Combinatorial Harmonic Coordinates

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Fractals 5 - Cornell University.

Perspective

Motivating questions

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Motivating answers

There are interesting cases in which both answers are "yes".

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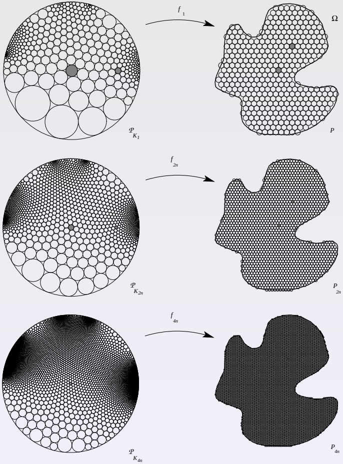

Rodin-Sullivan proved (87), important and beautiful extensions ever since by Schramm-He, Beardon-Stephenson, C.D Verdiere, Chow-Luo....

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Our work extends (with different methods) previous work by Schramm and Cannon-Floyd-Parry.

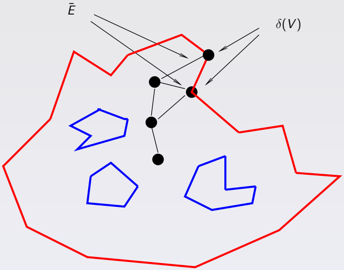
One main underlying idea of our work:

For an analytic function f of the complex plane one has

$$f = u + iv,$$

with u harmonic and v its harmonic conjugate.

Discrete Boundary value problems on graphs



We consider a planar, bounded, m-connected region Ω, and let $\partial\Omega$ be its boundary. Let

$$\partial\Omega = E_1 \sqcup E_2,$$

where E_1 is the outermost component of $\partial\Omega$.
Let $\mathcal{T}$ be a triangulation of $\Omega \cup \partial\Omega$. Invoke a *conductance function* on $\mathcal{T}^{(1)}$, i.e., each edge $(x, y) \in E$ is assigned a *conductance* $c(x, y) = c(y, x) > 0$, making it a simple *finite network*.

Let $u \in \mathcal{P}(\overline{V})$, the set of non-negative functions defined on $\bar{V}$, and $\overline{V} = V \cup \delta(V)$. Then for $x \in V$, the function

$$\Delta u(x) = \sum_{y \sim x} c(x,y)\big(u(x) - u(y)\big)$$

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For $x \in \delta(V)$, let $\{y_1, y_2, \ldots, y_m\} \in V$ be its neighbors enumerated clockwise. The *normal derivative* of u at a point $x \in \delta(V)$ with respect to a set V is

$$\frac{\partial u}{\partial n}(V)(x) = \sum_{y \sim x,\ y \in V} c(x,y)(u(x) - u(y)).$$

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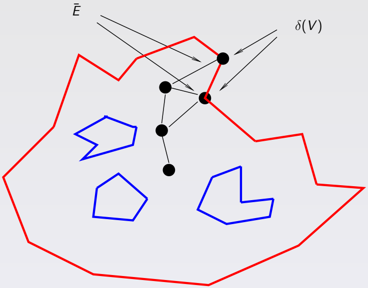
A function $u \in \mathcal{P}(\overline{V})$ is called *harmonic* in V if $\Delta u(x) = 0$, for all $x \in V$. The number

$$E(u) = \sum_{(x,y) \in \bar{E}} c(x,y)\big(u(x) - u(y)\big)^2$$

is called the *Dirichlet energy* of u.



Definition

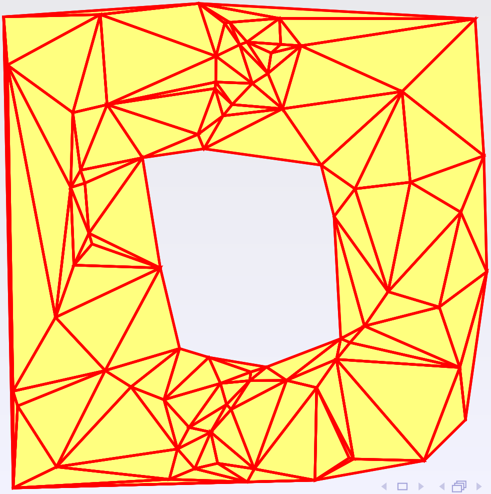
Let k be a positive constant. The Discrete *Dirichlet* Boundary Value Problem is determined by requiring that

1. $g|_{\mathcal{T}^{(0)} \cap E_1} = k$, $g|_{\mathcal{T}^{(0)} \cap E_2} = 0$, and
2. $\Delta g = 0$ at every interior vertex of $\mathcal{T}^{(0)}$.

These data will be called *Dirichlet data* for Ω.

A theorem.

We would like to give a concrete description of one of our new main theorems. Here is the setting.



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$$f : (\mathcal{A}, \partial\mathcal{A}, \mathcal{R}) \to (S_{\mathcal{A}}, \partial S_{\mathcal{A}}, T),$$

   such that f maps each quadrilateral in $\mathcal{R}^{(2)}$ onto a single annular shell in $S_{\mathcal{A}}$;

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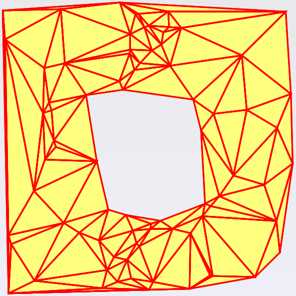
such that f maps each quadrilateral in $\mathcal{R}^{(2)}$ onto a single annular shell in $S_{\mathcal{A}}$; f preserves the measure of each quadrilateral, i.e.,

$$\nu(R) = \mu(f(R)), \text{ for all } R \in \mathcal{R}^{(2)}.$$



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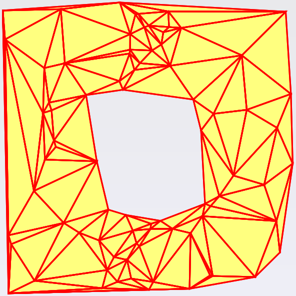

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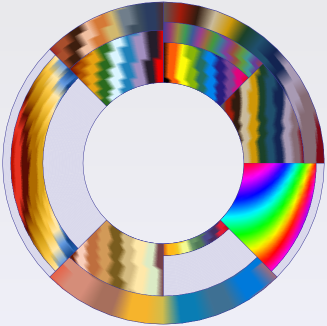

We will start the proof by extending g to the interior of the domain and affinely over edges in $\mathcal{T}^{(1)}$ and over triangles $\mathcal{T}^{(2)}$. We will (often) abuse notation and will not distinguish between a function defined on $\mathcal{T}^{(0)}$ and its extension over $|\mathcal{T}|$.

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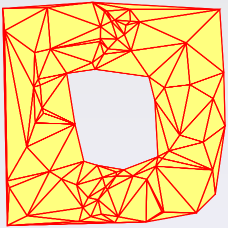

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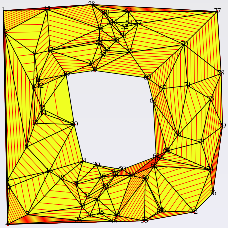

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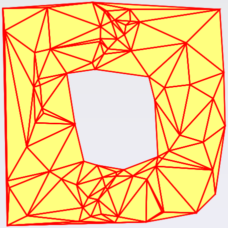

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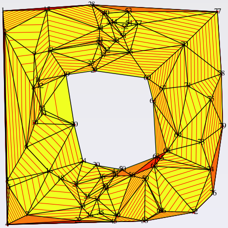

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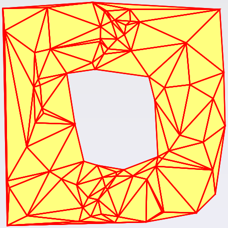

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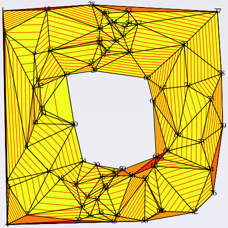

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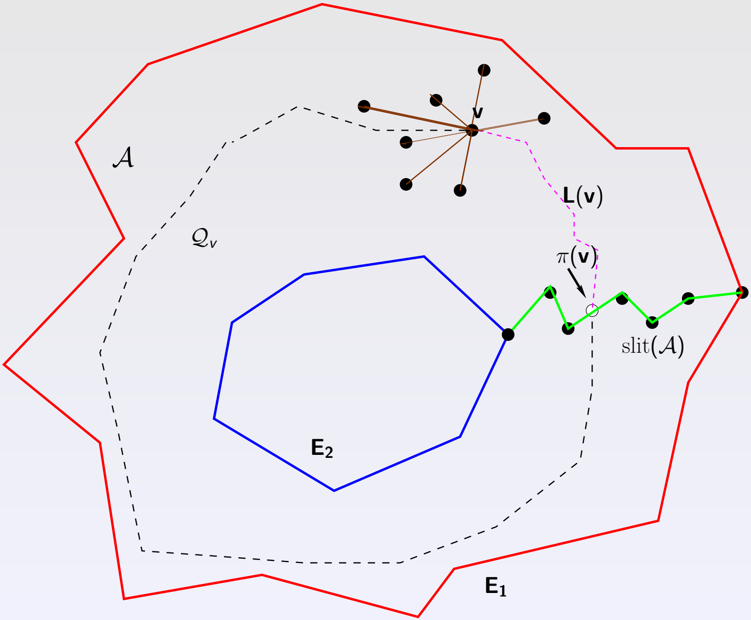
We define a *new* function, g^*, on $\mathcal{T}^{(0)}$. This function will actually be single-valued on an annulus minus a slit and will be called the *conjugate function* of g. It is obtained by integrating the *discrete normal derivative* of g along its level curves.


v
$\mathcal{A}$
L(v)
$\mathcal{Q}_v$
π(v)
slit($\mathcal{A}$)
E_2
E_1

What are the properties on g^*?

- The curve $\partial\mathcal{Q}_{\mathrm{top}}$ is a level curve of g^* in $\mathcal{Q}_{\mathrm{slit}}$
- Each level curve of g^* has no endpoint in the interior of $\mathcal{Q}_{\mathrm{slit}}$, is simple, and joins E_1 to E_2. Furthermore, any two level curves of g^* are disjoint.
- The number of intersections between any level curve of g^* and any level curve of g is equal to 1.

Definition

The *period* of g^* is defined by the g^* value on $\partial\mathcal{Q}_{\mathrm{top}}$, that is,

$$\mathrm{period}(g^*) = g^*|\partial\mathcal{Q}_{\mathrm{top}} = \int_{u\in\mathcal{T}^{(0)}\cap E_1} \frac{\partial g}{\partial n}(\mathcal{A}_{(E_2,E_1)})(u).$$