Berstein Seminar Notes - Length Spectrum of Compact M_κ-complex is Discrete

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This talk is based on the paper *The Length Spectrum of a Compact Constant Curvature Complex is Discrete* by Brady and McCammond [1]. At first we review some relevant results which can be found in the book *Metric Spaces of Non-Positive Curvature* by Bridson and Haefliger (BH).
Recall: An M_κ-complex is a polyhedral cell complex where each cell has a metric of constant curvature κ, with the metrics agreeing on cell intersections. From now on we denote by X an M_κ complex. Our aim will be to prove Theorem 1 below.

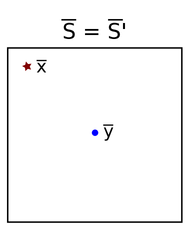
Definition. *For $x, y \in X$, the length spectrum from x to y is the set of lengths of all local geodesics from x to y. The length spectrum of X is the set of lengths of all closed geodesic loops in X.*

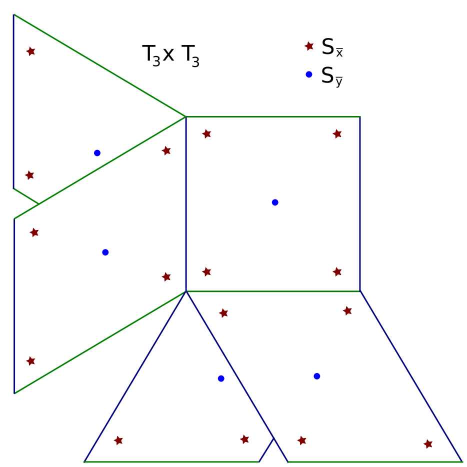
Theorem 1 (Brady–McCammond). *If X is compact, the length spectrum of X is discrete.*

Let Shapes(X) denote the set of isometry classes of cells in X. We assume Shapes(X) is finite, so that X is a geodesic space.
Let $\bar{S}, \bar{S}' \in$ Shapes(X) (we think of both as cells), and $\bar{x} \in \bar{S}, \bar{y} \in \bar{S}'$. Let

$$S_{\bar{x}} = \{x \in X | \text{there is an isometry} f \text{from } \bar{S} \text{ to an open cell in } X \text{ and } f(\bar{x}) = x\}$$

and let $S_{\bar{y}}$ be defined analogously.
For example, in case of the product of two 3-regular trees $T_3 \times T_3$, we have Shapes($T_3 \times T_3$)={point, segment, square}. Below are sets $S_{\bar{x}}$ and $S_{\bar{y}}$ in $T_3 \times T_3$ for a given $\bar{x}$ and $\bar{y}$ in $\bar{S} = \bar{S}' \in$ Shapes($T_3 \times T_3$).





Before we deal with the Theorem 1, we prove the following weaker result:

Theorem 2. *For all $\bar{S}, \bar{S}' \in$ Shapes(X) and $\bar{x} \in \bar{S}, \bar{y} \in \bar{S}'$, the set $\{d(x,y) | x \in S_{\bar{x}}, y \in S_{\bar{y}}\}$ is discrete.*

In order to prove this theorem, we use the proposition below.

Proposition 1. *For all $l > 0$, there is an integer N such that every geodesic of length l or less in X is contained in a union of at most N cells.*

Proof of theorem 2. Let T be an open cell in X. Note that $|S_{\bar{x}} \cap T| \leq |\mathrm{Isom} T| < \infty$ (and same for $S_{\bar{y}}$). For $V \subset X$ a finite connected subcomplex, and d_V the intrinsic metric on V, $|\{d_V(x,y) | x \in S_{\bar{x}} \cap V, y \in S_{\bar{y}} \cap V\}|$ is finite. Let $\Sigma_N = \bigcup_V \{d_V(x,y) | x \in S_{\bar{x}} \cap V, y \in S_{\bar{y}} \cap V\}$, where the union is over subsets V containing at most N cells. Note that Σ_N is finite since there are only finitely many isometry classes of such V, due to the finiteness of Shapes(X). By Proposition 1 above, we have

$$\{d(x,y) | x \in S_{\bar{x}}, y \in S_{\bar{y}}\} \cap [0,l) \subset \Sigma_N$$

since by the proposition there's a union of at most N cells in which the geodesic between x and y is contained, and in this union V, $d_V(x,y) = d(x,y)$. Since each Σ_N is finite, their union is discrete. □

Instead of proving Proposition 1 above, we'll look at a stronger statement given below.

Proposition 2. *For all $l > 0$ there is an integer N such that for every taut m-string of length l or less in X, $m \leq N$.*

An *m-string* between x and y in X is a $\Sigma = (x_0, x_1, \ldots, x_m)$, where $x_i \in X$, $x_0 = x$, $x_m = y$, and $\{x_i, x_{i+1}\}$ are contained in the same closed cell S_i. An m-string Σ is *taut* if for all i there's no cell containing $\{x_{i-1}, x_i, x_{i+1}\}$ and the concatenation of the geodesics $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ in S_{i-1} and S_i respectively is a geodesic in $S_{i-1} \cup S_i$. Note that proposition 2 implies proposition 1 since every geodesic is a taut string. Let $\overline{st(x)}$ be the union of every closed cell that contains the point $x \in X$, and let $st(x)$ denote the interior of $\overline{st(x)}$. There is a natural projection $p : \overline{st(x)} \setminus \{x\} \to Lk(x,X)$.

Lemma (I.7.23 in BH). *For x a vertex of X, $\Sigma = (x_0, \ldots, x_m)$ a taut m-string in $st(x) \setminus \{x\}$, and for $\kappa > 0$, $d_S(x, x_i) < \frac{\pi}{2\sqrt{\kappa}}$, then $p(\Sigma)$ is a taut m-string in $Lk(x,X)$ of length less than π.*

The proof of proposition 2, which you can find in BH on page 110, uses an induction argument together with the above lemma. We omit the details here.

Let γ be a local geodesic in X. The sequence of open cells that γ passes through is called the *linear gallery* of γ. Similarly, for γ a closed local geodesic, we get a *circular gallery.* We now turn to the proof of Theorem 1, divided into three parts.

Proof of Brady–McCammond theorem.
The idea behind the proof is to show that the lengths of closed geodesic loops form a set of measure zero and have only finitely many connected components in a suitable analogue of space of loops up to a given length; taking the limit as length increases, we arrive at the desired result.

Step 1:

Assume to the contrary that X is an M_κ-complex not satisfying the theorem. Then there is an infinite sequence of (closed, geodesic) loops $\gamma_1, \gamma_2, \ldots$ such that each is of distinct length less than l. By the compactness of X and Proposition 2, we can choose an infinite subsequence of $\{\gamma_i\}$ in which all geodesic loops share the same circular gallery G.

Step 2:

For γ a closed geodesic loop with circular gallery G, the dimension of the cells that γ passes through alternately increases and decreases. Open cells of a locally maximal (minimal) dimension along γ we call top (bottom) cells and denote $T_1, T_2, \ldots, T_m$ (and $B_0, B_1, \ldots, B_m$). Note that γ intersects each top cell in a segment and each bottom cell at a point (we consider a 0-cell both open and closed). Thus choosing a point in each bottom cell defines an m-string. Let $\mathrm{Loops}(G) = \prod_{i=1}^{m} B_i$, the space of m-strings which are geodesic in each simplex and pass through each B_i. Let $d : \mathrm{Loops}(G) \to \mathbb{R}$ be the distance map, measuring the length of m-strings in $\mathrm{Loops}(G)$. Recall the weak form of Morse-Sard theorem:

Theorem 3 (Morse-Sard). *For M, N smooth manifolds, and $f : M \to N$ a smooth map, the set of critical values of f has measure 0 (where a critical value is the image of a critical point).*

Note that the map d is smooth, since we can decompose d as the sum of d_i, where each $d_i : T_i \to \mathbb{R}$ is the distance map restricted to the top cell T_i having cells B_{i-1} and B_i in its boundary. These functions are smooth, hence is their sum d. Thus we can apply Morse-Sard to conclude that the measure of the critical values of d, which include the lengths of geodesic closed loops, is 0.

Step 3:

The finishing touch is showing that the set of critical values of d has only finitely many connected components, and hence, since we've shown it has measure zero, the length spectrum of G must be finite. This supplies the contradiction to Step 1.

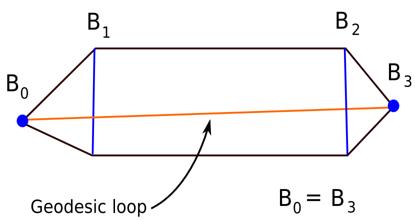
Definition. *A real semi-algebraic set is a subset of $\mathbb{R}^n$ which can be described by finite boolean combinations of sets defined by a polynomial equation or inequality.*

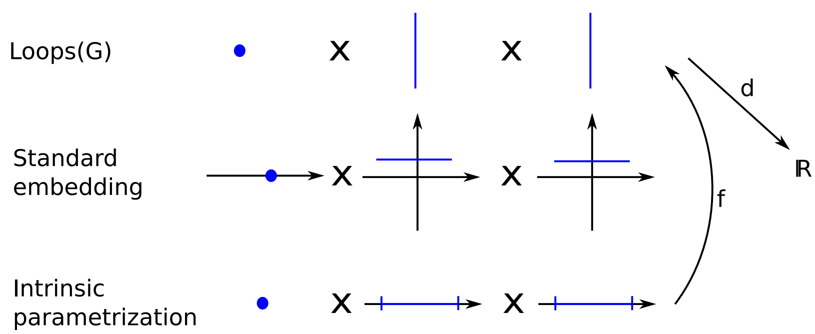
We use, without proof, the following standard result.

Theorem 4. *Real semi-algebraic sets have a finite number of connected components.*

We fix a standard isometric embedding of a bottom cell B_i of type $\mathbb{S}^n$, $\mathbb{R}^n$, or $\mathbb{H}^n$ in $\mathbb{R}^{n+1}$ as $k(x_1^2+x_2^2+\ldots+x_n^2)+x_{n+1}^2 = 1$ for $k = 1, 0, -1$ respectively. Such embeddings consist of polynomial equations and inequalities defining each B_i. Doing this to all bottom cells, we get an injection $g : \text{Loops}(G) \to \mathbb{R}^N$, for $N = \sum_{i=1}^{m} \dim(B_i) + m$. Once we fix a similar embedding of top cells, there are unique linear (Euclidean) transformations from the standard embeddings of bottom cells to the embeddings of top cells in which they are contained. In a top cell, the distance function can be written as $\cos^{-1}(x \cdot y)$, $\sqrt{x \cdot y - 1}$, or $\cosh^{-1}(x \circ y)$ for $\mathbb{R}, \mathbb{S}, \mathbb{H}$ cells respectively (where $\circ$ is the Lorentzian inner product). Recall that d is the sum of these restricted distance functions d_i. Note that the derivatives of these functions with respect to any variable are rational up to a square root of a polynomial in the denominator.

To show that the critical points of d in $\mathbb{R}^N$ are real semi-algebraic, we first reduce the number of variables by projecting to the first n coordinates in every standard embedding of a bottom cell (see illustration below). These n coordinates become the intrinsic parameters. Taking the product, we get a smooth homeomorphism f from the intrinsic parametrization to our standard embedding of Loops(G) in $\mathbb{R}^N$. Thus the critical points of d are the critical points of $d \circ f$ with respect to the intrinsic parameters. With a bit of calculation, these can be written as a sum of polynomials over square roots of polynomials, and finally converted to a system of equations and inequalities. Thus the critical points are a real semi-algebraic set. $\square$





References

[1] Brady, N. and McCammond, J. *The Length Spectrum of a Compact Constant Curvature Complex is Discrete.* Geometriae Dedicata. Vol 119, num. 1, pp 159-167. Springer 2006.