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The classical Szegö Theorem

If $P_n$ is projection onto the span of $\{e^{im\theta}, 0 \leq m \leq n\}$ in $L^2(\mathbb{T})$ and $[f]$ is multiplication by a positive $C^{1+\alpha}$ function for $\alpha > 0$ then

$$\lim_{n \to \infty} \frac{\log \det P_n[f]P_n}{n + 1} = \int_0^{2\pi} \log f(\theta) \, d\theta / 2\pi.$$ 

Equivalently, $(n + 1)^{-1} \text{Tr} \log P_n[f]P_n$ has the same limit.
Set up and notation

- $X$ is the Sierpiński gasket.
- $\mu$ is the standard measure
- $\Delta$ is the Dirichlet Laplacian on $X$ defined by the symmetric self-similar resistance on $X$. 
**Set up and notation**

- $X$ is the Sierpiński gasket.
- $\mu$ is the standard measure.
- $\Delta$ is the Dirichlet Laplacian on $X$ defined by the symmetric self-similar resistance on $X$.
- For $\lambda \in \text{sp}(-\Delta)$, let $E_\lambda$ be its eigenspace, $d_\lambda = \dim E_\lambda$, and $P_\lambda$ the projection onto $E_\lambda$.
- For $\Lambda > 0$, let $E_\Lambda$ be the span of all eigenfunctions corresponding to $\lambda \leq \Lambda$ and let $P_\Lambda$ be the projection onto $E_\Lambda$. 
The Szegö Theorem for the Sierpiński Gasket

Theorem (Okoudjou, Rogers, Strichartz, 2010)

Let $f > 0$ be a continuous function on $X$. Then

$$\lim_{\Lambda \to \infty} \frac{1}{d_{\Lambda}} \log \det P_{\Lambda}[f] P_{\Lambda} = \int_X \log f(x) d\mu(x).$$
Theorem (I., Okoudjou, Rogers, 2014)

Let $p : X \times (0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda_n)$ is continuous for all $n \in \mathbb{N}$. Assume that $\lim_{n \to \infty} p(x, \lambda_n) = q(x)$ is uniform in $x$. Then, for any continuous function $F$ supported on $[A, B]$, we have that

$$
\lim_{\Lambda \to \infty} \frac{1}{d_{\Lambda}} \text{Tr} \left( P_{\Lambda} p(x, -\Delta) P_{\Lambda} \right) = \int_X F(q(x))d\mu(x).
$$
The spectrum decomposes naturally into three sets called the 2-series, 5-series and 6-series eigenvalues.

Each eigenvalue has a generation of birth $j$.

2-series eigenfunctions have $j = 1$ and multiplicity 1.

Each $j \in \mathbb{N}$ occurs in the 5-series and the corresponding eigenspace has multiplicity $(3^{j-1} + 3)/2$.

Each $j \geq 2$ occurs in the 6-series with multiplicity $(3^j - 3)/2$.

There are 5 and 6-series eigenfunctions that are localized.
Pseudo-differential operators on the Sierpiński Gasket

**Definition**

If $p : (0, \infty) \to \mathbb{C}$ is measurable then

$$p(-\Delta)u = \sum_n p(\lambda_n) \langle u, \varphi_n \rangle \varphi_n$$

for $u \in \mathcal{D}$ gives a densely defined operator on $L^2(\mu)$ called a constant coefficient pseudo-differential operator.

If $p : X \times (0, \infty) \to \mathbb{C}$ is measurable, we define a variable coefficient pseudo-differential operator $p(x, -\Delta)$ via

$$p(x, -\Delta)u(x) = \sum_n \hat{X} p(x, \lambda_n) P_{\lambda_n}(x, y) u(y) d\mu(y).$$
Definition

If $p : (0, \infty) \to \mathbb{C}$ is measurable then

$$p(-\Delta)u = \sum_n p(\lambda_n)\langle u, \varphi_n \rangle \varphi_n$$

for $u \in D$ gives a densely defined operator on $L^2(\mu)$ called a constant coefficient pseudo-differential operator.
If \( p : (0, \infty) \to \mathbb{C} \) is measurable then

\[
p(-\Delta)u = \sum_n p(\lambda_n) \langle u, \varphi_n \rangle \varphi_n
\]

for \( u \in D \) gives a densely defined operator on \( L^2(\mu) \) called a constant coefficient pseudo-differential operator.

If \( p : X \times (0, \infty) \to \mathbb{C} \) is measurable we define a variable coefficient pseudo-differential operator \( p(x, -\Delta) \) via

\[
p(x, -\Delta)u(x) = \sum_n \int_X p(x, \lambda_n) P_{\lambda_n}(x, y) u(y) d\mu(y).
\]
Assumptions

Fact

\[ p : X \times (0, \infty) \rightarrow \mathbb{R} \text{ is measurable and that} \]

1. \( p(\cdot, \lambda) \) is continuous for all \( \lambda \in \text{sp}(-\Delta) \) and
2. \( \lim_{\lambda \in \text{sp}(-\Delta), \lambda \to \infty} p(x, \lambda) = q(x) \) uniformly in \( x \).
Some key lemmas

Theorem

The eigenvalues of $P_\Lambda p(x, -\Delta)P_\Lambda$ are contained in a bounded interval $[A, B]$ for all $\Lambda > 0$. 

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Spectral Properties of PSDO
Some key lemmas

Theorem

The eigenvalues of \( P_\Lambda p(x, -\Delta) P_\Lambda \) are contained in a bounded interval \([A, B]\) for all \( \Lambda > 0 \).

Theorem

Let \( \Lambda > 0 \). Then the map on \( C[A, B] \) defined by

\[
F \mapsto \frac{1}{d_\Lambda} \text{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda)
\]

is a continuous non-negative functional.
Fact

If $\lambda \in \text{sp}(-\Delta)$ then $\Gamma_\lambda := P_\lambda p(x, -\Delta)P_\lambda$ is a $d_\lambda \times d_\lambda$ matrix with entries

$$\gamma_\lambda(i, j) = \int p(x, \lambda)u_i(x)u_j(x)d\mu(x).$$
Let \( \{\lambda_j\} \) be an increasing sequence of 6- or 5-series eigenvalues where \( \lambda_j \) has generation of birth \( j \). Let \( N \geq 1 \) be fixed, and suppose \( f = \sum_{i=1}^{3^N} a_i \chi_{C_i} \) is a simple function. Then for all \( k \geq 0 \)

\[
\lim_{j \to \infty} \frac{\text{Tr}(P_j[f]P_j)^k}{d_j} = \int f(x)^k d\mu(x).
\]
Sketch of the proof.

The matrix $P_j[f]P_j$ has the following structure with respect to the basis $\{u_m\}_{m=1}^{d_j}$:

$$
\begin{bmatrix}
R_j & 0 \\
0 & N_j
\end{bmatrix}.
$$
Sketch of the proof.

The matrix $P_j[f]P_j$ has the following structure with respect to the basis $\{u_m\}_{m=1}^{d_j}$:

$$
\begin{bmatrix}
R_j & 0 \\
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$$

Moreover

$$
\text{Tr}(R_j)^k = \sum_{i=1}^{3^N} m_j^N a_i^k = d_j^N \sum_{i=1}^{3^N} \frac{a_i^k}{3^N} = d_j^N \int f(x)^k d\mu(x)
$$

and

$$
|\text{Tr}(N_j)^k| \leq (\alpha^N)^k \|f\|_{\infty}^k.
$$
Theorem

Let \( \{\lambda_j\} \) be an increasing sequence of 6- or 5-series eigenvalues where \( \lambda_j \) has generation of birth \( j \). Then

\[
\lim_{j \to \infty} \frac{1}{d_j} \text{Tr} \ F(P_j p(x, -\Delta) P_j) = \int_X F(q(x)) d\mu(x)
\]

for any continuous \( F \) supported on \([A, B]\).
It suffices to prove

$$\lim_{j \to \infty} \frac{1}{d_j} \text{Tr}(P_j p(x, -\Delta) P_j)^k = \int_X q(x)^k d\mu(x).$$
Sketch of the Proof

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It suffices to prove

\[
\lim_{j \to \infty} \frac{1}{d_j} \Tr(P_j p(x, -\Delta)P_j)^k = \int_X q(x)^k d\mu(x).
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It suffices to assume that \( p(x, \lambda) \geq C > 0 \) for all \((x, \lambda)\).
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\]

It suffices to assume that \( p(x, \lambda) \geq C > 0 \) for all \( (x, \lambda) \).

Approximate \( p(x, \lambda) \) with a simple function \( f_N \) such that

\[
0 \leq P_j[f_N - \delta] P_j \leq P_j p(x, -\Delta) P_j \leq P_j[f_N + \delta] P_j.
\]

Then

\[
\left| \frac{1}{d_j} \operatorname{Tr}(P_j p(x, -\Delta) P_j)^k - \frac{1}{d_j} \operatorname{Tr}(P_j[f_N] P_j)^k \right| < \varepsilon.
\]
Main Theorem

Theorem

Let $p : X \times (0, \infty) \rightarrow \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda_n)$ is continuous for all $n \in \mathbb{N}$. Assume that

$$\lim_{n \to \infty} p(x, \lambda_n) = q(x)$$

is uniform in $x$. Then, for any continuous function $F$ supported on $[A, B]$, we have that

$$\lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \text{Tr} F(P_\Lambda p(x, -\Delta) P_\Lambda) = \int_X F(q(x)) d\mu(x).$$
Sketch of the proof.

\[
\begin{align*}
&\left| \frac{\text{Tr}(P_\Lambda p(x, -\Delta)P_\Lambda)^k}{d_\Lambda} - \int q(x)^k d\mu(x) \right| \\
\leq \sum_{\lambda \in \tilde{\Gamma}_J(\Lambda)} \left| \frac{\text{Tr}(P_\lambda p(x, -\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)}{d_\Lambda} \right| \\
&\quad + \sum_{\lambda \in \Gamma_J(\Lambda)} \left| \frac{\text{Tr}(P_\lambda p(x, -\Delta)P_\lambda)^k - d_\lambda \int q(x)^k d\mu(x)}{d_\Lambda} \right|.
\end{align*}
\]
Examples

1. If $p: (0, \infty) \to \mathbb{R}$ is a bounded measurable map such that $\lim_{j \to \infty} p(\lambda_j) = q$, then for any continuous $F$ supported on $[\| p(\lambda - \Delta) \|, \| p(\lambda + \Delta) \|]$, we have $\lim_{\Lambda \to \infty} 1 d_{\Lambda} \text{Tr} F (P_{\Lambda} p(\lambda - \Delta) P_{\Lambda}) = F(q)$.

2. Riesz and Bessel Potentials: If $p(\lambda) = 1 + \lambda - \beta$ or $p(\lambda) = 1 + (1 + \lambda) - \beta$, $\lambda > 0$, $\beta > 0$, then $\lim_{\Lambda \to \infty} \text{Tr} F (P_{\Lambda} p(\lambda - \Delta) P_{\Lambda}) d_{\Lambda} = F(1)$. 

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If $p : (0, \infty) \to \mathbb{R}$ is a bounded measurable map such that $\lim_{j \to \infty} p(\lambda_j) = q$, then for any continuous $F$ supported on $[-\|p(-\Delta)\|, \|p(-\Delta)\|]$ we have

$$\lim_{\Lambda \to \infty} \frac{1}{d_{\Lambda}} \text{Tr} F(P_{\Lambda} p(-\Delta) P_{\Lambda}) = F(q).$$
Examples

1. If \( p : (0, \infty) \rightarrow \mathbb{R} \) is a bounded measurable map such that \( \lim_{j \to \infty} p(\lambda_j) = q \), then for any continuous \( F \) supported on \([ -\|p(-\Delta)\|, \|p(-\Delta)\| ]\) we have

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2. Riesz and Bessel Potentials: If \( p(\lambda) = 1 + \lambda^{-\beta} \) or \( p(\lambda) = 1 + (1 + \lambda)^{-\beta} \), \( \lambda > 0, \beta > 0 \), then

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\lim_{\Lambda \to \infty} \frac{\operatorname{Tr} F(P_\Lambda p(-\Delta)P_\Lambda)}{d_\Lambda} = F(1).
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If $p$ is a 0-symbol, then for any continuous $F$ supported on $[-\|p(x, -\Delta)\|, \|p(x, -\Delta)\|]$ we have

$$\lim_{\Lambda \to \infty} \frac{1}{d_{\Lambda}} \mathrm{Tr} F(P_{\Lambda} p(x, -\Delta) P_{\Lambda}) = \int_X F(q(x)) d\mu(x).$$
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Examples

1. If \( p \) is a 0-symbol, then for any continuous \( F \) supported on \([-\|p(x, -\Delta)\|, \|p(x, -\Delta)\|]\) we have

\[
\lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \text{Tr} \ F(P_\Lambda p(x, -\Delta)P_\Lambda) = \int_X F(q(x))d\mu(x).
\]

2. If \( p(x, \lambda) = f(x) \), then for any continuous \( F \) supported on \([-\|[f]\|, \|[f]\|]\) we have

\[
\lim_{\Lambda \to \infty} \frac{1}{d_\Lambda} \text{Tr} \ F(P_\Lambda[f]P_\Lambda) = \int_X F(f(x))d\mu(x).
\]
Definition

Let $p : (0, \infty) \rightarrow \mathbb{R}$ be a measurable function and let $\chi$ be a real-valued bounded measurable function on $X$. We call the operator $H = p(-\Delta) + [\chi]$ a generalized Schrödinger operator with potential $\chi$. 
Example

Assume that \( \lim_{\lambda \to \infty} p(\lambda) = l \) exists and \( \chi \) is a continuous function on \( X \). Let \( F \) be a continuous function supported on \([-\|H\|, \|H\|]\). Then, if \( \{\lambda_j\}_{j \geq 1} \) is an increasing sequence of 6-series or, respectively, 5-series eigenvalues, we have that

\[
\lim_{j \to \infty} \frac{\text{Tr} \, F(P_jHP_j)}{d_j} = \int F(l + \chi(x))d\mu(x).
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Example

Assume that \( \lim_{\lambda \to \infty} p(\lambda) = l \) exists and \( \chi \) is a continuous function on \( X \). Let \( F \) be a continuous function supported on \( [-\|H\|, \|H\|] \). Then, if \( \{\lambda_j\}_{j \geq 1} \) is an increasing sequence of 6-series or, respectively, 5-series eigenvalues, we have that

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\]

Hence

\[
\lim_{\Lambda \to \infty} \frac{\text{Tr} F(P_{\Lambda} H P_{\Lambda})}{d_{\Lambda}} = \int F(l + \chi(x)) d\mu(x).
\]
Theorem

Let \( \{\lambda_j\}_{j \in \mathbb{N}} \) be an increasing sequence of 6- or 5-series eigenvalues such that \( \lambda_j \) has generation of birth \( j \), for all \( j \geq 1 \). Assume that

\[
\lim_{j \to \infty} p(x, \lambda_j) = q(x) \quad \text{for all } x \in X.
\]

Suppose that \( p(\cdot, \lambda_j) \in \text{Dom}(\Delta) \) for all \( j \in \mathbb{N} \) and that both \( p(\cdot, \lambda_j) \) and \( \Delta_x p(\cdot, \lambda_j) \) are bounded uniformly in \( j \). Then there is a subsequence \( \{\lambda_{k_j}\} \) of \( \{\lambda_j\} \) such that

\[
\lim_{j \to \infty} \frac{1}{d_{k_j}} F(P_{k_j} p(x, -\Delta) P_{k_j}) = \int_X F(q(x)) d\mu(x).
\]
Let $H = p(-\Delta) + [\chi]$ be a Schrödinger operator, where $p : (0, \infty) \to \mathbb{R}$ is a continuous function, such that there is $\lambda > 0$ so that $p$ is increasing on $[\lambda, \infty)$ and

$$|p(\lambda) - p(\lambda')| \geq c|\lambda - \lambda'|^{\beta}$$

for all $\lambda, \lambda' \geq \lambda$, and $\chi$ is a continuous function on $X$. 
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for all $\lambda, \lambda' \geq \lambda$, and $\chi$ is a continuous function on $X$.

Let \( \{\lambda_j\} \) be a sequence of 6-series eigenvalues of $-\Delta$ such that the separation between $\lambda_j$ and the next higher and lower eigenvalues of $-\Delta$ grows exponentially in $j$. 
Let \( \tilde{\Lambda}_j \) be the portion of the spectrum of \( H \) lying in \( [\lambda_j(\min \chi, \max \chi)] \). For large \( j \), \( \tilde{\Lambda}_j \) contains exactly \( d_j \) eigenvalues \( \{ \nu_{ji} \} \), \( d_j = 1 \). We call this the \( \lambda_j \) cluster of the eigenvalues of \( H \). The characteristic measure of the \( \lambda_j \) cluster of \( H \) is

\[
\Psi_j(\lambda) = \frac{1}{d_j} \sum_{i=1}^{d_j} \delta(\lambda - (\nu_{ji} - \lambda_j)).
\]
Fact

Let $\tilde{\Lambda}_j$ be the portion of the spectrum of $H$ lying in $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$. The characteristic measure of the $p(\lambda_j)$ cluster of $H$ is

$$\Psi_j(\lambda) = \frac{1}{d_j} \sum_{i=1}^{d_j} \delta(\lambda - (\nu_{ji} - p(\lambda_j))).$$
Fact

Let $\tilde{\Lambda}_j$ be the portion of the spectrum of $H$ lying in $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.

For large $j$, $\tilde{\Lambda}_j$ contains exactly $d_j$ eigenvalues $\{\nu^j_i\}_{i=1}^{d_j}$. 
Fact

- Let $\tilde{\Lambda}_j$ be the portion of the spectrum of $H$ lying in $[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi]$.
- For large $j$, $\tilde{\Lambda}_j$ contains exactly $d_j$ eigenvalues $\{\nu^j_i\}_{i=1}^{d_j}$.
- We call this the $p(\lambda_j)$ cluster of the eigenvalues of $H$. 
Fact

- Let $\tilde{\Lambda}_j$ be the portion of the spectrum of $H$ lying in $\left[p(\lambda_j) + \min \chi, p(\lambda_j) + \max \chi \right]$.
- For large $j$, $\tilde{\Lambda}_j$ contains exactly $d_j$ eigenvalues $\{\nu^j_i\}_{i=1}^{d_j}$.
- We call this the $p(\lambda_j)$ cluster of the eigenvalues of $H$.
- The characteristic measure of the $p(\lambda_j)$ cluster of $H$ is

$$\Psi_j(\lambda) = \frac{1}{d_j} \sum_{i=1}^{d_j} \delta(\lambda - (\nu^j_i - p(\lambda_j))).$$
The sequence \( \{ \Psi_j \}_{j \geq 1} \) converges weakly to the pullback of the measure \( \mu \) under \( \chi \) defined for all continuous functions \( f \) supported on \([\text{min } \chi, \text{max } \chi]\) by

\[
\langle \Psi_0, f \rangle = \int_{\chi} f(\chi(x)) \, d\mu(x).
\]
Theorem

If \( p : (0, \infty) \to \mathbb{C} \) is continuous then \( \text{sp} p(-\Delta) = \overline{p(\text{sp}(-\Delta))} \).
Sketch of the Proof: some lemmas

**Theorem**

If \( p : (0, \infty) \to \mathbb{C} \) is continuous then \( \text{sp} \, p(-\Delta) = \overline{p(\text{sp}(-\Delta))} \).

**Theorem**

Let \( p : (0, \infty) \to \mathbb{R} \) be a continuous function such that there is \( A \in \mathbb{R} \) with \( p(\lambda) \geq A \) for all \( \lambda \geq \lambda_1 \), where \( \lambda_1 \) is the smallest positive eigenvalue of \( -\Delta \). For \( i = 1, 2 \), let \( \chi_i \) be real-valued bounded measurable functions on \( X \). Let \( H_i = p(-\Delta) + [\chi_i] \) denote the corresponding generalized Schrödinger operators. For \( n \geq 1 \), the \( n \)th eigenvalues \( \nu_n^i \) of \( H_i \), \( i = 1, 2 \), satisfy the following inequality:

\[
|\nu_n^1 - \nu_n^2| \leq \|\chi_1 - \chi_2\|_{L^\infty}.
\]
Theorem

Assume that $N > 0$ and that $\chi_N = \sum_{i=1}^{N} a_i \chi_{C_i}$ is a simple function. Let $H_N = p(-\Delta) + [\chi_N]$ be the corresponding generalized Schrödinger operator, $\tilde{\Lambda}_j^N$ the $p(\lambda_j)$ cluster of $H_N$, and let $\overline{P}_j^N$ be the spectral projection for $H_N$ associated with the $p(\lambda_j)$ cluster. Then

$$\lim_{j \to \infty} \frac{\text{Tr}(\overline{P}_j^N (p(-\Delta) + [\chi_N] - p(\lambda_j))\overline{P}_j^N)^k}{d_j} = \lim_{j \to \infty} \frac{\text{Tr}(P_j[\chi_N]P_j)^k}{d_j} = \int_{\chi} \chi_N(x)^k \, d\mu(x),$$

for all $k \geq 0$. 