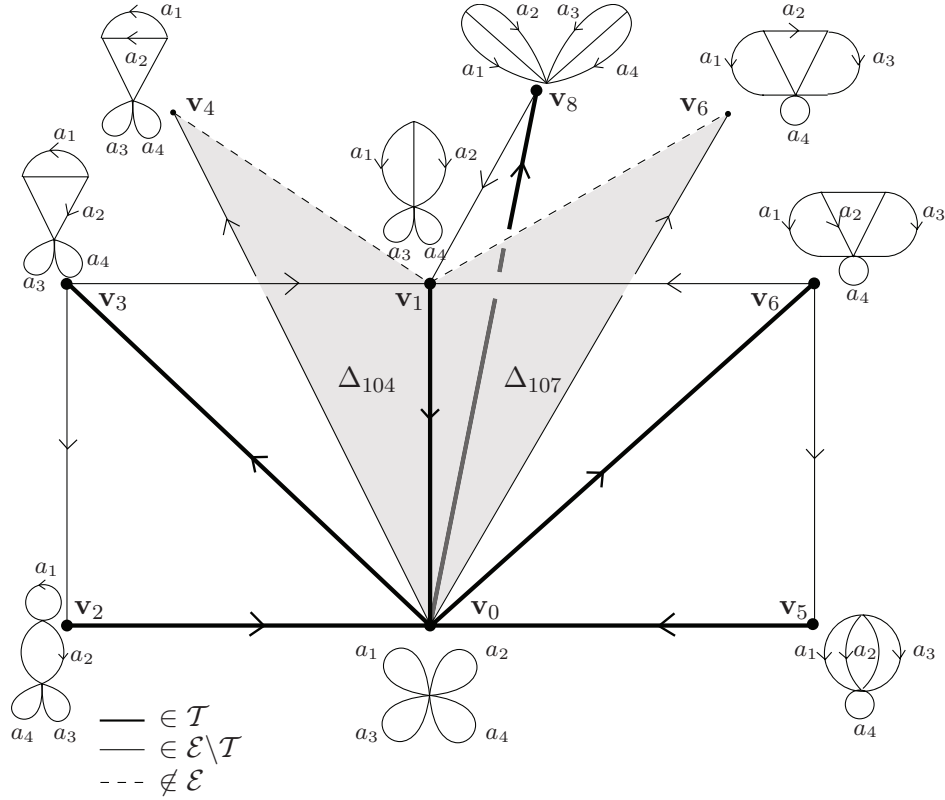


FIGURE 3. A lift of the quotient $K_2/Aut(F_4)$. For $n > 4$, add loops to the basepoint.

The stabilizer of a vertex in the spine of Auter space can be identified with the automorphism group of the marked combinatorial graph associated to that vertex [8]. We compute:

$$G_0 = stab(v_0) = W_n.$$

$$G_1 = stab(v_1) = \Sigma_3 \times W_{n-2}, \text{ where } \Sigma_3 \text{ is the symmetric group on the three vertical edges, which is generated by } \sigma_{12} \text{ and } \eta.$$

$$G_4 = stab(v_4) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \times W_{n-2}. \text{ Here one } \mathbb{Z}_2 \text{ is generated by } \sigma_{12} \text{ and the other } \mathbb{Z}_2 \text{ is generated by } \tau_1\tau_2.$$

Since $v_3 = \eta v_4$, $G_3 = stab(v_3) = \eta G_4 \eta$, which is generated by $\eta\tau_1\tau_2\eta$ and $\eta\sigma_{12}\eta$.

$$G_2 = stab(v_2) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \times W_{n-2}. \text{ The first } \mathbb{Z}_2 \text{ is generated by } \eta\tau_1\tau_2\eta \text{ and the second by } \tau_1.$$

$G_7 = \text{stab}(v_7) = D_8 \times W_{n-3}$, where D_8 is the dihedral group generated by $\eta\sigma_{12}\eta$ and $\tau_2\sigma_{13}$.

$G_6 = \text{stab}(v_6) = D_8 \times W_{n-3}$. Here D_8 is the dihedral group generated by σ_{12} and $\eta\sigma_{13}\tau_2\eta$. Note that $G_7 = \eta G_6 \eta$, since $v_7 = \eta v_6$.

$G_5 = \text{stab}(v_5) = \Sigma_4 \times W_{n-3}$. The symmetric group Σ_4 corresponds to permuting the edges which are not loops and is generated by the involutions $\sigma_{12}, \eta\sigma_{13}\tau_2\eta$ and σ_{23} . Note that G_6 is a subgroup of G_5 .

$G_8 = \text{stab}(v_8) = ((\Sigma_3 \times \Sigma_3) \rtimes \mathbb{Z}_2) \times W_{n-4}$. The factor \mathbb{Z}_2 is generated by $\omega = \sigma_{13}\sigma_{24}$, the first Σ_3 is equal to G_1 and the second Σ_3 is $\omega G_1 \omega$.

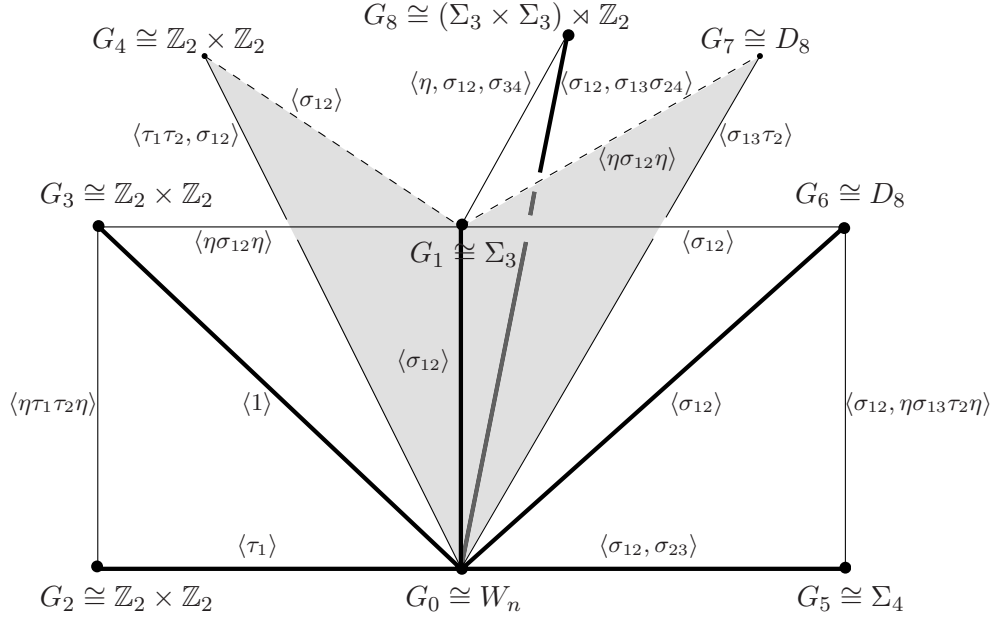


FIGURE 4. Edge and vertex stabilizers with W_{n-k} factors omitted, except at G_0 . The vertex stabilizers are generated by the incoming edge stabilizers.

By Brown's theorem, $\text{Aut}(F_n)$ is generated by the vertex stabilizers G_i corresponding to vertices of \mathcal{T} , i.e. $G_0, G_1, G_2, G_3, G_5, G_6$ and G_8 , together with a generator t_e for each of the 13 edges of \mathcal{E} . We denote the oriented edge from v_i to v_j by e_{ij} .

3.4. Tree relations. If $e \in \mathcal{T}$ (i.e. $e = e_{0k}$ for $k \in \{3, 6, 8\}$ or e_{k0} for $k \in \{1, 2, 5\}$), then the tree relations set $t_e = 1$.

3.5. Face relations. If all edges of a triangle Δ are in \mathcal{E} and two of the edges lie in \mathcal{T} , then $h = 1$ and $g_e = 1$ for all edges in the boundary of Δ , so the face relation associated to Δ reduces to $t_e = 1$ for the third edge e . We now have $t_e = 1$ for all edges except e_{04} and e_{07} .

The only faces which do not have two edges in \mathcal{T} are the shaded faces labeled Δ_{104} and Δ_{107} in Figure 3. The boundary of Δ_{107} is given by the edge-path loop $e_{10}e_{07}e_{71}$. The first edge e_{10} is in \mathcal{T} , giving $h_{10} = 1$ and $g_{e_{10}} = 1$, so $g_{10} = 1$. The second edge e_{07} is in \mathcal{E} , so $h_{07} = 1$; this edge has $t(e_{07}) = v_7 = \eta v_6$, so $w(e_{07}) = v_6$ and $g_{e_{07}} = \eta$, giving $g_{07} = \eta$. The last edge e_{71} is equal to ηe_{61} , and $e_{61} \in \mathcal{E}$. Thus $h_{71} = 1$, and $g_{e_{71}} = 1$, giving $g_{71} = 1$. We have $g_{10}g_{07}g_{71} = \eta$, and the relation associated to Δ_{107} is now

$$1 \cdot t_{e_{10}} \cdot 1 \cdot t_{e_{07}} \cdot 1 \cdot t_{e_{71}} = \eta$$

which reduces to $t_{e_{07}} = \eta$. An identical calculation for Δ_{104} gives $t_{e_{04}} = \eta$, since $\eta v_4 = v_3$.

3.6. Edge relations. The edge relations identify generators of the vertex groups with the appropriate products of the τ_i , σ_{ij} and η . If $e \in \mathcal{T}$, the edge relations identify all of the generators written as products of τ_i and σ_{ij} in our descriptions of the G_i with the corresponding elements of $G_0 = W_n$; in particular, the subgroups W_{n-k} of the stabilizers are all identified with the corresponding subgroup of G_0 . The edge relation associated to e_{04} identifies the generators of G_3 with $\eta\tau_1\tau_2\eta$ and $\eta\sigma_{12}\eta$, since $\eta v_4 = v_3$. The edge relation associated to e_{23} then identifies the first generator of G_2 with $\eta\tau_1\tau_2\eta$.

The edge relation associated to e_{07} identifies the second generator of G_6 with $\eta\tau_2\sigma_{13}\eta$, since $\eta v_7 = v_6$. The edge relation associated to e_{56} identifies G_6 with the subgroup of G_5 generated by $\eta\tau_2\sigma_{13}\eta$ and σ_{12} .

The edge relation associated to e_{18} identifies the first Σ_3 with the corresponding Σ_3 subgroup of G_1 .

3.7. Stabilizer relations. We will not list the relations in $G_0 = W_n$. The relations in G_1 which do not come from G_0 are those involving η , i.e.

- (1) $\eta^2 = 1$
- (2) $(\sigma_{12}\eta)^3 = 1$
- (3) $(\eta\tau_i)^2 = 1 \quad i > 2$
- (4) $(\eta\sigma_{ij})^2 = 1 \quad i, j > 2.$

Since $G_3 = \eta G_4 \eta$, and G_4 is a subgroup of W_n , G_3 does not contribute any new relations. The fact that the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2 \leq G_2$ commute contributes the relation $(\eta\tau_1\tau_2\eta\tau_1)^2 = 1$, which looks a little nicer if we conjugate by $\eta\tau_1$:

$$(5) \quad ((\eta\tau_1)^2\tau_2)^2 = 1.$$

In the dihedral group $D_8 \leq G_6$, the fact that the product of our generators has order 4 contributes a new relation

$$(6) \quad (\eta\sigma_{13}\tau_2\eta\sigma_{12})^4 = 1.$$

The symmetric group $\Sigma_4 \leq G_5$ is generated by the involutions σ_{12}, σ_{23} and $\phi = \eta\sigma_{13}\tau_2\eta$, with relations $(\sigma_{12}\sigma_{13})^3 = 1, (\sigma_{12}\phi)^4 = 1$ and finally $\sigma_{12}\phi\sigma_{12}(\sigma_{23}\phi)^2 = 1$. The first relation comes from G_0 and the second from G_6 , so G_5 adds only the third relation, i.e.

$$(7) \quad \sigma_{12}\eta\sigma_{13}\tau_2\eta\sigma_{12}(\sigma_{23}\eta\sigma_{13}\tau_2\eta)^2 = 1.$$

The fact that the two copies of Σ_3 which are contained in G_8 commute produces the relation $(\sigma_{14}\sigma_{23}\eta\sigma_{23}\sigma_{14}\eta)^2 = 1$, i.e.

$$(8) \quad (\sigma_{14}\sigma_{23}\eta)^4 = 1.$$

All other relations in G_8 are consequences of this and relations in G_0 and G_1 ; for example the fact that $\eta\sigma_{12}\eta = \sigma_{34}$ commutes with η is a consequence of relations already accounted for in G_1 .

This completes the proof of Theorem 1 and its corollaries.

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