Trace and extension results for a class of domains with self-similar boundary

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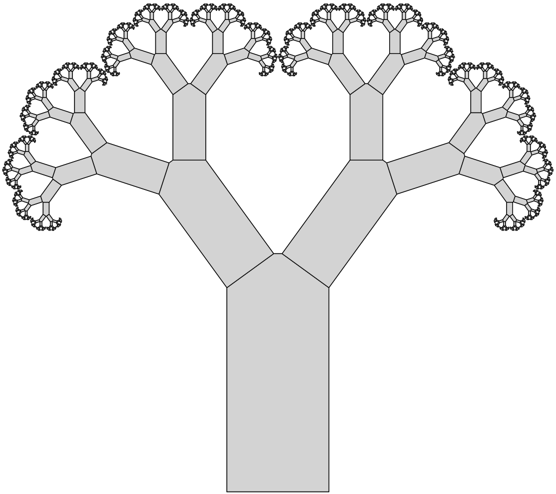
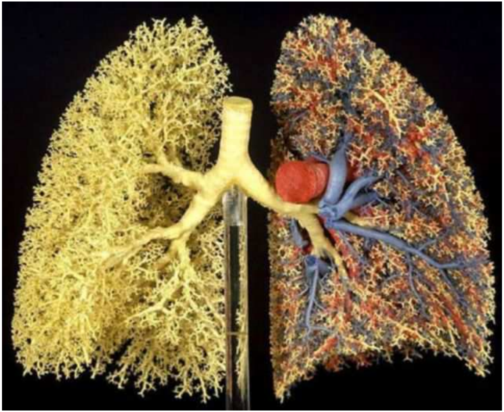
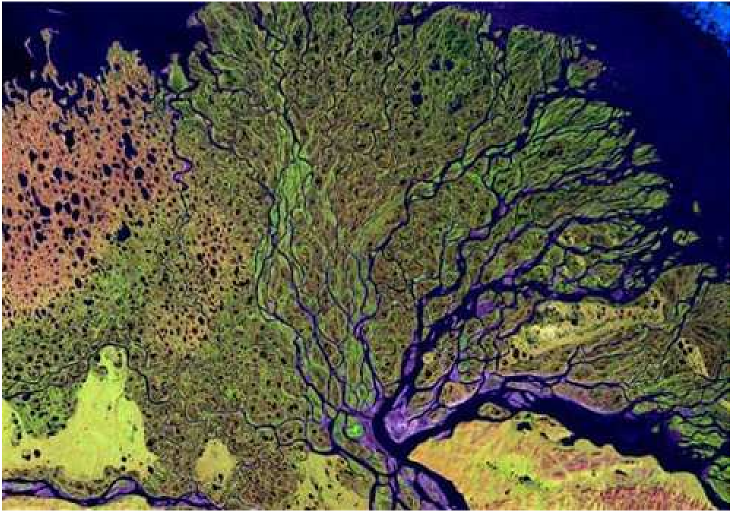


An example of ramified domain

Motivations



Human lungs (E.R. Weibel)



Lena river delta

Outline

1. Presentation
   - The self-similar boundary
   - The class of ramified domains
2. Trace and extension results
   - General results
   - Trace theorems in the critical case
   - Extension theorem in the critical case
3. Comparison of the notions of trace

A class of self-similar sets

Consider an iterated function system (f_1, f_2), where

- f_1 and f_2 have ratio $a < 1$,
- f_1 and f_2 have opposite angles $\pm\theta$ $(0 \leqslant \theta < \frac{\pi}{2})$.

Denote Γ the invariant set associated with (f_1, f_2) : $\Gamma \; = \; f_1(\Gamma) \cup f_2(\Gamma)$.

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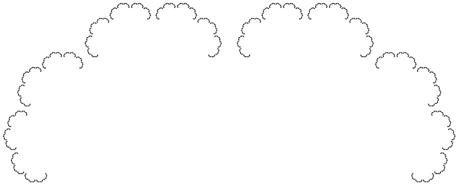

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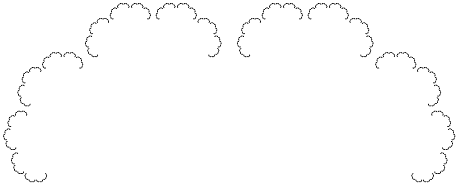


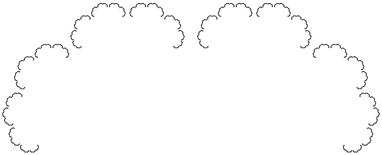
An example of self-similar set Γ

- B. Mandelbrot, M. Frame *The canopy and shortest path in a self-contacting fractal tree*, 1999.

The critical ratio $a^\star$

There exists a critical ratio $a^\star$ depending only on the angle θ such that

- if $a < a^\star$, then Γ is totally disconnected,



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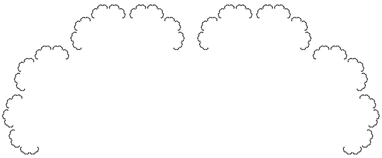
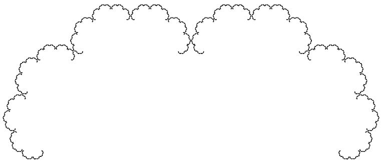


- if $a = a^\star$, then Γ is connected and has multiple points.



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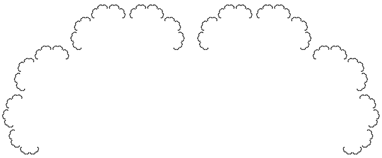

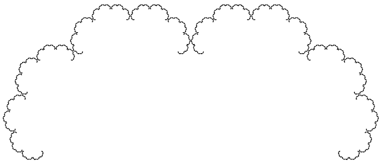

If $a \leqslant a^\star$, Hausdorff dimension of Γ:

$$d := \dim_H(\Gamma) = -\frac{\log 2}{\log a}$$

Sobolev spaces on Γ

The self-similar set Γ is endowed with its invariant measure, *i.e.* the only probability measure μ on Γ satisfying

$$\mu(B) \;=\; \frac{1}{2}\mu(f_1^{-1}(B)) + \frac{1}{2}\mu(f_2^{-1}(B))$$

for every Borel set $B \subset \Gamma$.

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The spaces $W^{s,p}(\Gamma)$ (Jonsson, Wallin, 1984)

For $0 < s < 1$ and $1 \leqslant p < \infty$, if $v \in L^p_\mu(\Gamma)$, then $v \in W^{s,p}(\Gamma)$ if and only if

$$|v|^p_{W^{s,p}(\Gamma)} \;:=\; \iint_{|x-y|<1} \frac{|v(x)-v(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \;<\; \infty.$$

Outline

The class of ramified domains

Γ^0

First cell Y^0

The class of ramified domains

First cell Y^0

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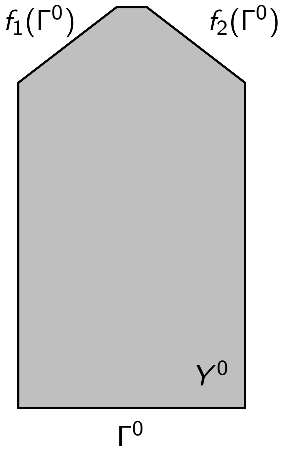

First cell Y^0

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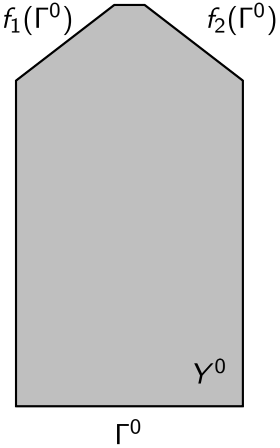

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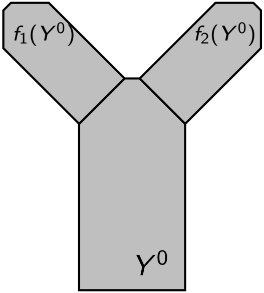


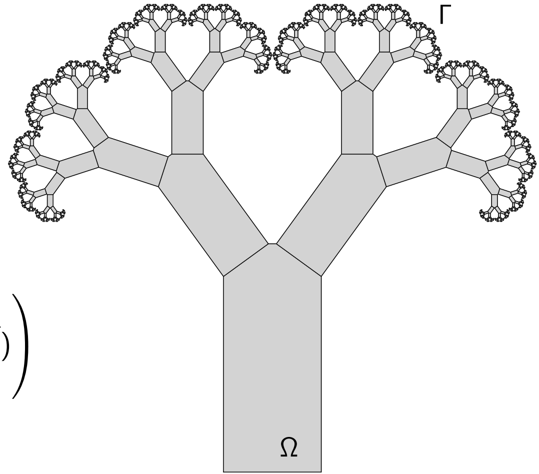
Second iteration

The class of ramified domains

If $\sigma = (\sigma_1, \ldots, \sigma_k) \in \{1,2\}^k$, write

$$f_\sigma := f_{\sigma_1} \circ \ldots \circ f_{\sigma_k}.$$

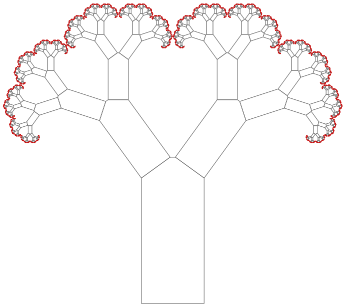
$$\Omega = \text{Interior}\left(\bigcup_{k\geqslant 0} \bigcup_{\sigma\in\{1,2\}^k} f_\sigma(\overline{Y^0})\right)$$



The set Ξ of multiple points of Γ

$a = a^\star$

Case ❶



$$\theta \notin \left\{\frac{\pi}{2k},\ k \in \mathbb{N}^\star\right\}$$

Ξ is countable

where Ξ is the set of multiple points of Γ.

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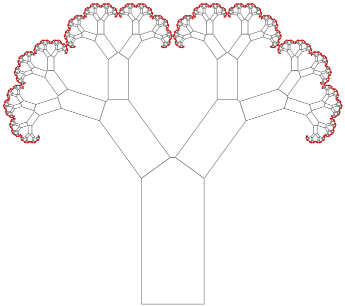

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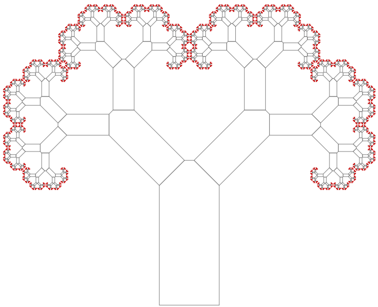

$\theta = \frac{\pi}{2k},\ k \in \mathbb{N}^\star$

$\dim_H \Xi = \frac{d}{2}$

where Ξ is the set of multiple points of Γ.

Outline

Questions

- **Trace**:
  - Notion of trace for functions in $W^{1,p}(\Omega)$ on Γ ?
  - Sobolev regularity of the trace on Γ ?

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- **Trace**:
  - Notion of trace for functions in $W^{1,p}(\Omega)$ on Γ ?
  - Sobolev regularity of the trace on Γ ?
- **Extension**:
  - for which values of p is the domain Ω a $W^{1,p}$-extension domain, *i.e.* there exists a linear continuous extension operator

$$W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)?$$

If Ω is a $W^{1,p}$-extension domain for every $p \in [1, \infty]$, Ω is said to be a Sobolev extension domain.

Trace theorems

Theorem (Gagliardo, 1957)

If $\omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary and $1 < p < \infty$, one has the trace result:

$$W^{1,p}(\omega)_{|\partial\omega} \; = \; W^{1-\frac{1}{p},p}(\partial\omega).$$

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Theorem (A. Jonsson, H. Wallin, 1984)

If $1 < p < \infty$, and $1 - \frac{2-d}{p} > 0$, then

$$W^{1,p}(\mathbb{R}^2)_{|\Gamma} \;=\; W^{1-\frac{2-d}{p},p}(\Gamma)$$

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If $1 < p < \infty$, *and* $1 - \frac{2-d}{p} > 0$, *then*

$$W^{1,p}(\mathbb{R}^2)_{|\Gamma} \ = \ W^{1-\frac{2-d}{p},p}(\Gamma)$$

Sense of the trace: u is strictly defined at $x \in \mathbb{R}^2$ if the limit

$$\overline{u}(x) \ := \ \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, \mathrm{d}y$$

exists. Trace of u on Γ : $\bar{u}_{|\Gamma}$.

Extension theorems

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Every Lipschitz domain in $\mathbb{R}^n$ *is a Sobolev extension domain.*

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Jones domains (P.W. Jones, 1981) :

A domain $\omega \subset \mathbb{R}^n$ is a Jones domain if there exist $\varepsilon, \delta > 0$ such that for every $x, y \in \omega$ satisfying $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset \omega$ joining x to y such that

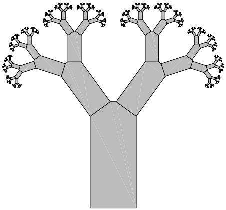
- $L(\gamma) \leqslant \frac{1}{\varepsilon} |x - y|$ where $L(\gamma) =$length of γ,
- $d(z, \partial\omega) \geqslant \varepsilon \min(|x - z|, |y - z|)$ for all $z \in \gamma$.

Theorem (P.W. Jones, 1981)

Jones domains are Sobolev extension domains.

The subcritical case ($a < a^\star$)

- **Extension**



If Ω is a ramified domain with $a < a^\star$, then Ω is a Jones domain, so it has the p-extension property for all $1 \leqslant p \leqslant \infty$.

- **Traces**

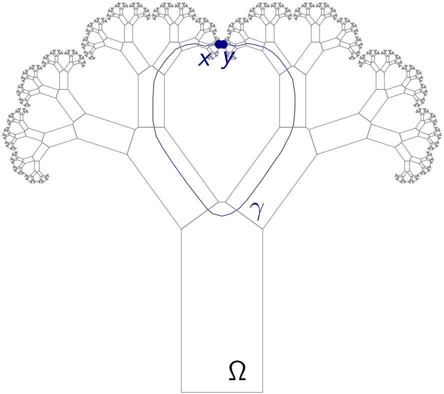
Jones theorem combined with Jonsson and Wallin's trace operator yields a trace operator:

$$W^{1,p}(\Omega) \;\to\; W^{1-\frac{2-d}{p},p}(\Gamma)$$

Outline

The ramified domains with $a = a^\star$

The case $a = a^\star$

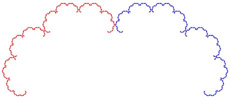


In this case, Ω cannot be a $W^{1,p}$-extension domain for $p > 2$.

Haar wavelets on Γ

Haar wavelets on Γ:

$$\left\{ \begin{array}{rcl} g_0 & = & \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\ & & \end{array} \right.$$



Mother wavelet g_0

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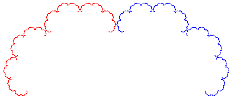
$$\begin{cases} g_0 = \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\ g_{\sigma\,|f_\sigma(\Gamma)} = 2^{\frac{k}{2}} g_0 \circ {f_\sigma}^{-1} \text{ and } g_{\sigma\,|\Gamma \setminus f_\sigma(\Gamma)} = 0 \text{ for } \sigma \in \{1,2\}^k \end{cases}$$



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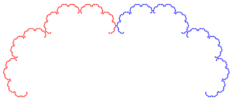

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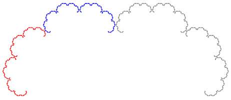

g_σ for $\sigma = (1)$

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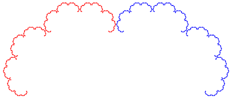

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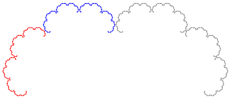
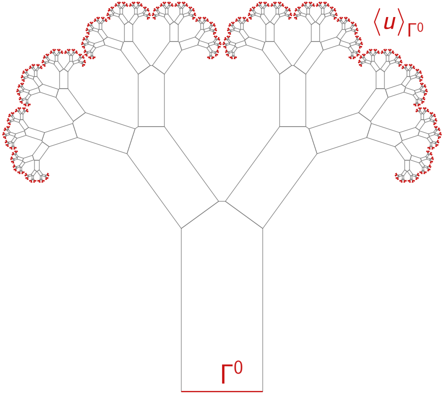


g_σ for $\sigma = (1)$

Every function $v \in L^p_\mu(\Gamma)$, $1 \leqslant p < \infty$ can be expanded in the Haar wavelet basis (g_σ):

$$v = \langle v \rangle_\Gamma + \sum_{k\geqslant 0} \sum_{\sigma \in \{1,2\}^k} \beta_\sigma g_\sigma.$$

Construction of a self-similar trace operator



For $u \in W^{1,p}(\Omega)$, define:

$$\ell^0(u) \equiv \langle u \rangle_{\Gamma^0},$$

where $\langle u \rangle_{\Gamma^\sigma} = \dfrac{1}{|\Gamma^\sigma|} \displaystyle\int_{\Gamma^\sigma} u.$

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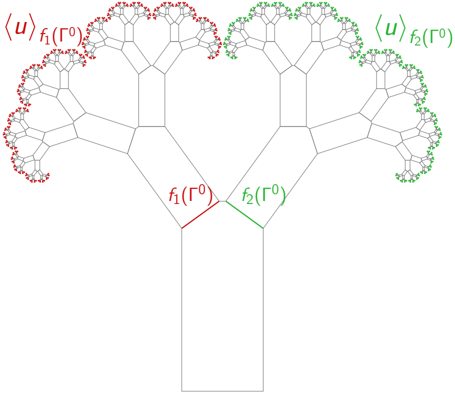

$$\ell^1(u) = \langle u \rangle_{f_1(\Gamma^0)} \mathbb{1}_{f_1(\Gamma)} + \langle u \rangle_{f_2(\Gamma^0)} \mathbb{1}_{f_2(\Gamma)}$$

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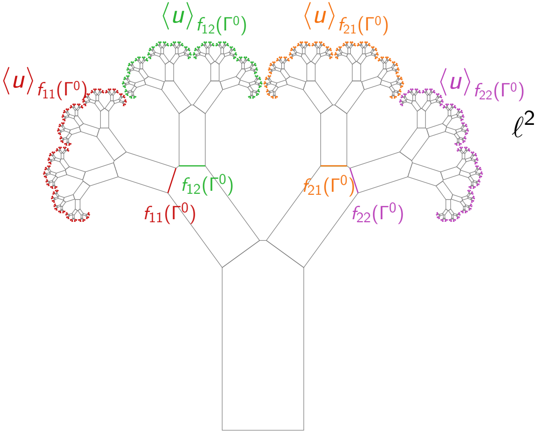
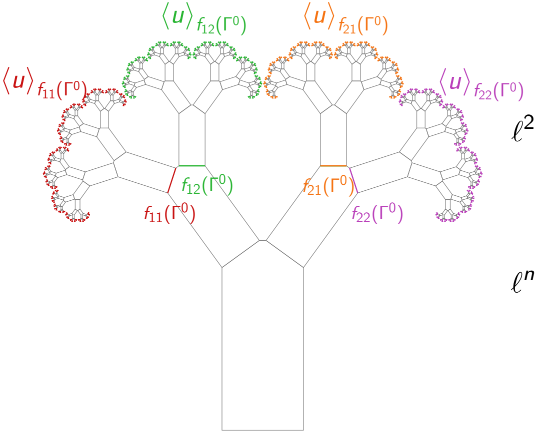


$$\ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)}$$

Construction of a self-similar trace operator



$$\ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)}$$

$$\ell^n(u) = \sum_{\sigma \in \{1,2\}^n} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)}$$

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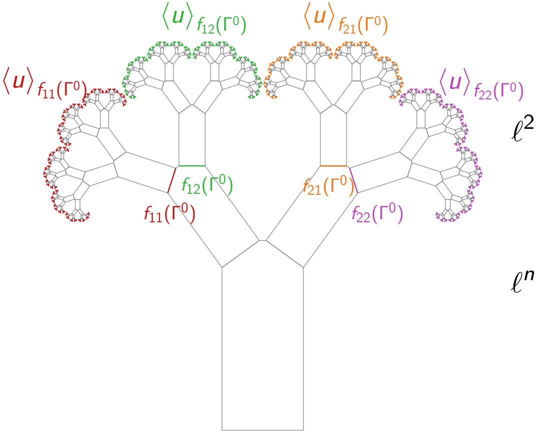

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The sequence (ℓ^n) converges in $\mathcal{L}(W^{1,p}(\Omega), L^p_\mu(\Gamma))$ to an operator ℓ^∞.

Theorem (Y. Achdou, T.D., N. Tchou, 2012)

Set $p^\star = \begin{cases} 2 & \text{in case ❶} \\ 2 - \frac{d}{2} & \text{in case ❷} \end{cases} = 2 - \dim_H \Xi$

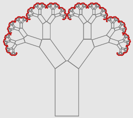
If $a = a^\star$, *then*

- *if* $p < p^\star$,

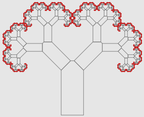
$$\ell^\infty(W^{1,p}(\Omega)) \;=\; W^{1-\frac{2-d}{p},p}(\Gamma)$$

- *if* $p \geqslant p^\star$, $\quad \ell^\infty(W^{1,p}(\Omega)) \;\subset\; W^{s,p}(\Gamma)$

  for every $s < \begin{cases} \frac{d}{p} & \text{in case ❶} \\ \frac{d}{2p} & \text{in case ❷} \end{cases} = \frac{d - \dim_H \Xi}{p}$



$p^\star = 2$



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If $a = a^\star$, *then*

- *if* $p < p^\star$,

$$\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma)$$

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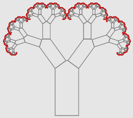

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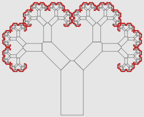
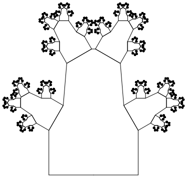


$p^\star = 2 - \frac{d}{2}$

In the case of a ramified domain with 4 similitudes and $\dim \Xi = \frac{\dim_H \Gamma}{4}$, the result holds.



Outline

Extension theorem

In the case $a = a^\star$,

- we know that a ramified domain Ω is not a $W^{1,p}$-extension domain for $p > 2$.
- the trace theorem suggests that Ω is not a $W^{1,p}$-extension domain for $p > p^\star$.
- $p < p^\star$?

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- $p < p^\star$?

Theorem (T.D., 2013)

If Ω is a critical ramified domain and $p^\star = 2 - \dim \Xi$, then for all $p < p^\star$, Ω is a $W^{1,p}$-extension domain

Outline

Comparison of the notions of trace

The following theorem justifies *a posteriori* the use of several notions of trace on Γ.

Theorem (Y. Achdou, T.D., N. Tchou, 2013)

For $1 < p < \infty$, every function $u \in W^{1,p}(\Omega)$ is strictly defined μ-almost everywhere on Γ, and

$$\overline{u}_{|\Gamma} \;=\; \ell^{\infty}(u)$$

μ-almost everywhere on Γ.

The proof uses as a key ingredient the extension operator for $p < p^{\star}$. In this case, the trace on Γ does not depend on the extension operator: $(\mathcal{E}u)_{|\Gamma} =: \overline{u}_{|\Gamma}$.

Corollary

If $p > p^{\star}$, Ω is not a $W^{1,p}$-extension domain.

Thank you for your attention!