Berstein Seminar - Trees in Geometric Group Theory.

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1 Length Functions

Reference: "Abstract Length Functions in Groups", I.M. Chiswell.

Definition. *A* length function *on a group G is a function $G \to \mathbb{Z}$ which we denote $g \mapsto |g|$ satisfying the following properties. It gives rise to the "overlap" function $d(x, y) := \frac{1}{2}(|x| + |y| - |xy^{-1}|)$.*

1. $|1| = 0$
2. For all $x \in G$: $|x| = |x^{-1}|$
3. For all $x, y, z \in G$: $d(x, y) < d(x, z) \Rightarrow d(y, z) = d(x, y)$

   Sometimes we add:
4. $d(x, y) \in \mathbb{Z}$.

Note: Axiom (1) implies $d(x, x) = |x|$ for all $x \in G$. Axiom (3) then shows that $|x| \geq 0$.

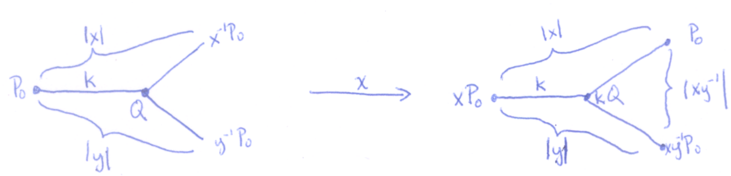
Example. F_n

$G = F_n$ and $|g|$ is the length of the reduced word representing g. Here $d(x, y)$ is the length of the largest common final segment of the reduced words representing x and y. The axioms are easily verified and left as an exercise.

Example. *G acts on a simplicial tree T.*

Fix P_0 in $V(T)$, the vertices of T. Let $|g|$ be the length of the shortest path from P_0 to gP_0. Let γ_g denote this path. Axioms (1) and (2) hold trivially.

Let $x, y \in G$. If the paths $\gamma_{x^{-1}}$ and $\gamma_{y^{-1}}$ bifurcate at a point Q a distance k from P_0, then the shortest path from $x^{-1}P_0$ to $y^{-1}P_0$ goes through Q and has length $(|x|-k)+(|y|-k)$. But this path has the same length as $\gamma_{xy^{-1}}$, so $|xy^{-1}| = |x|+|y|-2k$. Therefore, $d(x, y) = \frac{1}{2}(|x|+|y|-(|x|+|y|-2k)) = k$. That is, $d(x, y)$ is the length of the overlap of $\gamma_{x^{-1}}$ and $\gamma_{y^{-1}}$.



Axiom (3) now follows easily, for if $d(x, y) < d(x, z)$ then γ_x bifurcates from γ_z strictly after it does from γ_y, so γ_y bifurcates from γ_x and γ_z at the same point, and thus $d(x, y) = d(y, z)$. Axiom (4) also holds easily.

2 Realizing Length Functions

Question. *Can any length function $G \to \mathbb{Z}$ be realized by an action of G on some tree?*

Answer: Yes. In what follows, $|\cdot|$ is a fixed length function.

Let $S = \{(x,m) | x \in G, m \in \mathbb{Z}, 0 \le m \le |x|\}$ and define a relation $\sim$ on S by $(x,m) \sim (y,n)$ if $m = n$ and $d(x^{-1}, y^{-1}) \ge m$. We easily verify that $\sim$ is an equivalence relation:

$\sim$ is reflexive since $d(x^{-1}, x^{-1}) = |x| \ge m$ for any $(x,m) \in S$.

$\sim$ is symmetric since d is.

Finally, $\sim$ is transitive: suppose $(x,m) \sim (y,m) \sim (z,m)$. Then $d(x^{-1}, y^{-1}), d(y^{-1}, z^{-1}) \ge m$. If $d(x^{-1}, z^{-1}) < d(x^{-1}, y^{-1})$ then axiom (3) yields $d(x^{-1}, z^{-1}) = d(z^{-1}, y^{-1}) = d(y^{-1}, z^{-1})$, the last equality by axiom (2). So in any case, $d(x^{-1}, z^{-1}) \ge \min\{d(x^{-1}, y^{-1}), d(y^{-1}, z^{-1})\} \ge m$, so $(x,m) \sim (z,m)$.

Let $[x,m]$ denote the equivalence class of (x,m) and V the set of equivalence classes.

We now construct a tree T with vertex set V: Connect vertices $[x,m], [y,m+1]$ with a (unique) edge if $(y,m) \in [x,m]$. Let T be the resulting metric space (each edge has length 1.) Observe that $[x,0] = [e,0]$ for all $x \in G$ (e is the identity), so we define this to be our basepoint P_0 for T. T is connected since every $[x,m]$ is connected to P_0 via the sequence $[x,0], [x,1], \ldots, [x,m]$ of edges.

To verify that T is indeed a tree, we need to show its fundamental group is trivial. Observe that $\lambda : [x,m] \mapsto m$ is a (well-defined) "level function" $V \to \mathbb{Z}$ in the following sense:

- There is a point P with minimal $\lambda(P)$, namely P_0 with value 0.
- For each $Q \neq P_0 \in V$, there is a unique P adjacent to Q such that $\lambda(P) = \lambda(Q) - 1$. Indeed, if $Q = [x,m]$ this unique point is $[x, m-1]$.
- For each $Q \in V$, all adjacent points P' except the aforementioned one satisfy $\lambda(P') > \lambda(Q)$.

T is therefore a tree. (If the fundamental group were nontrivial, there would be a non-backtracking loop. We may assume it to start at a point of minimal value under λ among all such loops, by the first property above. By the third property, the values of λ along the path would initially increase and thus eventually decrease. Finally, the second property contradicts the assumption the path does not backtrack.)

Now we define an action of G on T. Let's first motivate the definition. We have $P_0 = [e,0]$ as our basepoint, and we want gP_0 to be the point $[g, |g|]$. Since $[x,m]$ is a distance m from P_0 along the path γ_x from P_0 to xP_0, we want $g[x,m]$ to be the corresponding point along the path from gP_0 to gxP_0. The length of the overlap of the paths from gP_0 to gxP_0 and to P_0 should be the same as the length of the overlap of the paths from P_0 to xP_0 and to $g^{-1}P_0$, which should be $d(x^{-1}, g)$. So the bifurcation point Q for the paths γ_g and γ_{gx} should be a distance $|g| - d(x^{-1}, g)$ from P_0.

If $m \le d(x^{-1}, g)$, then $g[x,m]$ should lie on γ_g a distance m from the end, so we want $g[x,m] = [g, |g| - m]$.

If $m \ge d(x^{-1}, g)$, then $g[x,m]$ should lie on γ_{gx} a distance $m - d(x^{-1}, g)$ from Q. The total distance from P_0 to $g[x,m]$ should then be $(|g| - d(x^{-1}, g)) + (m - d(x^{-1}, g)) = |g| + m - 2d(x^{-1}, g)$. Thus we want $g[x,m] = [gx, |g| + m - 2d(x^{-1}, g)]$.

So we define our action of G on V by:

$$g[x,m] := \begin{cases} [g, |g|-m] & d(x^{-1},g) \geq m \\ [gx, |g|+m-2d(x^{-1},g)] & d(x^{-1},g) \leq m \end{cases}$$

We first check this is well-defined, and then we will show it is an action. These verifications are messy, but straightforward.

If $(x,m) \sim (y,m)$ then $d(x^{-1},y^{-1}) \geq m$. We consider first the case that $d(g,x^{-1}) = d(g,y^{-1})$. Call the common value d. (So $|gx| = |x|+|g|-2d$ and $|gy| = |y|+|g|-2d$.) Then the applicable line of the definition is the same for both (x,m) and (y,m).

There is nothing to show for the first line. In the case of the second line, we have

$$\begin{aligned} d((gx)^{-1},(gy)^{-1}) &= \frac{1}{2}(|gx|+|gy|-|x^{-1}y|) = \frac{1}{2}(|x|+|g|-2d+|y|+|g|-2d-|x^{-1}y|) \\ &= |g|-2d+\frac{1}{2}(|x|+|y|-|x^{-1}y|) = |g|-2d+d(x^{-1},y^{-1}) \geq |g|+m-2d. \end{aligned}$$

So $g[x,m] = g[y,m]$. [Note: taking $y = x$ shows that $|gx| \geq |g|+m-2d(x^{-1},g)$ so that the second line of the definition actually gives a point in V.]

Next we consider the case $d(g,x^{-1}) \neq d(g,y^{-1})$. WLOG, $d(g,x^{-1}) < d(g,y^{-1})$, so axiom (3) says that $d(g,x^{-1}) = d(x^{-1},y^{-1})$. So $d(g,y^{-1}) > d(g,x^{-1}) = d(x^{-1},y^{-1}) \geq m$, and thus the first line of the definition applies for both (x,m) and (y,m). This completes the verification the map $G \times V \to V$ is well-defined.

Now we verify it is an action. $1[x,m] = [x,m]$ by the second line of the definition, so we need only verify $h(g[x,m]) = (hg)[x,m]$. There are four cases to consider depending on which line applies for h and which for g. We demonstrate one case and leave the remaining three similar cases as exercises.

So let us suppose $d(x^{-1},g) \geq m$ and $d(g^{-1},h) > |g|-m$. Then $h(g[x,m]) = h[g,|g|-m] = [h,|h|-|g|+m]$ since the first line is used in both instances. The inequality $d(g^{-1},h) > |g|-m$ can be rewritten as $|g|+|h|-|hg| > 2(|g|-m)$ or $|hg| < |h|-|g|+2m$. Therefore $d(g,hg) = \frac{1}{2}(|g|+|hg|-|h|) < \frac{1}{2}(|g|+(|h|-|g|+2m)-|h|) = m$. So $d(g,hg) < m \leq d(x^{-1},g)$. Axiom (3) yields $d(hg,x^{-1}) = d(g,hg) < m$. Thus line 2 of the definition applies to show $(hg)[x,m] = [hgx, |hg|+m-2d(x^{-1},hg)]$.

We must verify $[h,|h|-|g|+m] = [hgx,|hg|+m-2d(x^{-1},hg)]$. The second coordinates agree since $|hg|+m-2d(x^{-1},hg) = |hg|+m-2d(hg,g) = |hg|+m-|g|-|hg|+|h| = |h|-|g|+m$.

Finally, we need to verify that $d((hgx)^{-1},h^{-1}) \geq |h|-|g|+m$. Using our identity $d(hg,x^{-1}) = d(g,hg)$, we obtain $|hg|+|x|-|hgx| = |g|+|hg|-|h|$, so that $|hgx| = |x|+|h|-|g|$. This gives the second equality in:

$$\begin{aligned} d((hgx)^{-1},h^{-1}) &= \frac{1}{2}(|hgx|+|h|-|gx|) = \frac{1}{2}((|x|+|h|-|g|)+|h|-|gx|) \\ &= |h|-|g|+\frac{1}{2}(|x|+|g|-|gx|) = |h|-|g|+d(x^{-1},g) \geq |h|-|g|+m \end{aligned}$$

completing the verification (for this case.)

Now we may finally extend the action of G on V to an action on T. We need only verify that adjacency of vertices is preserved. This follows immediately from the definition of the action, since $g[x,m]$ and $g[x,m+1]$ can be computed using the same line of the definition. (This is where we use Axiom (4): for the case $m = d(x^{-1},g)$, both lines agree.)

By construction, the length of the shortest path from $P_0 = [g,0]$ to $gP_0 = [g,|g|]$ is $|g|$, so the action we have constructed indeed realizes the given length function.

3 Additional Axioms

If our length function satisfies more axioms, we can say more about the action of G on T.

5. $d(x,y) + d(x^{-1},y^{-1}) > |x| = |y| \Rightarrow x = y$.

6. $|g^2| > |g|$ if $g \neq 1$.

Theorem. *If Axiom (5) holds, the action of G on T has trivial edge stabilizers.*

Theorem. *If Axiom (6) holds, G acts freely on T.*

4 The Culler-Morgan Theorem

References: "Group Actions on $\mathbb{R}$-trees", Culler-Morgan.
"Complete Trees for Groups with a Real Valued Length Function", Alperin-Moss.
"On Metric Properties of Treelike Space", Imrich.

Work of Alperin & Moss and Imrich shows that Chiswell's construction above can be generalized to construct $\mathbb{R}$-trees realizing a given real-valued length function.

Now, fix an action $G \curvearrowright T'$, T' an $\mathbb{R}$-tree, and fix $x_0 \in T'$. Let $\|g\|$ denote the length of the shortest path $[x_0, gx_0]$ from x_0 to gx_0. Given this length function, build T as in the Chiswell construction.

Define $\varphi : T \to T'$ sending $[x, m]$ to $\alpha(m)$ where α is the isometric embedding $[0, \|g\|] \to [x_0, gx_0]$.

Then φ is an injective, G-invariant map. There was some confusion about this statement in class, but it follows immediately from the definitions. In fact, injectivity essentially motivated the definition of the equivalence relation $\sim$, while G-invariance motivated our definition of $G \curvearrowright T$. Now, since $\varphi(T)$ is invariant under the action of G, we see that φ is surjective and hence a G-equivariant isometry if $G \curvearrowright T'$ is minimal. Composing two such isometries, we obtain:

Proposition. *Suppose $G \curvearrowright T_i$ for $i = 1, 2$ are two minimal actions on $\mathbb{R}$-trees. Fix $x_i \in T_i$ and let $\|\cdot\|_i$ be the corresponding length function (i.e., $\|g\|_i$ is the length of the shortest path from x_i to gx_i in T_i.) If $\|g\|_1 = \|g\|_2$ for all $g \in G$, then T_1 and T_2 are isometric by a G-equivariant isometry.*

We now state some definitions needed for the statement of the Culler-Morgan theorem.

Definition. *A group action $G \curvearrowright T$ is* reducible *if any of the following hold. Otherwise it is* irreducible.

(i.) Every element of G fixes a point; or

(ii.) There is some line contained in T that is invariant under the action of G; or

(iii.) There is a fixed end of T.

Actions of type (ii) that preserve the orientation of the invariant line are called shifts*, while those that do not preserve orientation are called* dihedral.

Definition. *A group action $G \curvearrowright T$ is* semisimple *if any of the following hold:*

(I.) The action is irreducible; or

(II.) The action has a global fixed point; or

(III.) In the definition of reducible, (ii) holds.

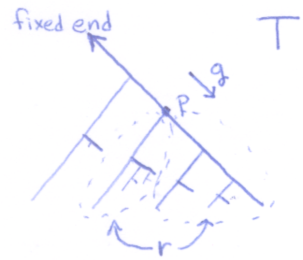
In other words, an action is semisimple unless (iii) holds and (i) and (ii) do not.

Theorem (Culler-Morgan). *Suppose $G \curvearrowright T_1, G \curvearrowright T_2$ are minimal semisimple actions of G on $\mathbb{R}$-trees. If $G \curvearrowright T_1, G \curvearrowright T_2$ have the same translation length function, then there exists an equivariant isometry from T_1 to T_2.*

Warning: The translation length function is not a length function in our sense.

Note: The semi-simplicity condition in the statement of the theorem cannot be removed, as the following example shows:

Example. *Let $G = \langle g, r \rangle$ act on the trivalent tree T as follows: g shifts by one unit along some chosen line. Choose a point p on this line. The complement of p has three components: two that each contain a ray of the chosen line, and one that does not. Let r flip one of the components containing a ray with the one that does not. Then G fixes an end of T (the one determined by the ray r fixes) but no line of T, and has no global fixed point. This action thus fails to be semisimple, but it is minimal since G acts transitively on T. The translation length function is $g \mapsto 1, r \mapsto 0$, which is the same as the translation length function of the (minimal, semisimple) action $G \curvearrowright \mathbb{R}$ where g is unit translation and r acts trivially. (Image below.)*



Observe that actions with nontrivial translation length functions must be irreducible, dihedral, or fixed end. In the first two cases, the action must be semisimple. A fact obtained from several technical lemmata, but that we will use without proof, shows that the particular case depends only on the length function:

Fact. *Let $G \curvearrowright T$, T an $\mathbb{R}$-tree, with nontrivial translation length function l. Then the action is:*

$$\begin{cases} \text{fixed end} & \iff \quad l \text{ is trivial on } G' = [G, G] \\ \text{dihedral} & \iff \quad l \text{ is nontrivial on } G' \text{ and } l([g, h]) = 0 \text{ for all } g, h \in G \text{ hyperbolic} \\ \text{irreducible} & \iff \quad l([g, h]) \neq 0 \text{ for some hyperbolic } g, h. \end{cases}$$

In particular, if $G \curvearrowright T_1, G \curvearrowright T_2$ have the same translation length function, they must be of the same type.

The proof of the Culler-Morgan Theorem proceeds by cases. Let l be the common translation length function of $G \curvearrowright T_i$, $i = 1, 2$. First, consider the case that l is identically zero. Then (I) cannot hold by (i). If (III) were to hold, then T_i would be a line, and the elements of G can only act as the identity or reflections. If there are two reflections about distinct points, we get a nontrivial translation. So all reflections are about the same point, which is then a global fixed point. So (II) holds, and we see that T_i is a point. So T_1 and T_2 are equivariantly isometric.

Next consider the case that $G \curvearrowright T_i$ is reducible and the translation length function is nontrivial. Then case (ii) holds, so the T_i are lines, and we can detect, by the Fact above, whether the action is dihedral or a shift. If it's a shift, then $\|g\|_i = l(g)$ for every $g \in G$ (for any choice of basepoint), so the above Proposition gives us the desired equivariant isometry. If it's dihedral, there are at least two distinct reflections (by nontriviality of l.) That is to say, we have $s, t \in G$ such that $l(s) = l(t) = 0$ but $l(st) \neq 0$. Let p_i, q_i be the fixed points of s, t respectively in T_i. Choose the basepoint x_i in T_i to be halfway between p_i and q_i. Then the length of a translation can be detected from l as above, while the length $\|r\|_i$ of a reflection about a point r_i in T_i is detected by l as follows: since $l(rs), l(rt)$ are twice the distances between r_i and p_i, respectively s_i and p_i, we can determine where r_i is on T_i and thus the distance from r_i to x_i. $\|r\|_i$ is twice this distance, so the Proposition again gives an equivariant isometry between T_1 and T_2.

Finally, we have the case that $G \curvearrowright T_i$ is irreducible. We use the following lemma without proof. It is the synthesis of several technical lemmata.

Lemma. *If $G \curvearrowright T_1, G \curvearrowright T_2$ are irreducible minimal actions with identical translation length functions, then there exist $g, h \in G$ such that g, h, gh^{-1} are hyperbolic in T_i and such that the intersection of the translation axes $C_g \cap C_h \cap C_{gh^{-1}}$ is a single point.*

Assuming this lemma, let x_i be the intersection point. It will be our basepoint. For $k \in G$, let L be the geodesic from x_i to kx_i. Observe that at least one of $C_g \cap L$, $C_h \cap L$, $C_{gh^{-1}} \cap L$ will be the single point $\{x_i\}$, since the intersection of all three sets is the smallest one, and is contained in $C_g \cap C_h \cap C_{gh^{-1}} = \{x_i\}$. Similarly, at least one of $kC_g \cap L$, $kC_h \cap L$, $kC_{gh^{-1}} \cap L$ is the single point $\{kx_i\}$. As C ranges over $\{C_g, C_h, C_{gh^{-1}}\}$ and P ranges over $\{kC_g, kC_h, kC_{gh^{-1}}\}$, the distance between C and P is at most the distance between x_i and kx_i, and we have just shown that for some pair (C, P), the distance is actually realized. By the "Long Lemma" of Marisa's lecture, it follows that for a, b hyperbolic, the distance between C_a and C_b is $\frac{1}{2}\max(0, l(ab) - l(a) - l(b))$, so we can compute these distances (recall that $kC_a = C_{kak^{-1}}$) and hence $\|k\|_i$ from l. The Proposition again yields the desired equivariant isometry, completing the sketch of the proof.

5 Questions

1. Can these ideas be generalized to any 1-dimensional or even 2-dimensional simplicial complex?
2. Do semisimple actions relate to semisimple groups?
3. Can you classify the $\mathbb{R}$-trees on which a fixed group acts?