Quantum symmetry breaking

Scale anomaly and fractals

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PAZY
EXCELLING IN SCIENCE

ISRAEL SCIENCE FOUNDATION

6th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals, June 13-17, 2017

Benefitted from discussions and collaborations with:

Technion:

Evgeni Gurevich (KLA-Tencor)
Dor Gittelman
Eli Levy (+ Rafael)
Ariane Soret (ENS Cachan)
Or Raz (HUJI, Maths)
Omrie Ovdat
Ohad Shpielberg
Tal Goren
Alex Leibenzon

Rafael:
Assaf Barak
Amnon Fisher

NRCN:
Ehoud Pazy

Elsewhere:

Gerald Dunne (UConn.)
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Jacqueline Bloch (LPN, Marcoussis)
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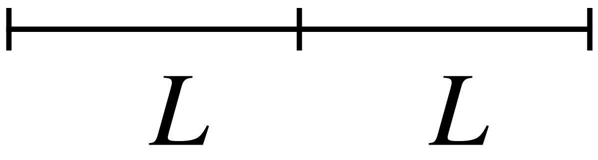
Continuous *vs*. discrete scale symmetry

Homogeneous string (uniform mass per unit length)

$d=1$ $m(L)$ Expect : $m(L)\propto L$

How to obtain this result ?



$$m(2L)=2\ m(L)$$

or more generally, $m(aL)=b\,m(L)$ $\forall\, a\in\mathbb{R}$

Continuous scale invariance (CSI)

Scaling relation: $f(ax) = bf(x)$

If this relation is **satisfied for all** a and $b(a)$, the system has a continuous scale invariance (CSI).

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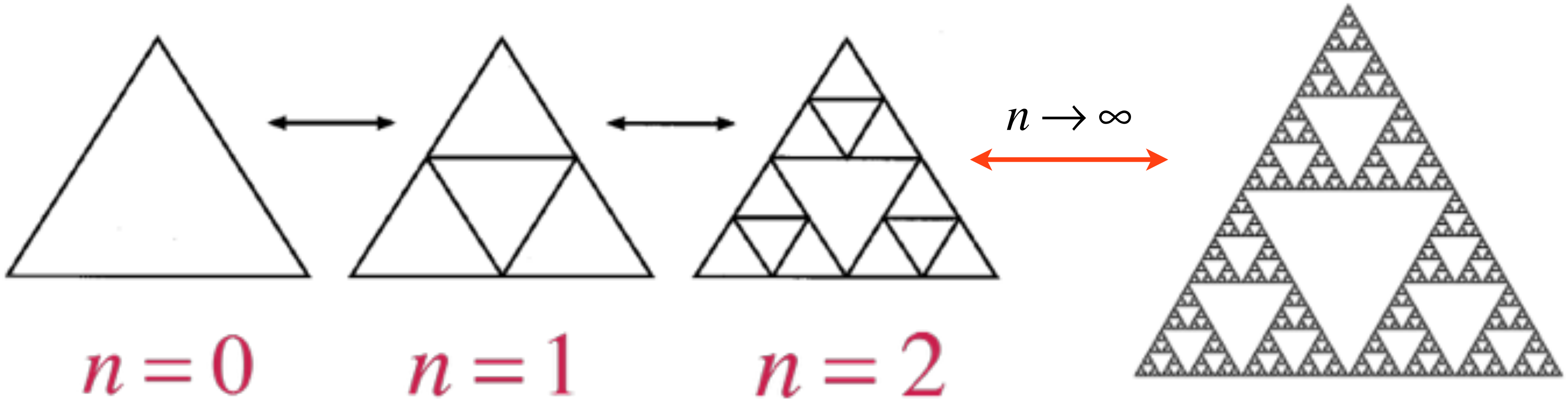
Discrete scale invariance (DSI)

discrete scale invariance is a weaker version of scale invariance, *i.e.*,

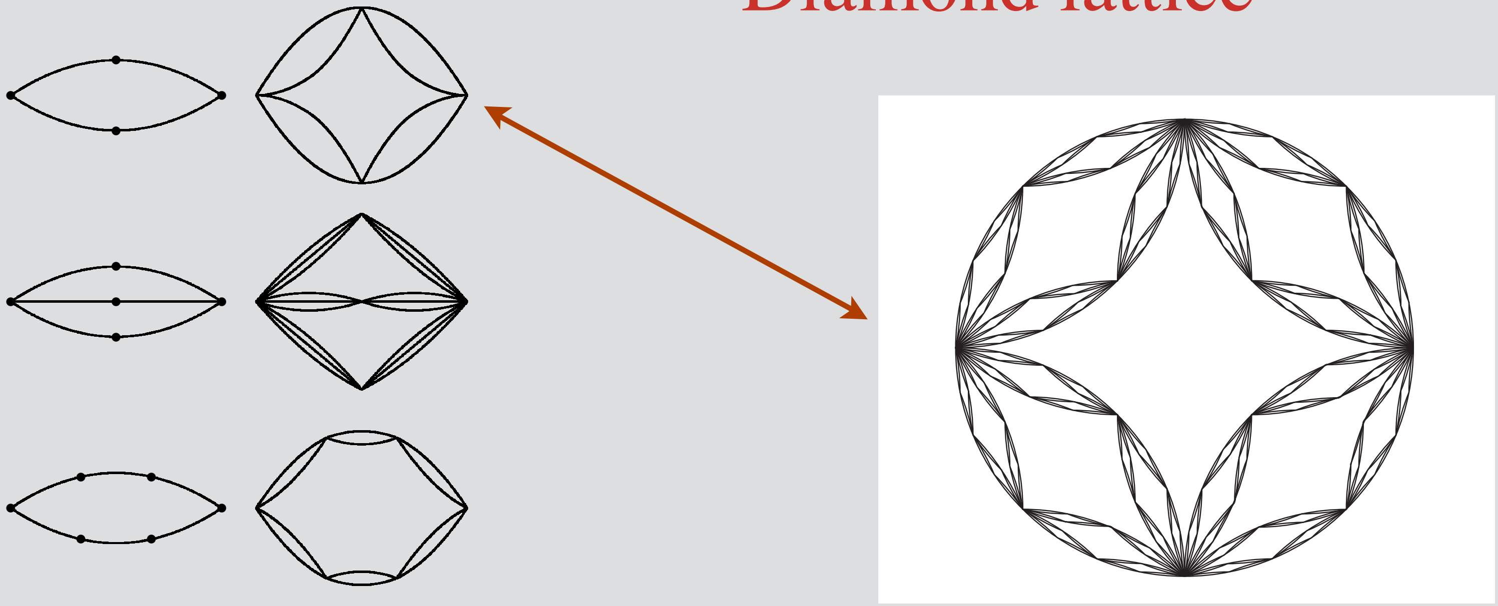
$$f(ax) = b\,f(x),\ \boxed{\mathit{with\ fixed}\,(a,b)}$$

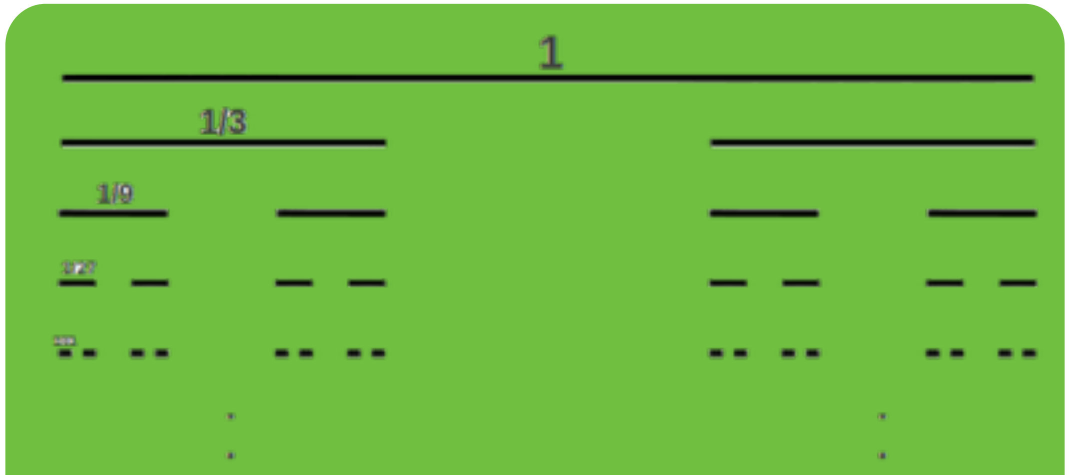
Iterative lattice structures (fractals)



Sierpinski gasket

Diamond lattice

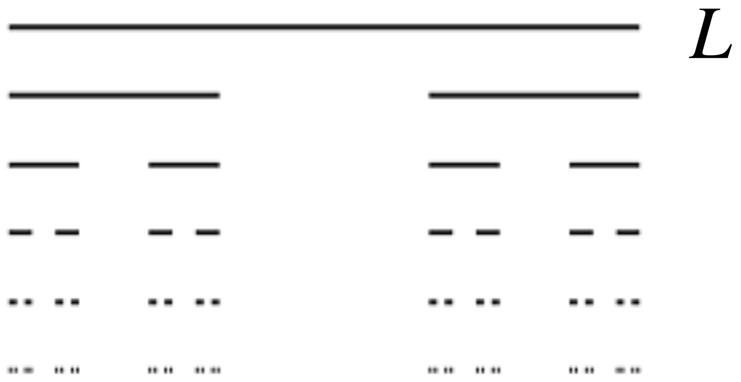




Triadic Cantor set

Fractals are self-similar objects

Cantor set



$$M_n = 2^n M$$

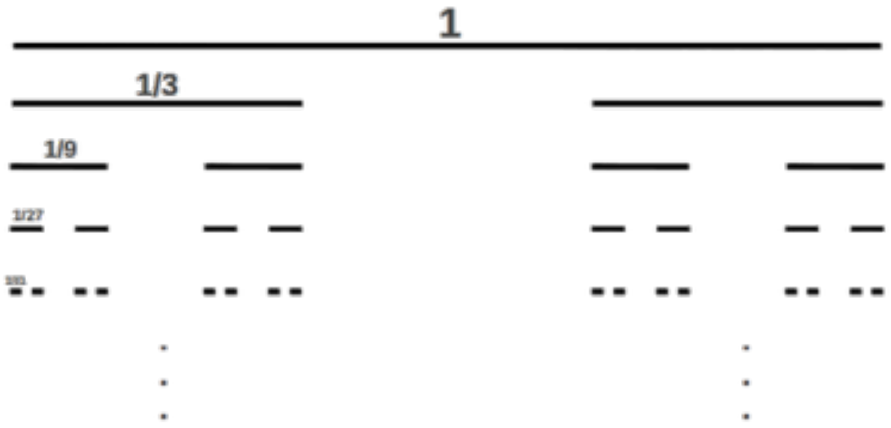
$$L_n = 3^n L$$

$$\frac{\ln M_n}{\ln L_n} \xrightarrow[n\to\infty]{} d_h = \frac{\ln 2}{\ln 3}$$

Alternatively, define the mass density $m(L)$ of the Cantor set

$$2m(L) = m(3L)$$

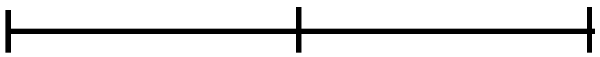
Relation between the different cases :



$$m(3L) = 2m(L)$$

$$(a,b) = (3,2)$$

Cantor set



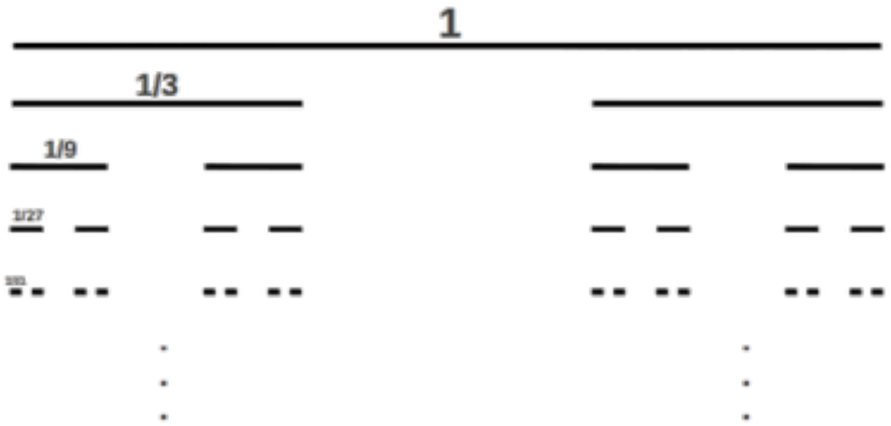
$$d = 1$$

$$m(2L) = 2\ m(L)$$

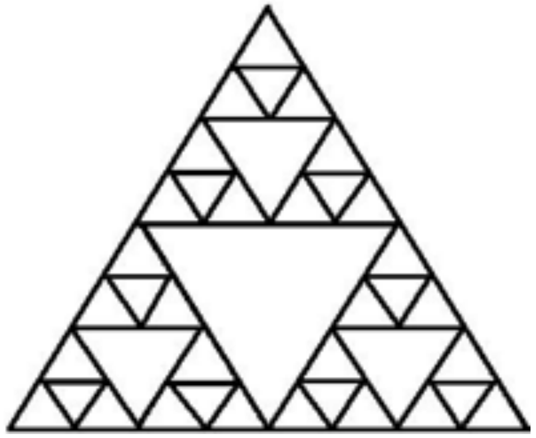
$$\forall\, b(a) \in \mathbb{R}$$

Euclidean lattice

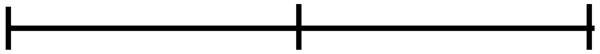
Relation between the two cases : discrete *vs*. continuous



$m(3L)=2m(L)$ $(a,b)=(3,2)$

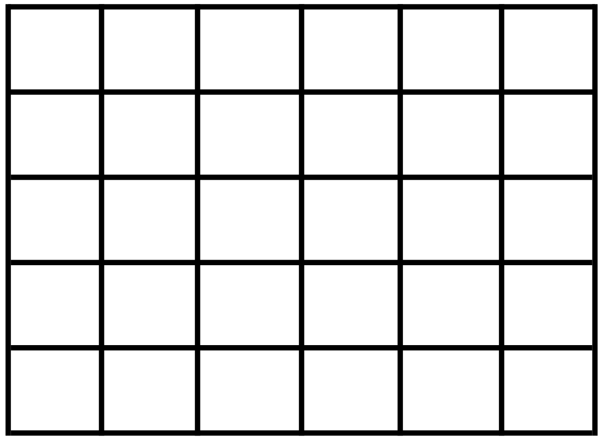


$m(2L)=3m(L)$ $(a,b)=(2,3)$



$d=1$

$m(2L)=2\ m(L)$ $\forall\, b(a)\in\mathbb{R}$



$d=2$

Both satisfy $f(a\,x)=b\,f(x)$ but with fixed (a,b)
for the fractals.

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f\left(a\,x\right)=b\,f\left(x\right)$$

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f(ax) = b\,f(x)$$

If satisfied $\forall\, b(a) \in \mathbb{R}$ (CSI),

General solution :

$$f(x) = C\,x^{\alpha}$$

with $\alpha = \dfrac{\ln b}{\ln a}$

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If satisfied *with fixed* (a,b) (DSI),

General solution:

$$f(x) = x^{\alpha}\,G\left(\frac{\ln x}{\ln a}\right)$$

where $G(u+1) = G(u)$ is a periodic function of period unity

Complex fractal exponents and oscillations

For a discrete scale invariance, $f(x) = x^{\alpha} G\left(\frac{\ln x}{\ln a}\right)$

and $G(u+1) = G(u)$ is a periodic function of period unity

Fourier expansion: $f(x) = \sum_{n=-\infty}^{\infty} c_n x^{\alpha + i\frac{2\pi n}{\ln a}}$

The scaling quantity $f(x)$ is characterised by an infinite set of complex valued exponents,

$$d_n = \alpha + i\frac{2\pi n}{\ln a}$$

Power laws with complex valued exponents are signature of discrete scale invariance (DSI)

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f\left(a\,x\right)=b\,f\left(x\right)$$

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General solution :

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Break CSI into DSI ?

If satisfied *with fixed* $\left(a,b\right)$ (DSI),

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Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f(ax) = b\,f(x)$$

If satisfied $\forall\, b(a) \in \mathbb{R}$ (CSI),

General solution (by direct inspection)

$$f(x) = C\,x^{\alpha}$$

If satisfied *with fixed* (a,b) (DSI),

General solution is

$$f(x) = x^{\alpha}\,G\left(\frac{\ln x}{\ln a}\right)$$

Break CSI into DSI ?

Claim : breaking of CSI into DSI occurs at the quantum level : quantum phase transition (scale anomaly)

A simple example of continuous scale invariance in quantum physics

An illustration of continuous scale invariance in (simple) quantum mechanics

Schr$\ddot{o}$dinger equation for a particle of mass μ in d-dimensions in an attractive potential :

$$V\left(r\right)=-\frac{\xi}{r^2}$$

An illustration of continuous scale invariance in (simple) quantum mechanics

Schrödinger equation for a particle of mass μ in d-dimensions in an attractive potential :

$$V(r) = -\frac{\xi}{r^2}$$

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

Redefining $k^2 = -2\mu E$

$$\zeta = 2\mu\xi - l(l+d-2)$$

$$\psi''(r) + \frac{d-1}{r}\psi'(r) + \frac{\zeta}{r^2}\psi(r) = k^2\psi(r)$$

orbital angular momentum

$$\boxed{\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)}$$

The only parameter ζ in the problem is dimensionless : no characteristic length (energy) scale, $e.g.$ Bohr radius $a_0 = \hbar^2/\mu e^2$ for the Coulomb potential.

Radial Schrödinger eq.

$$\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)$$

The only parameter $\zeta=2\mu\xi-l(l+d-2)$ is dimensionless : no characteristic length (energy) scale.

Consequence: Schrödinger eq. displays continuous scale invariance : it is invariant under:

$$\begin{cases} r\rightarrow\lambda r \\ k\rightarrow\frac{1}{\lambda}k \end{cases} \qquad \forall\lambda\in\mathbb{R}$$

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To every normalisable wave function $\psi(r,k)$ corresponds a family of wave functions $\psi(\lambda r,\, k/\lambda)$ of energy $(\lambda k)^2$

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The existence of one bound state implies those of a continuum of related bound states. No ground state. Problem !

It is a problem, but a well known (textbook) one.

It results essentially from :

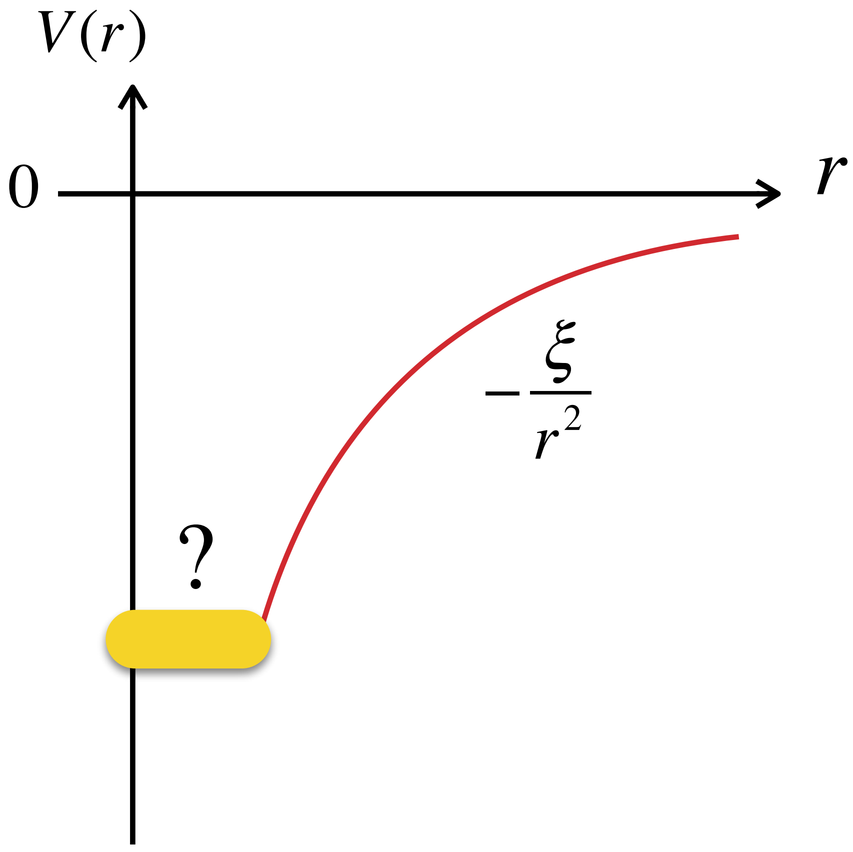
- the ill-defined behaviour of the potential $V(r) = -\frac{\xi}{r^2}$ for $r \rightarrow 0$
- the absence of characteristic length/energy.

Technically : non hermitian (self-adjoint) Hamiltonian.
To cure it : need to properly define boundary conditions (somewhere)

$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$ is scale invariant (CSI) : $r \to \lambda r \Rightarrow \hat{H} \to \frac{1}{\lambda^2}\hat{H}$

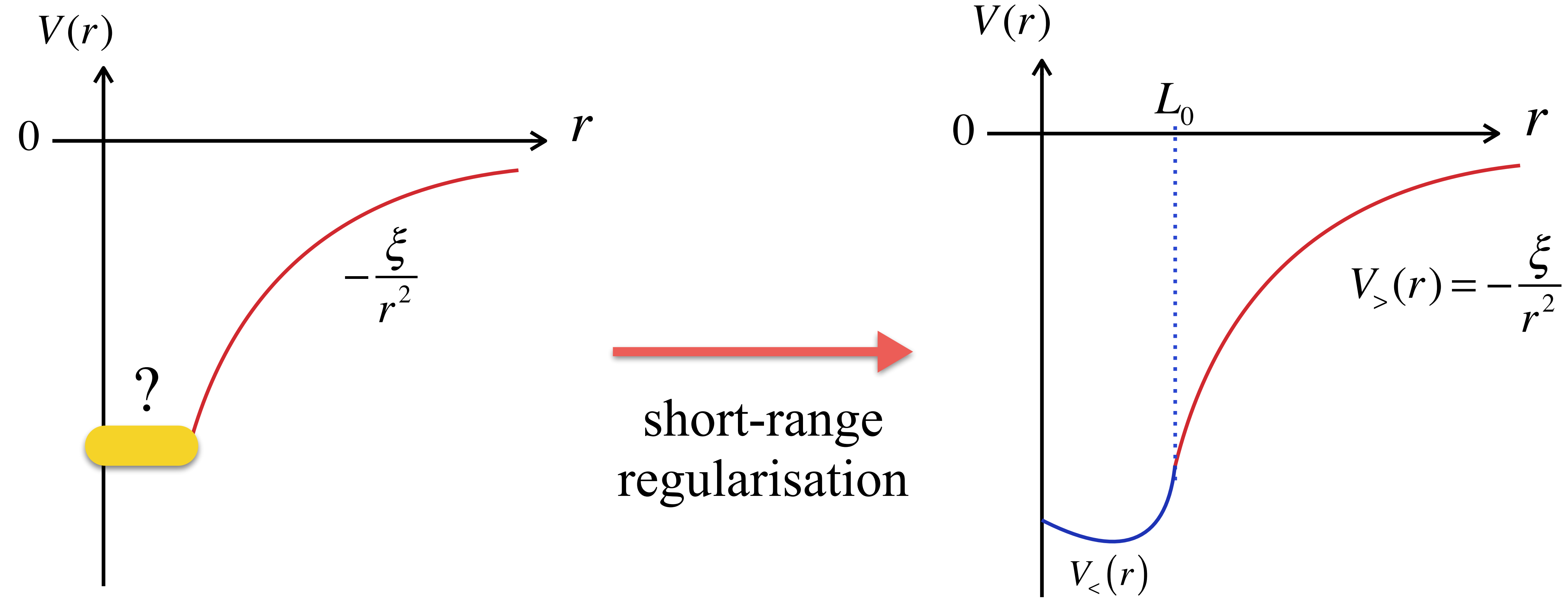
Any type of boundary conditions needed to find a well defined hermitian Hamiltonian break CSI.

Outline of the main results



No characteristic scale

Outline of the main results



short-range regularisation

No characteristic scale

Some potential $V_<(r)$: accounts for “real” short-range physics.

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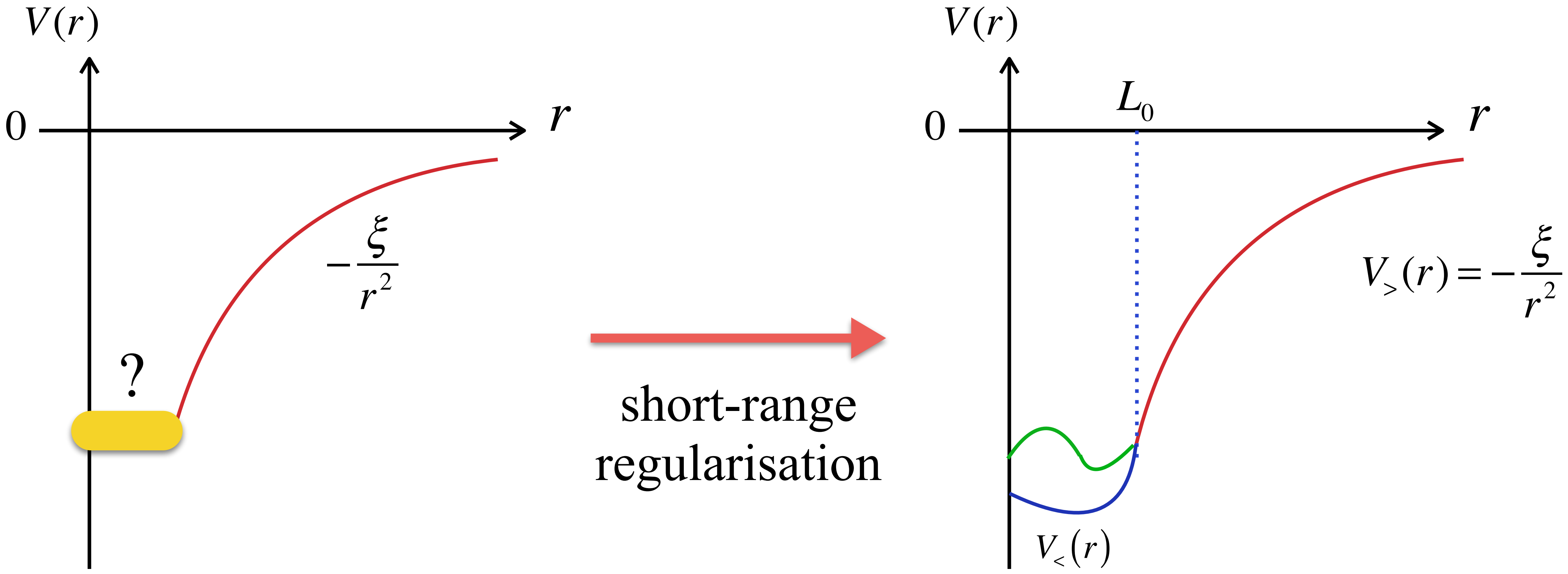

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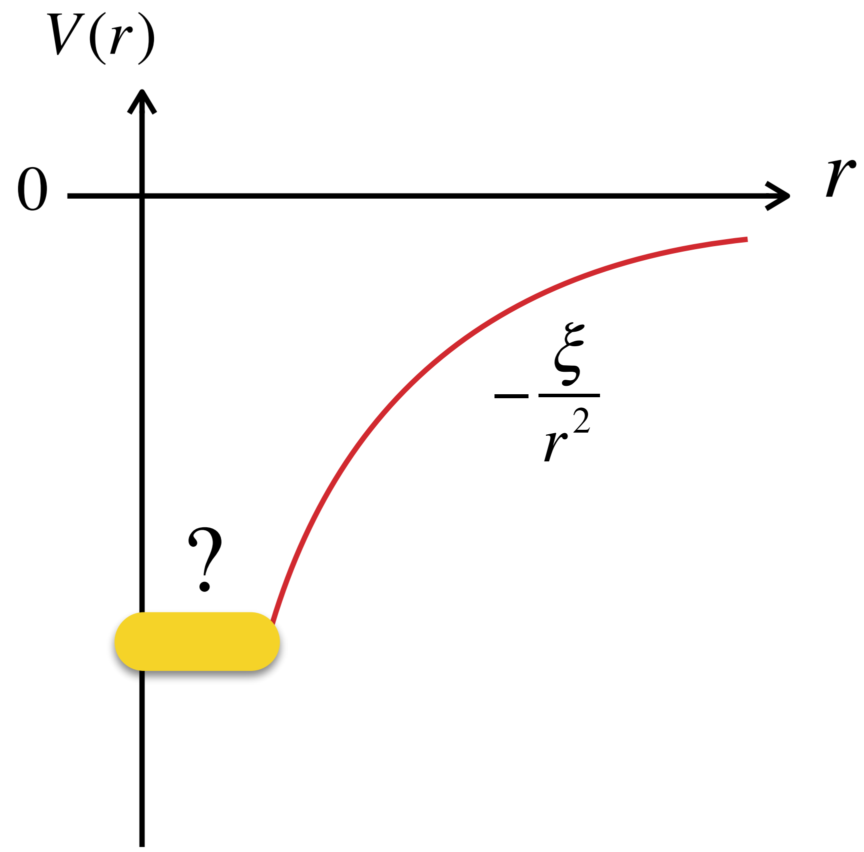
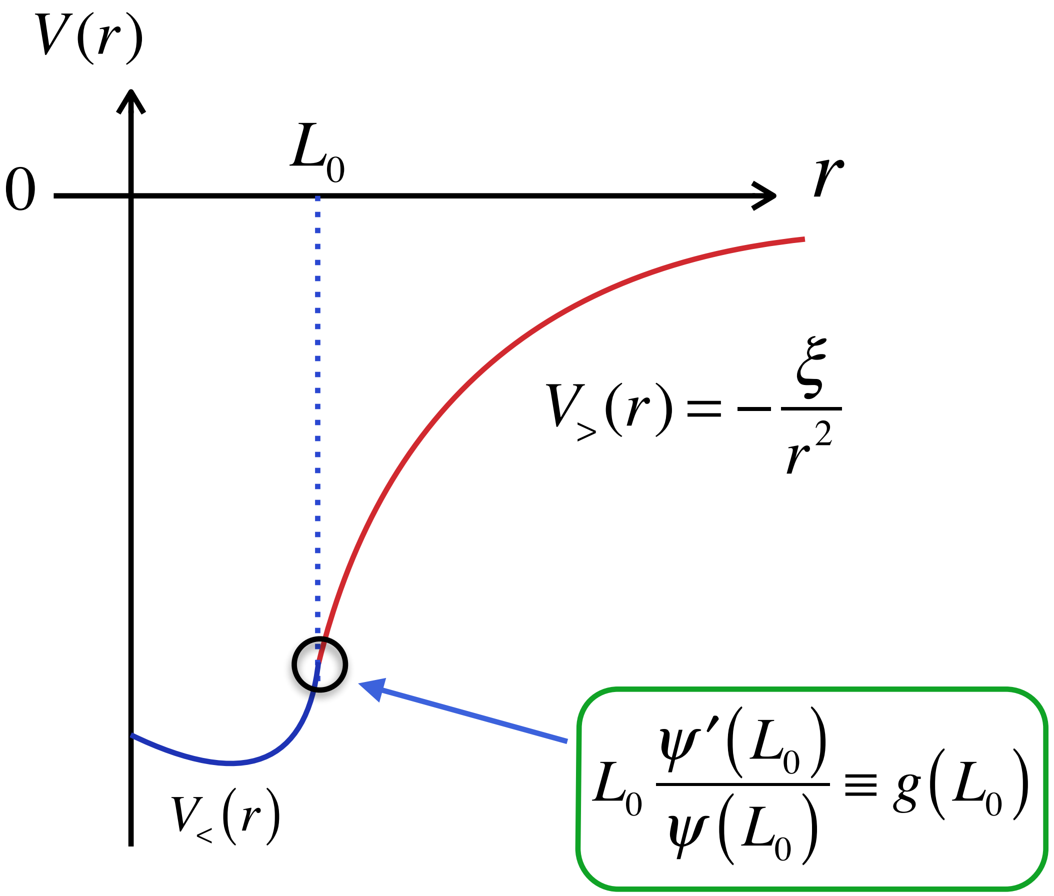
Exact expression is not important .

Outline of the main results





$$L_0 \frac{\psi'(L_0)}{\psi(L_0)} \equiv g(L_0)$$

Problem becomes well-defined :

- characteristic length L_0
- continuity of ψ and ψ' at L_0 (boundary condition)

$\Rightarrow$ energy spectrum

How the energy spectrum looks like ?

At low enough energies $\left(E \simeq 0\right)$,the spectrum has a “universal” behaviour.

- It depends on the parameter $\zeta = 2\mu\xi - l\left(l + d - 2\right)$
- It exists a singular value $\boxed{\zeta_{cr} = \frac{\left(d-2\right)^2}{4}}$

Universal part of the energy spectrum

It depends on the parameter $\zeta = 2\mu\xi - l(l+d-2)$

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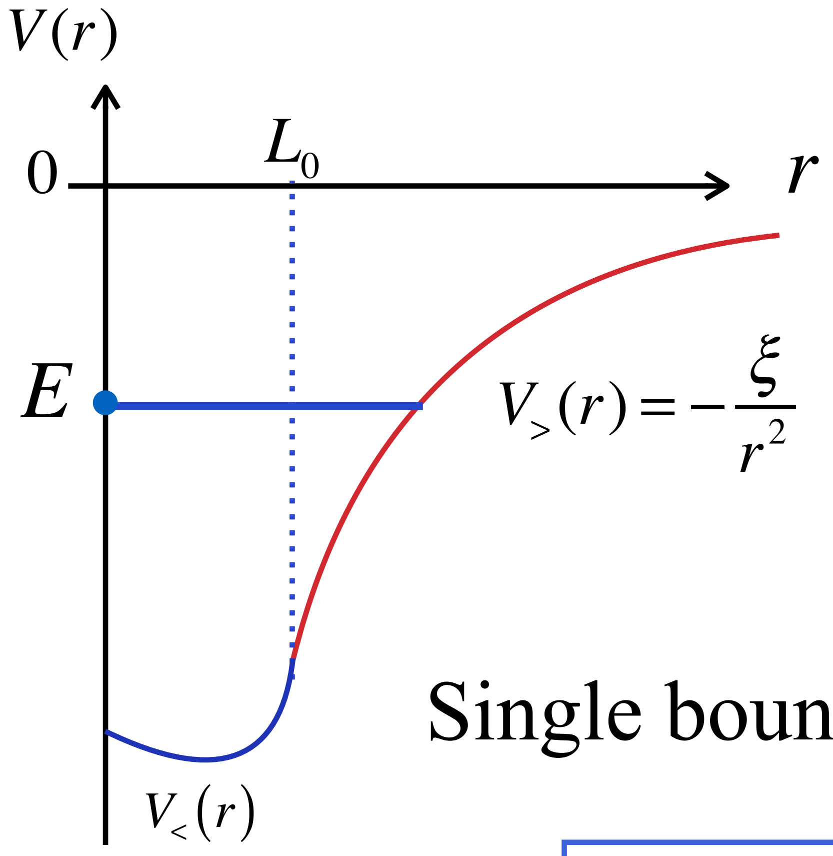
$\zeta < \zeta_{cr}$



Single bound state

$$\boxed{E = -\frac{1}{L_0^2} f(g)}$$

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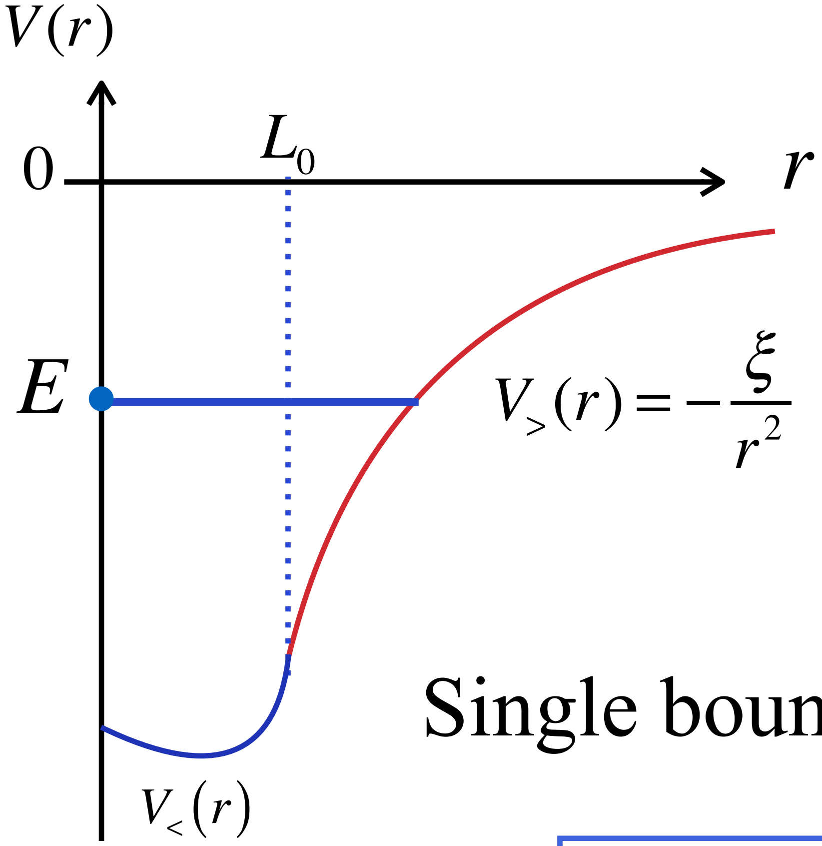

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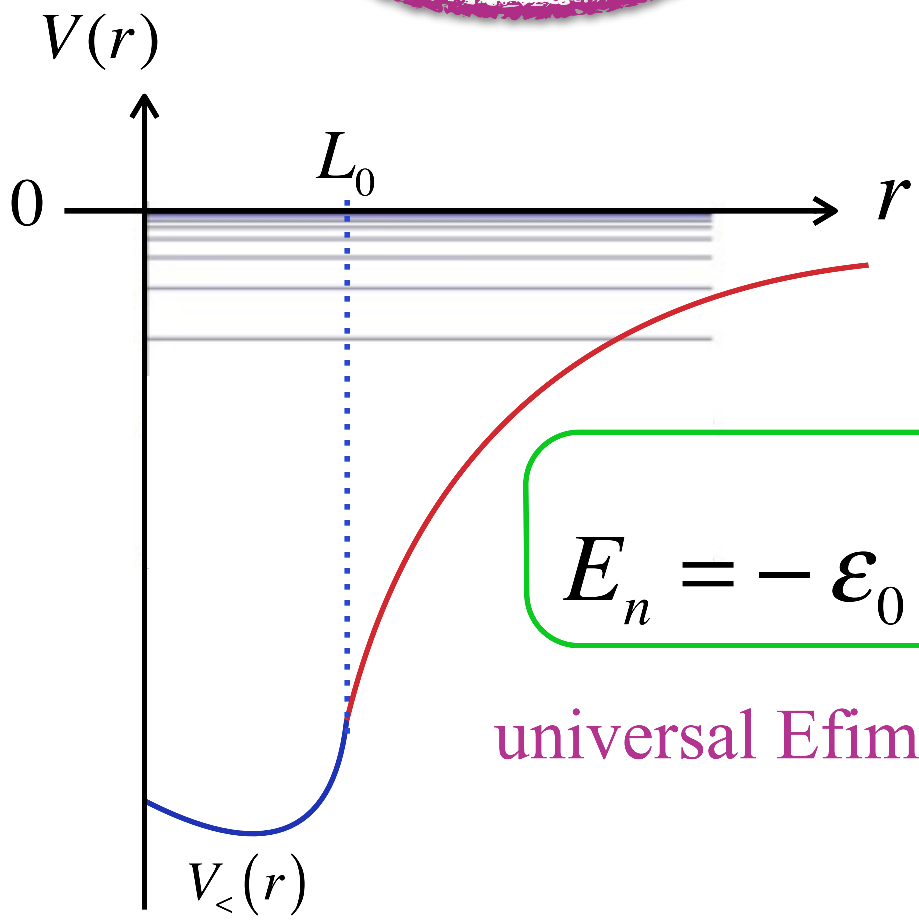

$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

universal Efimov spectrum

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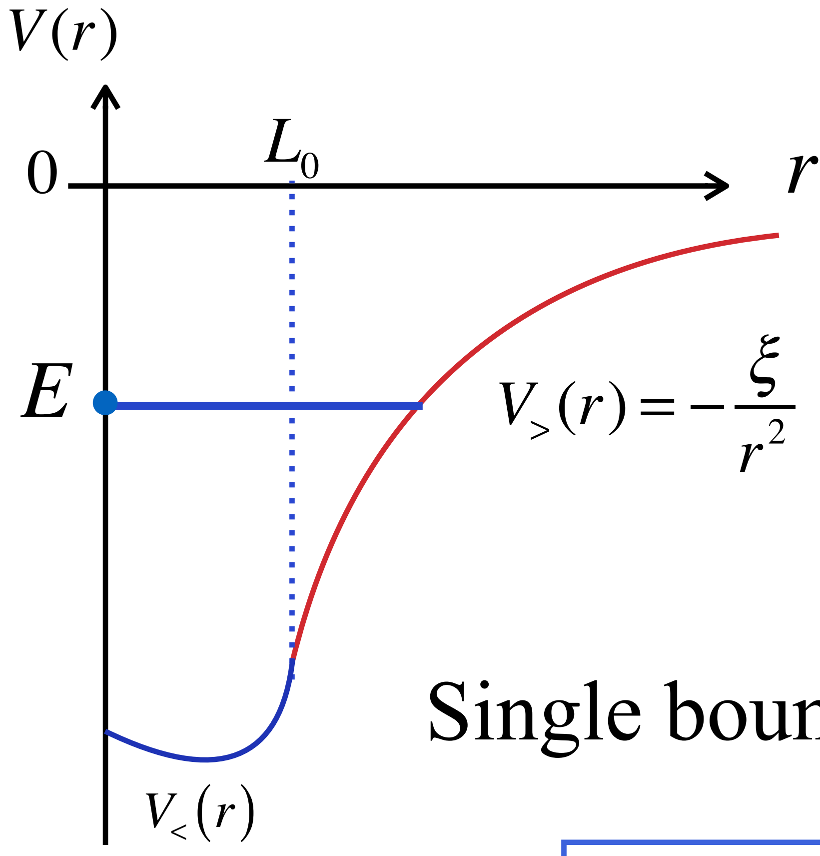

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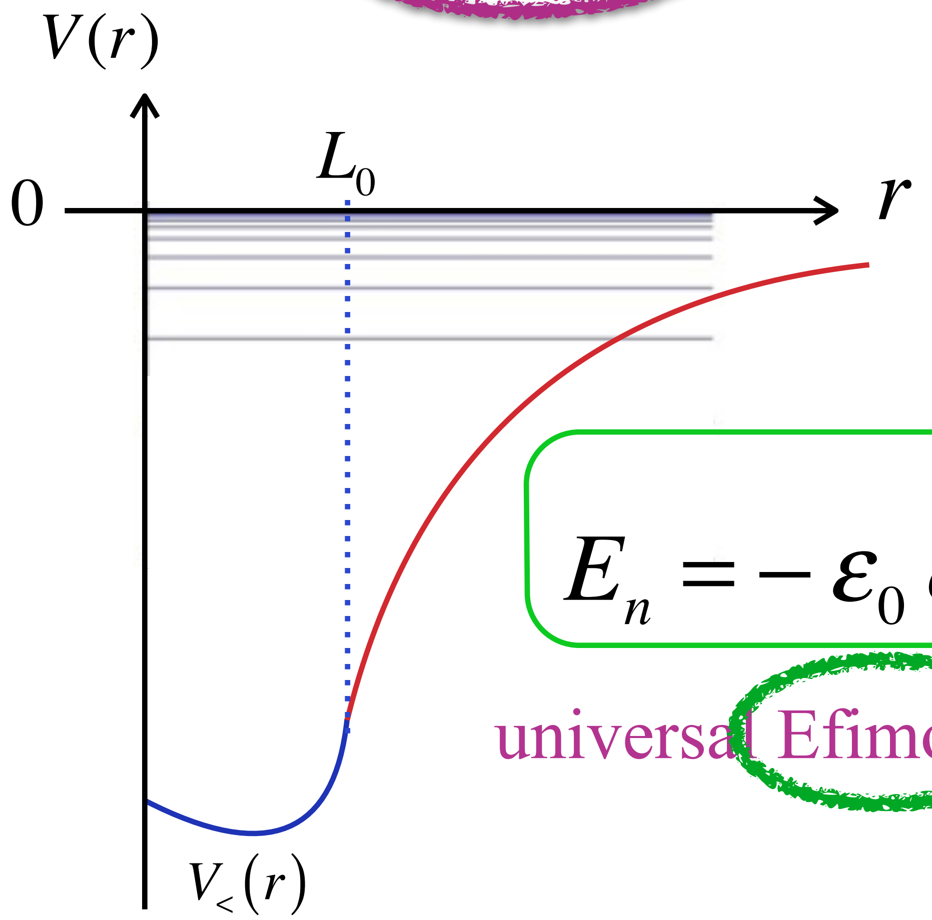

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Just a name for the moment

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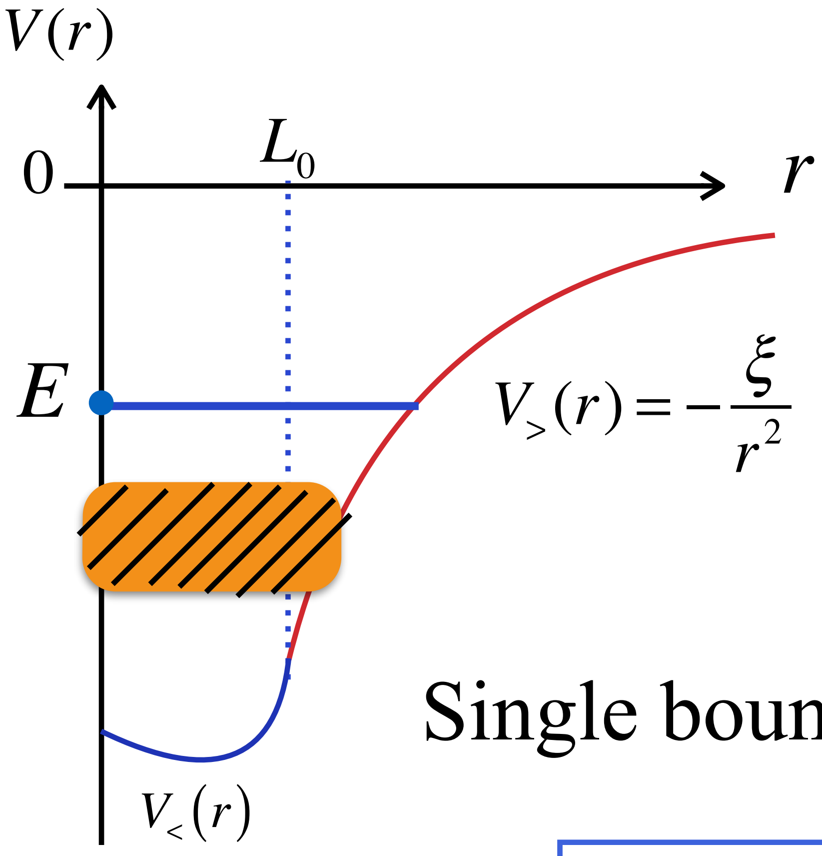

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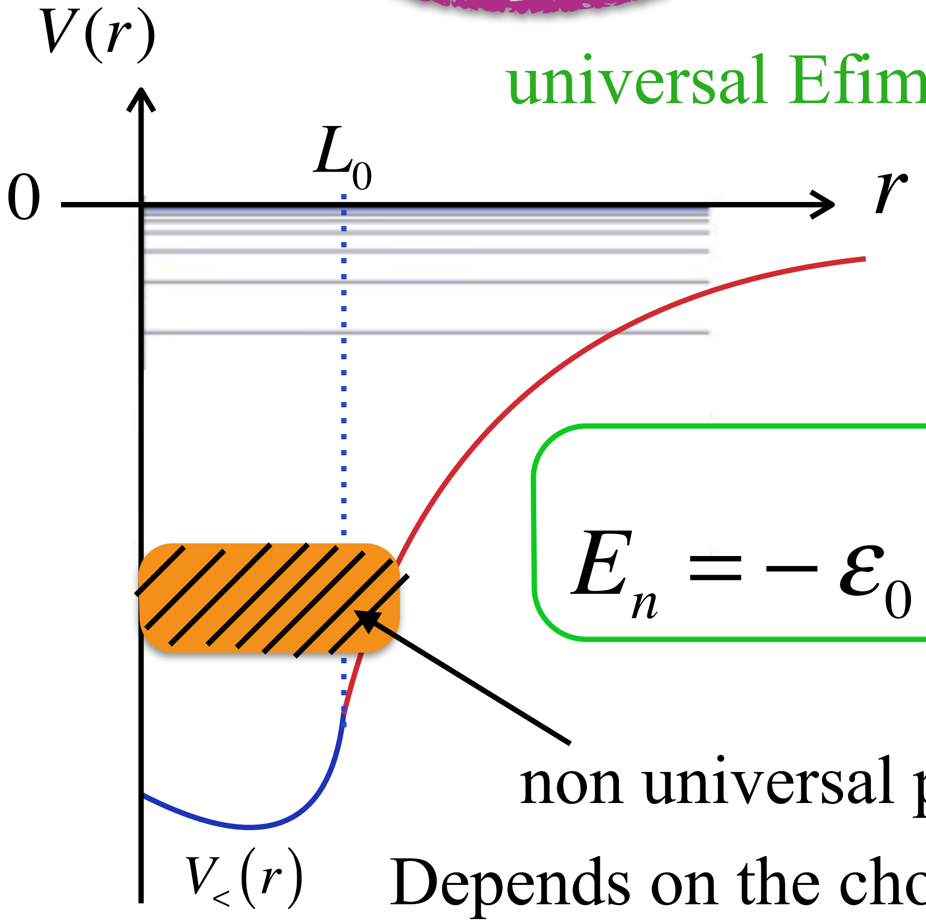


universal Efimov spectrum

$$\boxed{E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

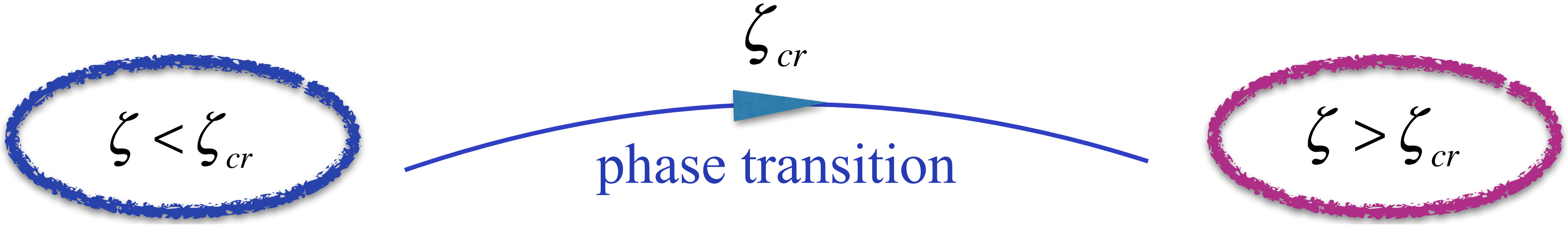
non universal part

Depends on the choice of $V_<(r)$

A quantum phase transition

It exists a singular value $\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$

Take the limit $L_0 \to \infty$
with EL_0^2 fixed



ζ_{cr}

$\zeta < \zeta_{cr}$

phase transition

$\zeta > \zeta_{cr}$

A quantum phase transition

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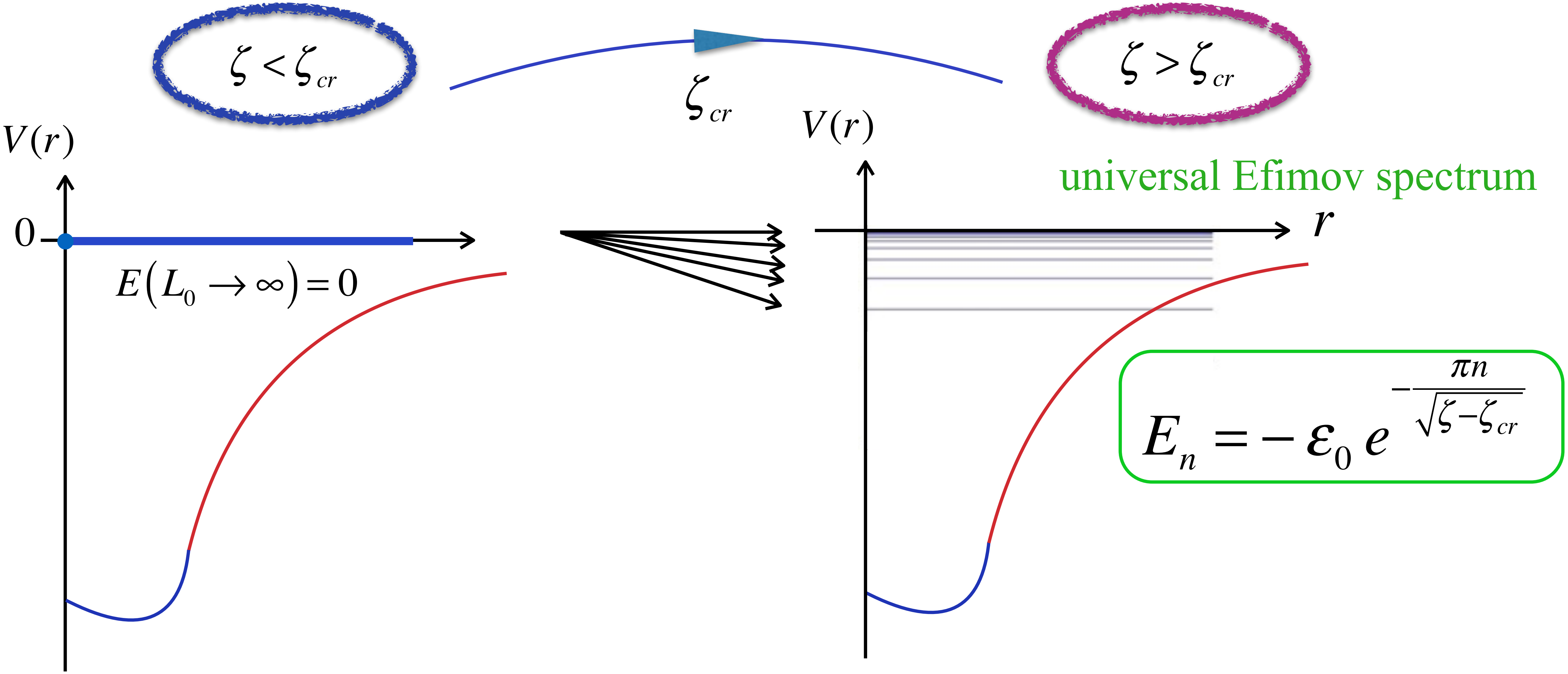
Take the limit $L_0 \to \infty$



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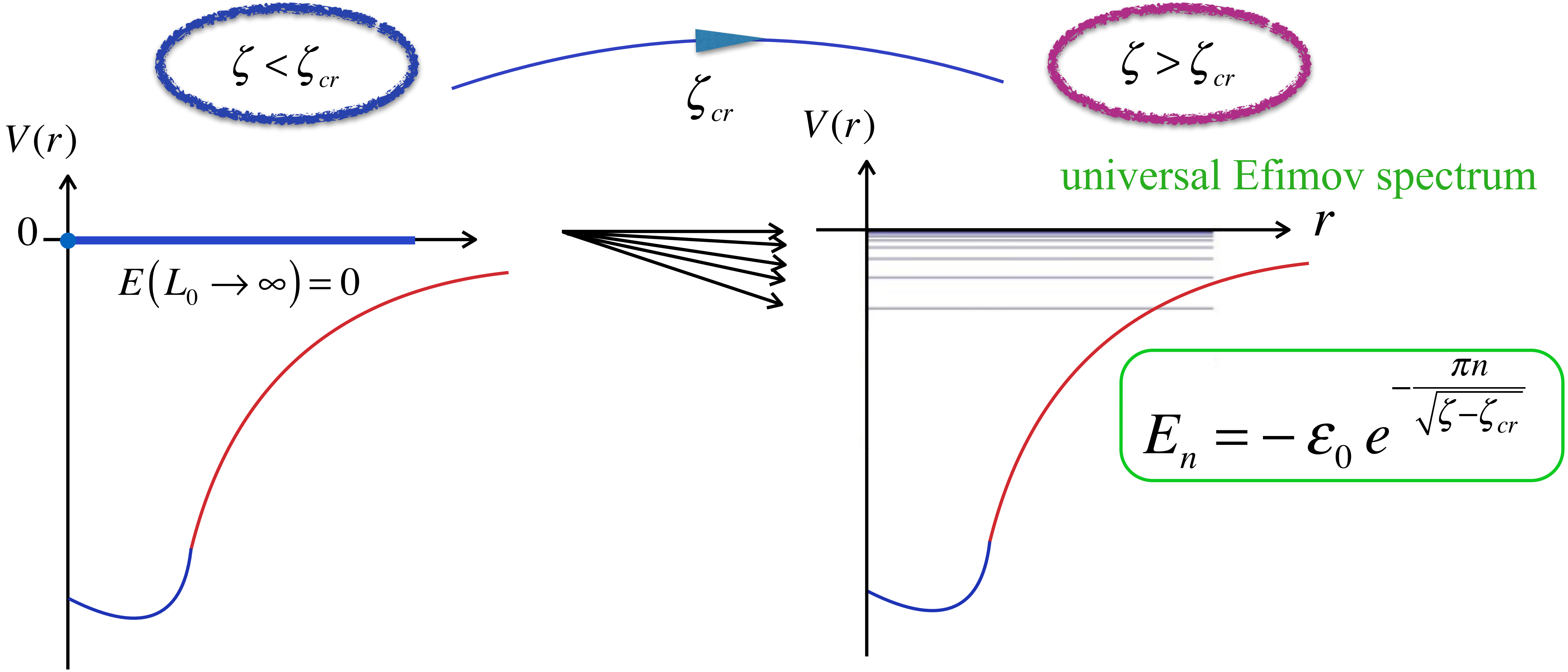

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continuous scale invariance (CSI)

but trivial : $\lambda E = 0 \quad \forall \lambda$

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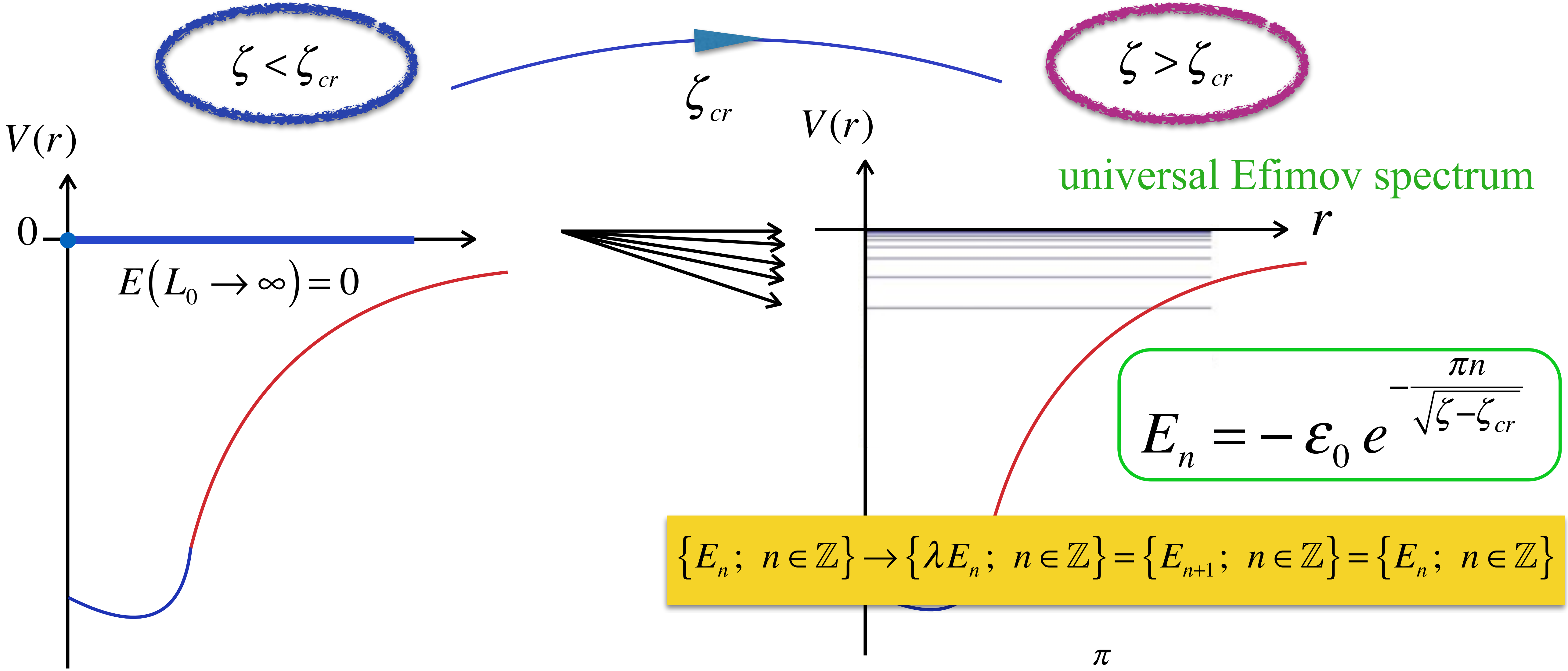

universal Efimov spectrum

$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

$$\{E_n;\ n \in \mathbb{Z}\} \to \{\lambda E_n;\ n \in \mathbb{Z}\} = \{E_{n+1};\ n \in \mathbb{Z}\} = \{E_n;\ n \in \mathbb{Z}\}$$

continuous scale invariance (CSI)

but trivial : $\lambda E = 0 \quad \forall \lambda$

$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$ is fixed :

discrete scale invariance (DSI)

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0 e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}} \equiv -\varepsilon_0 \lambda^n$$

Non universal energy parameter

Universal Efimov energy spectrum

$$\boxed{E_n = -\varepsilon_0 e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}} \equiv -\varepsilon_0 \lambda^n}$$

- The Efimov spectrum is invariant under a discrete scaling *w.r.t.* the parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$$

where $\zeta = 2\mu\xi - l(l + d - 2)$

$$\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$$

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\Lambda}} \equiv -\varepsilon_0 \, \lambda^n$$

- The Efimov spectrum is invariant under a discrete scaling *w.r.t.* the parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$$

where $\zeta = 2\mu\xi - l(l+d-2)$

- Density of states $\rho(E) = \sum_{n\in\mathbb{Z}} \delta(E - E_n)$

$$\rho\left(\lambda^2 E\right) = 2\mu \sum_{n\in\mathbb{Z}} \delta\left(\lambda^2 k^2 - k_n^2\right) = \cdots = \lambda^{-2} \rho(E)$$

so that $\rho(E) = \frac{1}{E} G\left(\frac{\ln E}{\ln \lambda}\right)$ where $G(u+1) = G(u)$

Universal Efimov energy spectrum

$$E_n = -\varepsilon_0\, e^{-}$$

- The Efimov spectrum is invaria
parameter :

$$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta-\zeta_{cr}}}}$$

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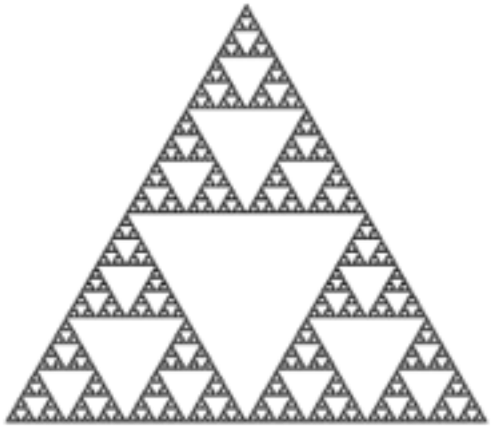

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Dirac equation + Coulomb

Dirac equation + Coulomb potential

Continuous scale invariance (CSI) of the Hamiltonian :

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

A immediate question : What about the Dirac eq. with a Coulomb potential ?

Dirac eq. $$i\sum_{\mu=0}^{d}\gamma^{\mu}\left(\partial_{\mu}+ieA_{\mu}\right)\Psi\left(x^{\nu}\right)=0$$ is linear with momentum and

Coulomb potential $$e A_0 = V(r) = -\frac{\xi}{r}, \quad \xi \equiv Z\alpha$$ $$A_i = 0, \quad i = 1,\ldots,d$$

fine
structure
constant

These two problems share the same continuous scale invariance (CSI).

Dirac equation + Coulomb potential

Continuous scale invariance (CSI) of the Hamiltonian :

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

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$$eA_0 = V(r) = -\frac{\xi}{r}, \quad \xi \equiv Z\alpha$$

$$A_i = 0, \quad i = 1,\ldots,d$$

$$\alpha = {e^2}/{\hbar c} \approx \frac{1}{137} \ll 1$$

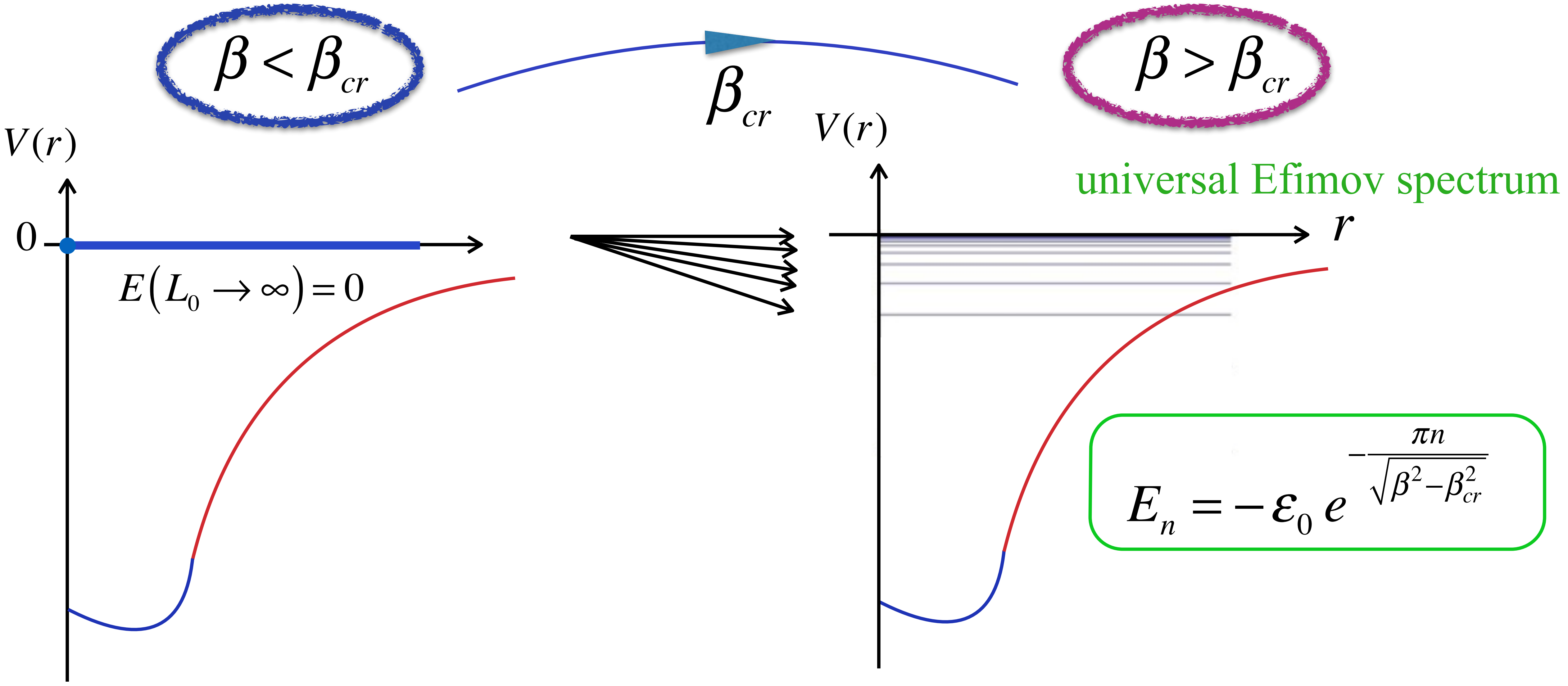
These two problems share the same continuous scale invariance (CSI).

The instability in the Dirac + Coulomb problem is an example of the breaking of CSI into DSI.

Dirac quantum phase transition

Dimensionless coupling $\beta \equiv Z\alpha$

Singular value $\boxed{\beta_{cr} = \frac{d-1}{2} = \frac{1}{2}}$



universal Efimov spectrum

$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\beta^2-\beta_{cr}^2}}}}$$

Continuous scale invariance (CSI)

Discrete scale invariance (DSI)

Problem : to observe this instability, we need $Z \geq 1/\alpha \simeq 137$

No such stable nuclei have been created.

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Idea: consider analogous condensed matter systems with a "much larger effective fine structure constant".

Graphene : Effective massless Dirac excitations with a Fermi velocity $v_F \simeq 10^6 \, m/s$ so that

$$\boxed{\alpha_G = e^2/\hbar v_F \approx 2.5}$$

and $\boxed{Z_c \geq 1/\alpha_G \simeq 0.4}$

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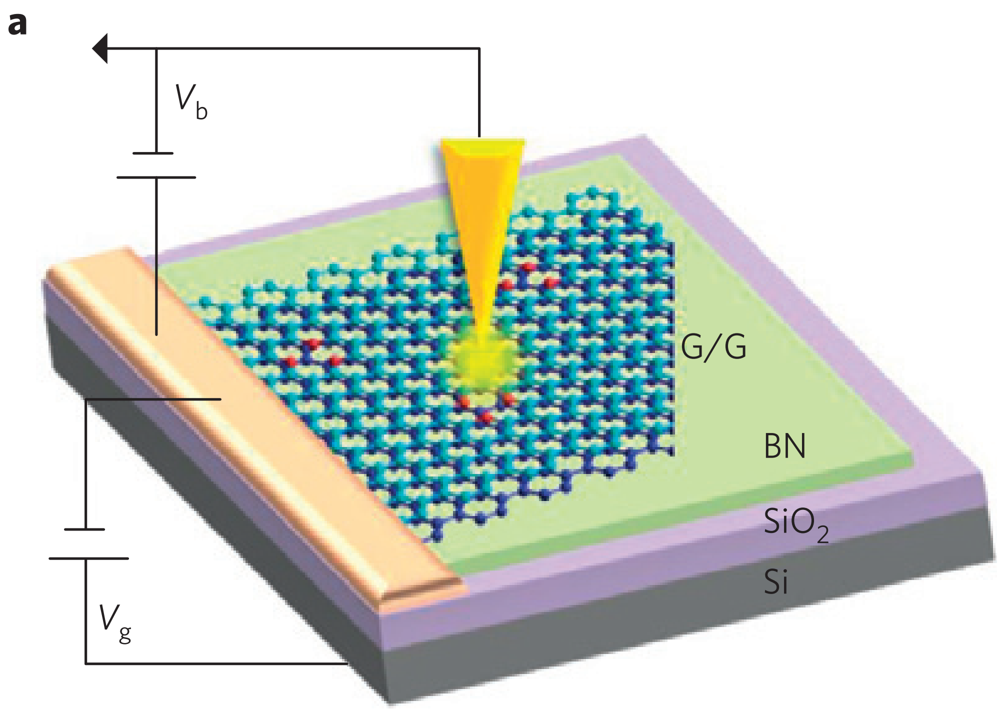
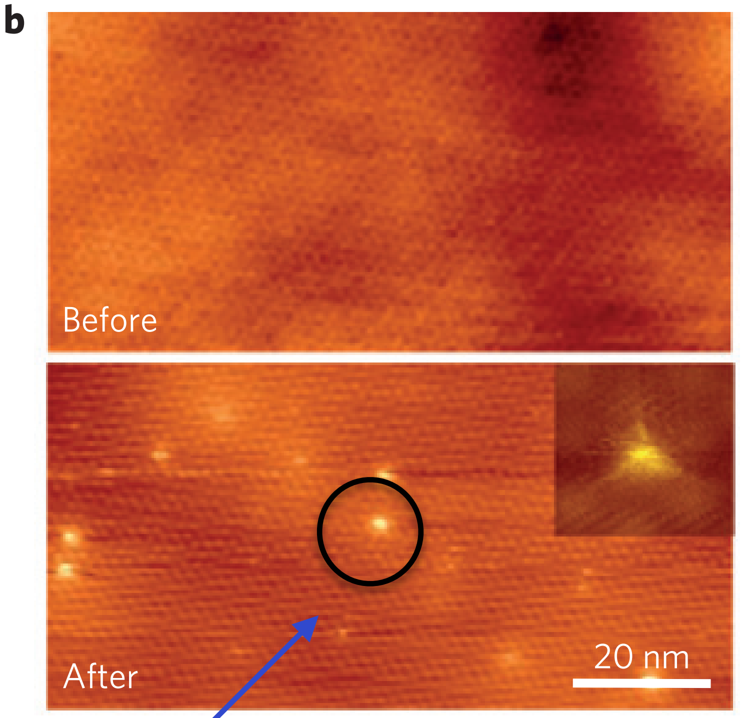
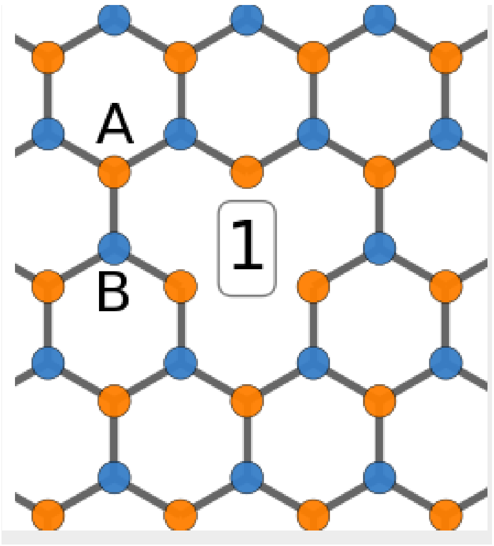
$$\boxed{\alpha = e^2/\hbar c \approx \frac{1}{137} \ll 1}$$

Dirac equation + Coulomb :
The experiment

Building an artificial atom in graphene





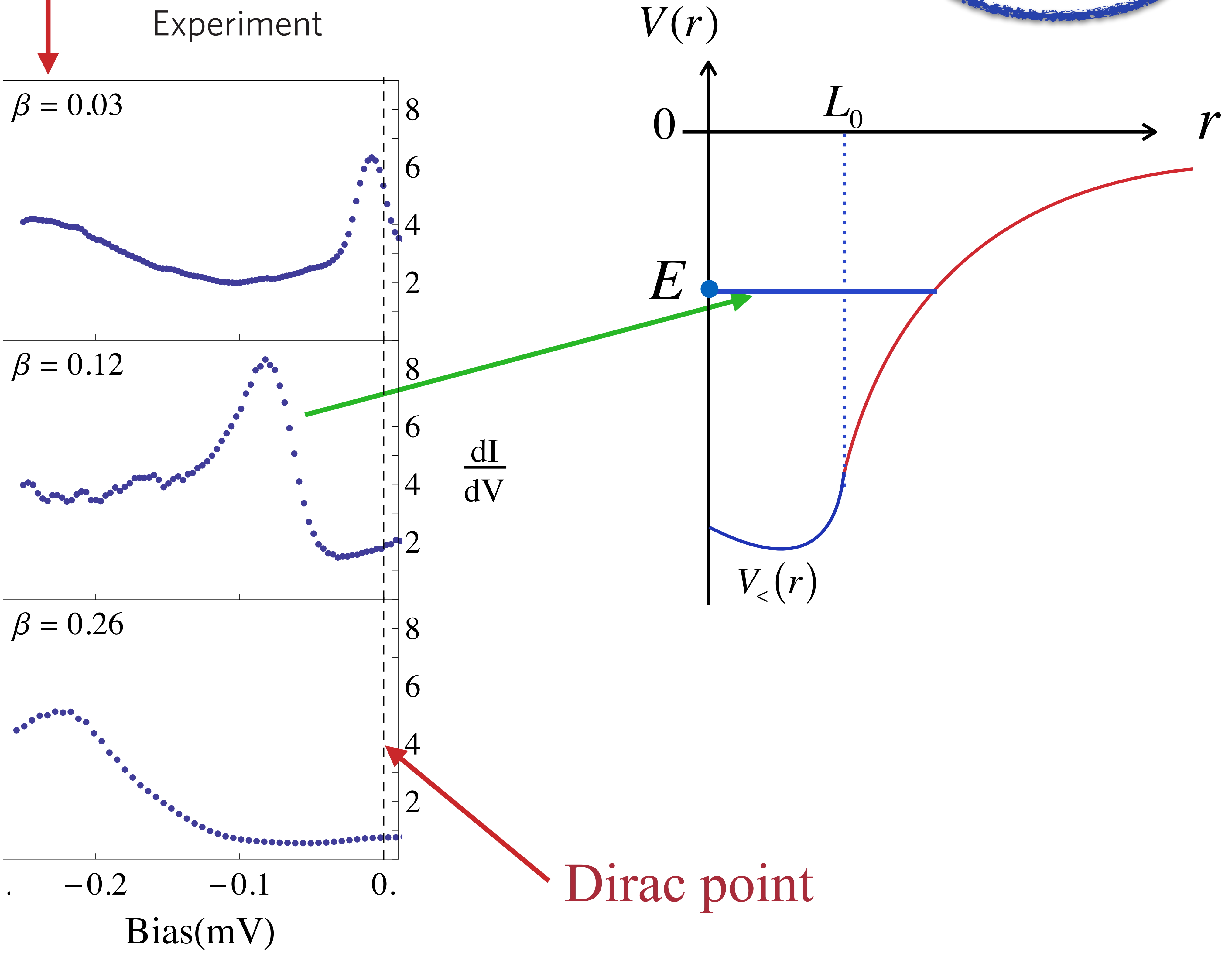


Local vacancy. Local charge is changed by applying voltage pulses with the tip of an STM

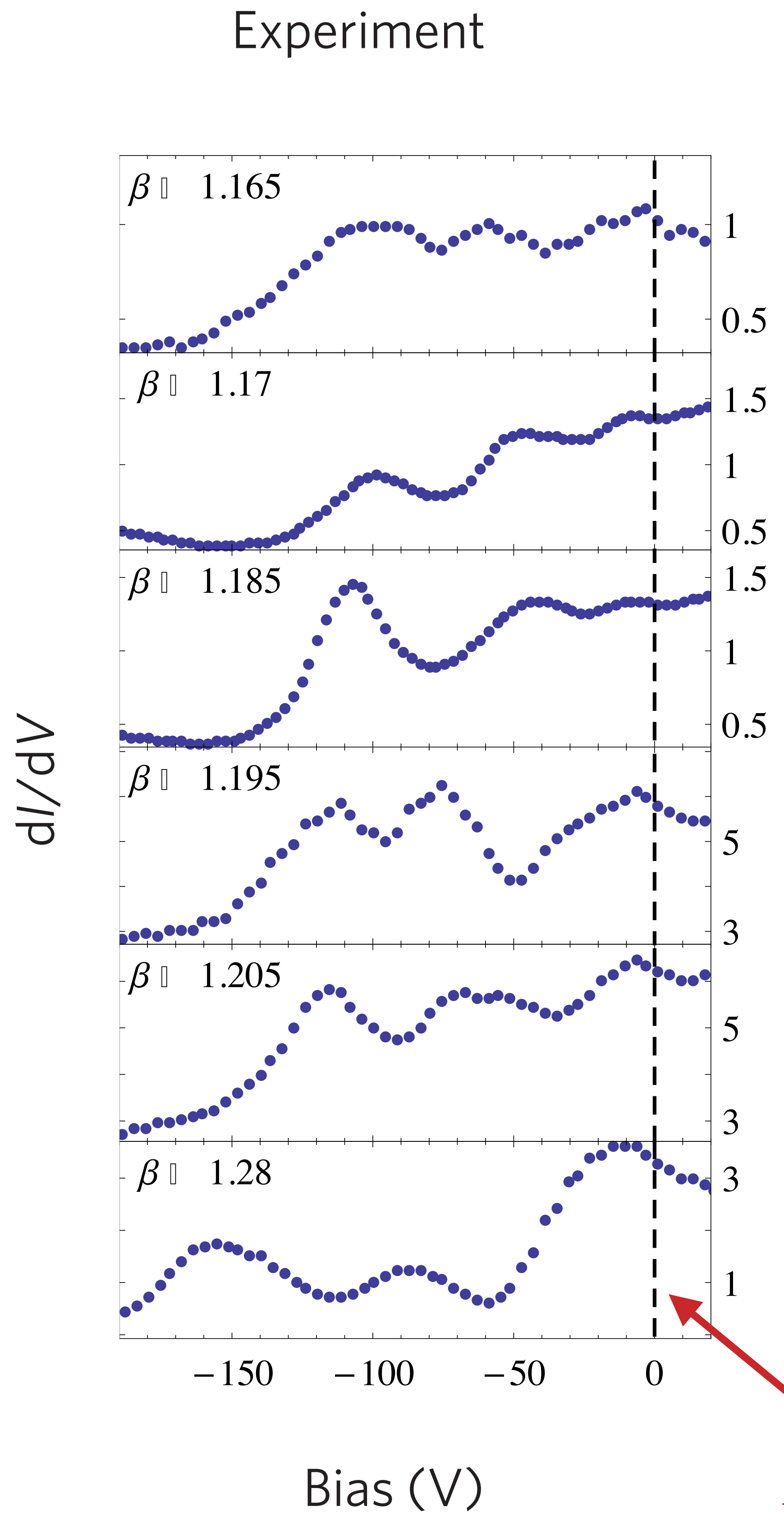
$$\beta < \beta_{cr} \equiv \frac{1}{2}$$

Increasing the charge, quasi-bound states are trapped.

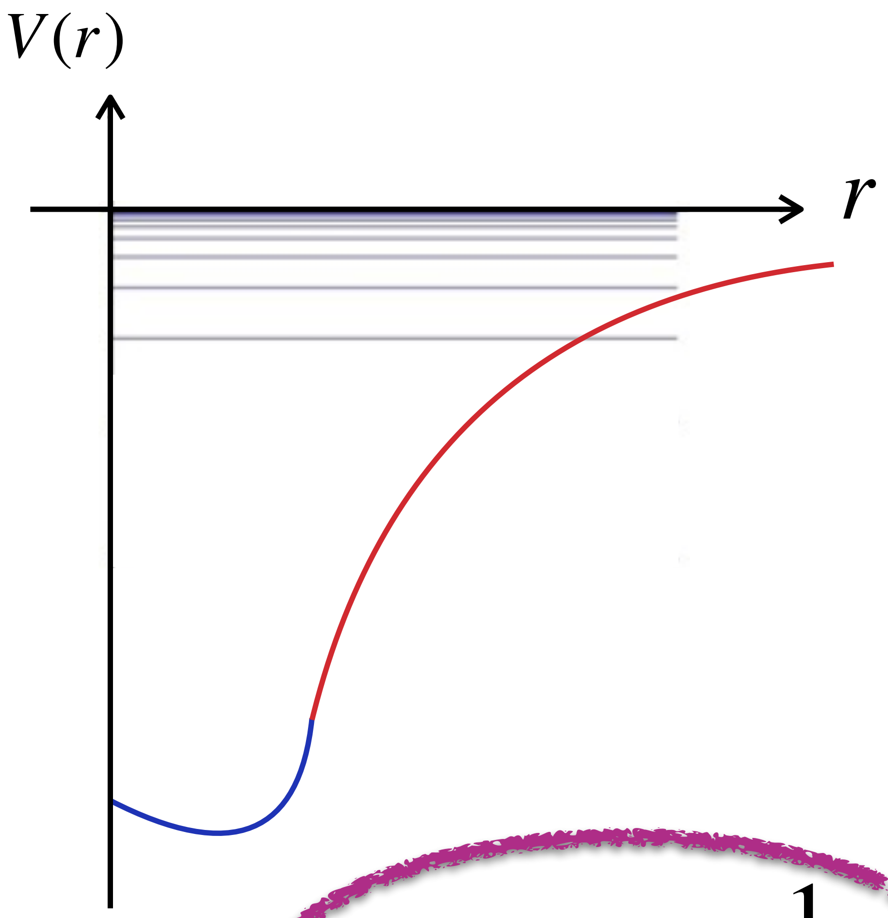
$$\beta < \beta_{cr}$$



Increasing the charge, quasi-bound states are trapped. For a large enough coupling, a discrete set of Efimov states shows up.



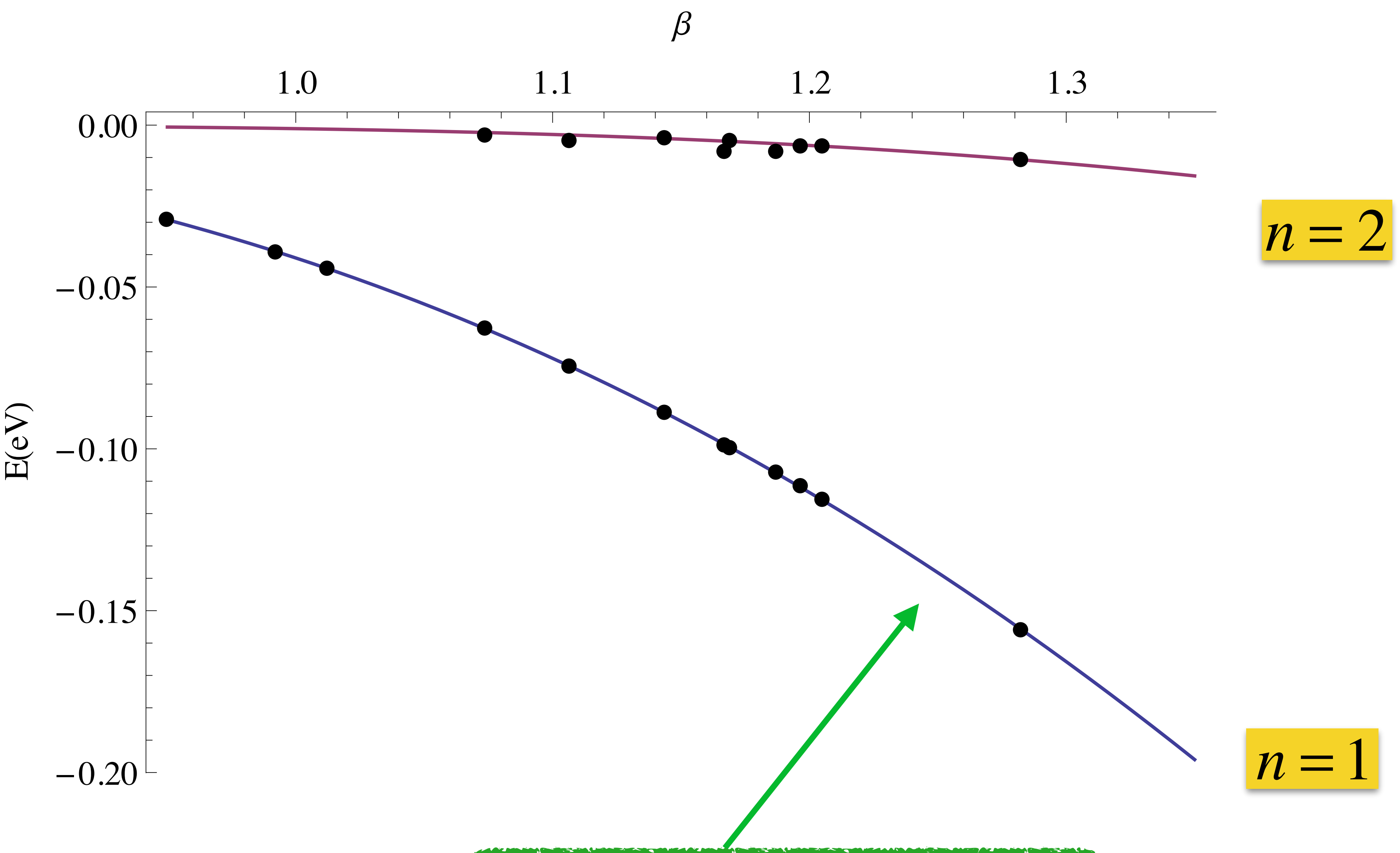
Dirac point



$$\beta > \beta_{cr} \equiv \frac{1}{2}$$

The Efimov universal spectrum

$$\beta > \beta_{cr} \equiv \frac{1}{2}$$



$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\beta^2-\beta_{cr}^2}}}$$

What is Efimov physics ?

Universality in cold atomic gases

DSI in the non relativistic quantum 3-body problem

Universality in cold atomic gases

non relativistic quantum 3-body problem

3-body (nucleon) system interacting through zero-range interactions (r_0)
Existence of universal physics at low energies, $E \ll \hbar^2 / m r_0^2$

When the scattering length a of the 2-body interaction becomes $a \gg r_0$
there is a sequence of 3-body bound states whose binding energies are
spaced geometrically in the interval between $\hbar^2 / m a^2$ and $\hbar^2 / m r_0^2$

As $|a|$ increases, new bound states appear according to

$$\boxed{E_n = -\varepsilon_0\, e^{-\frac{2\pi n}{s_0}}}$$

Efimov spectrum

where $s_0 \approx 1.00624$ is a universal number

The corresponding 3-body problem reduces to an <u>effective</u> Schr$\ddot{o}$dinger equation with the attractive potential :

$$V(r) = -\frac{s_0^2 + {}^1\!/_4}{r^2}$$

Efimov physics is always super-critical :

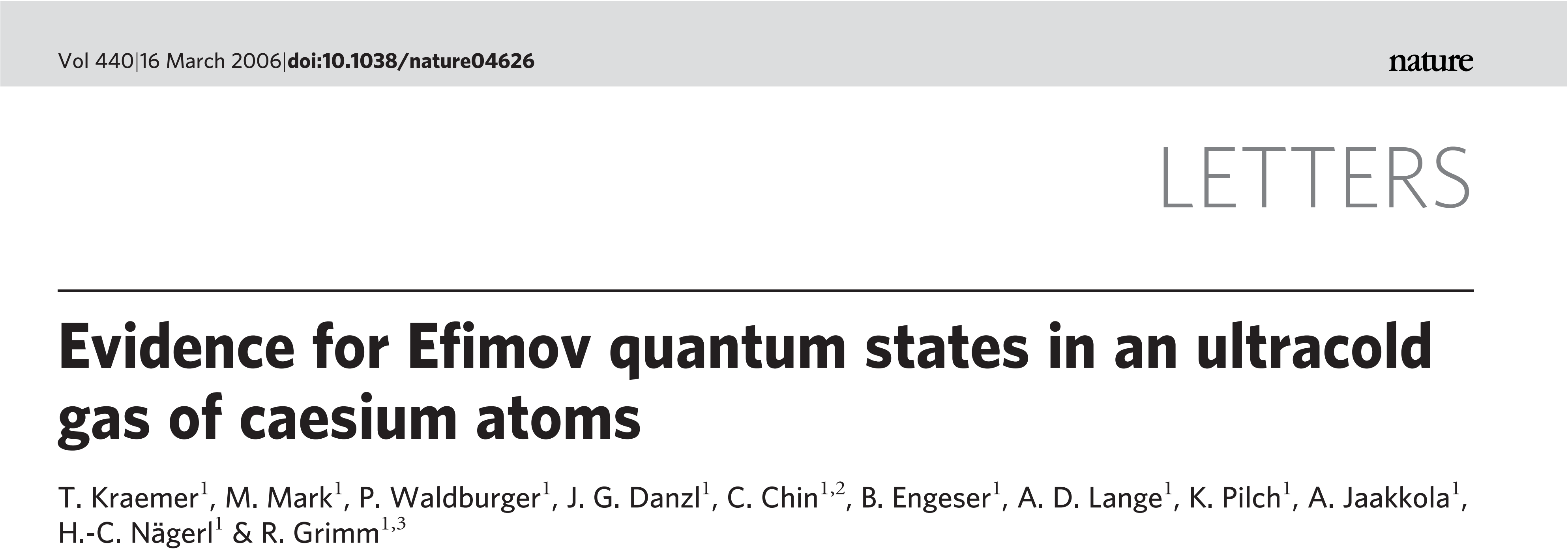
Schrodinger equation with an effective attractive potential $(d=3)$:

$$V(r) = -\frac{s_0^2 + 1/4}{r^2} \qquad s_0 \approx 1.00624$$

$$\zeta_{cr} = \frac{(d-2)^2}{4} = \frac{1}{4} \qquad \Longrightarrow$$ Efimov physics occurs at :

$$\boxed{\zeta_E = s_0^2 + 1/4 = 1.26251 > \zeta_{cr}}$$

ζ_E is <u>fixed</u> in Efimov physics. It cannot be changed !


Vol 440|16 March 2006|**doi:10.1038/nature04626**

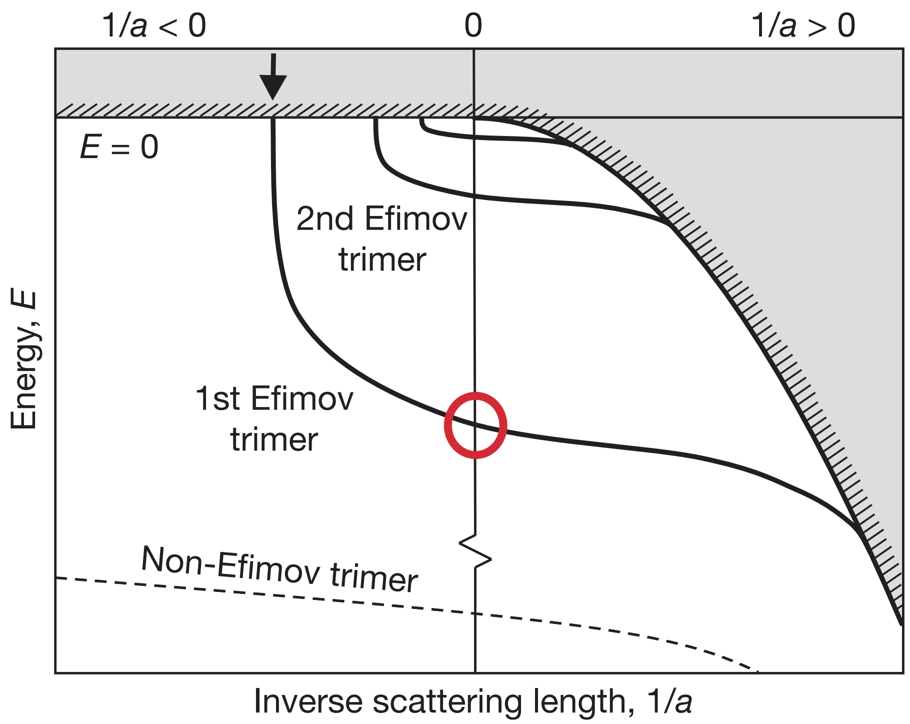
nature

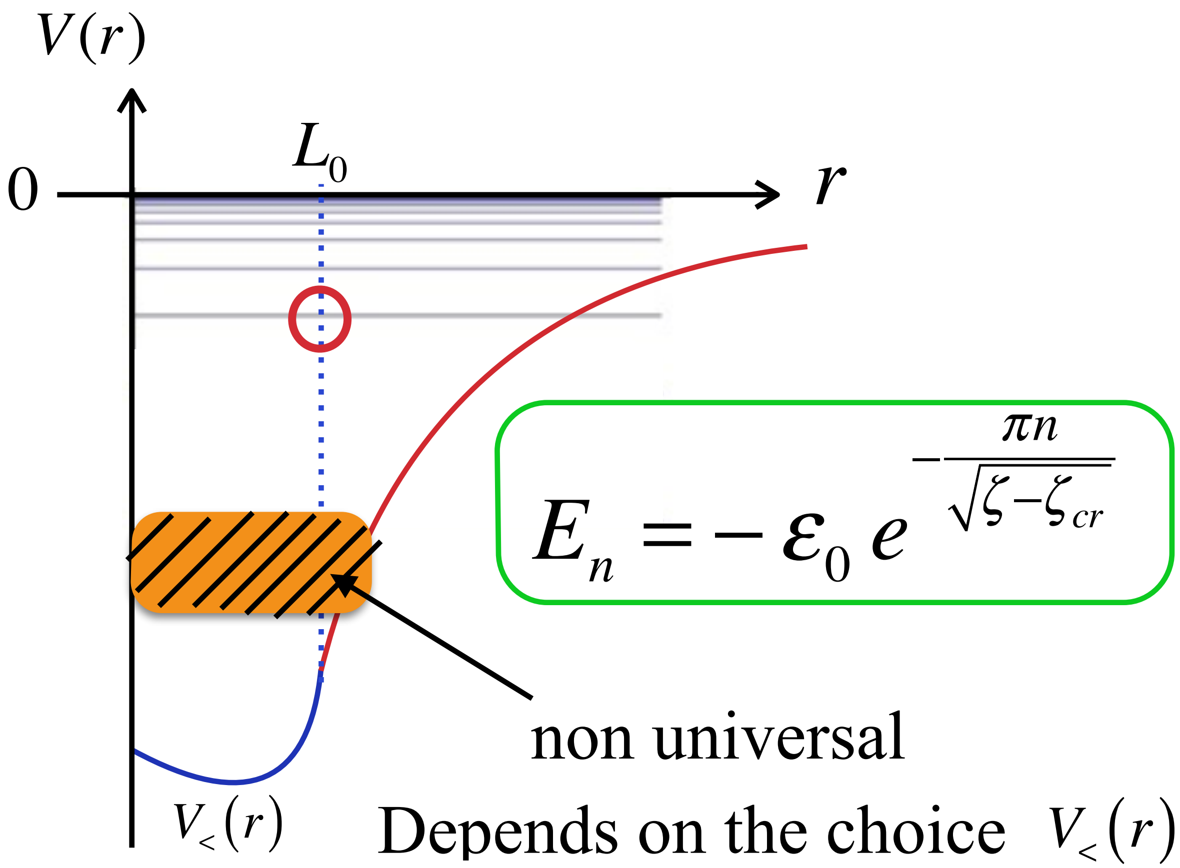
LETTERS

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer[1], M. Mark[1], P. Waldburger[1], J. G. Danzl[1], C. Chin[1,2], B. Engeser[1], A. D. Lange[1], K. Pilch[1], A. Jaakkola[1],
H.-C. Nägerl[1] & R. Grimm[1,3]

Measurement of a single Efimov state : n=1

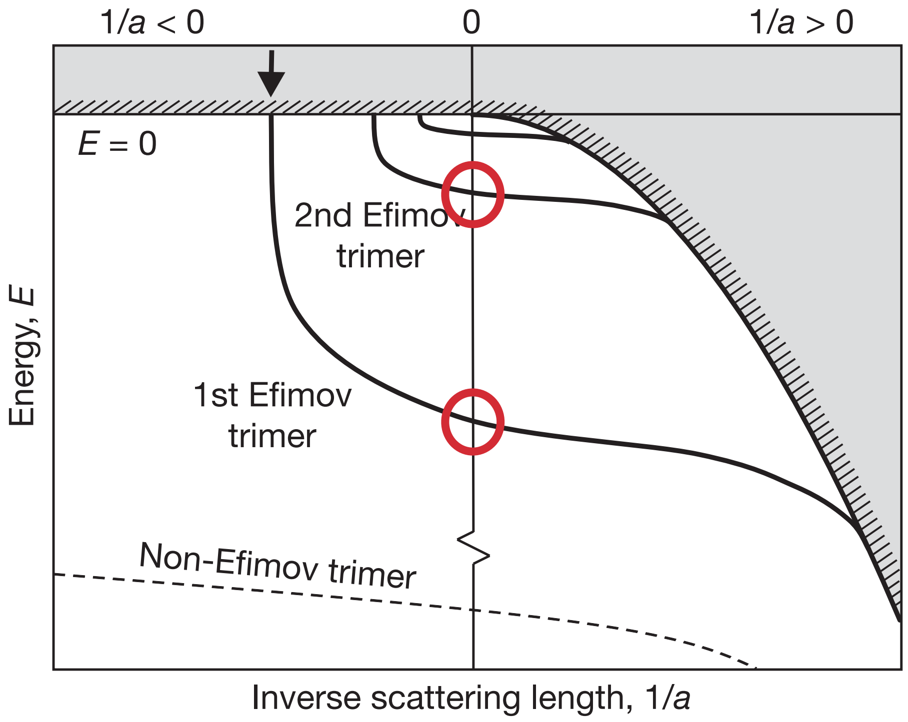


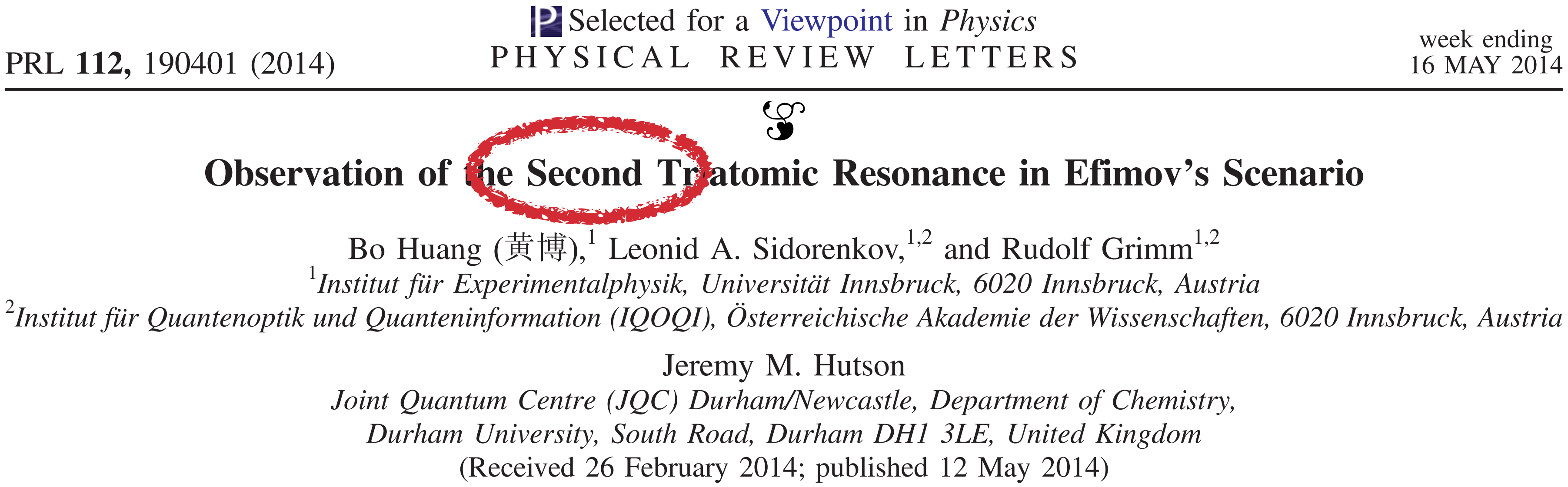


$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta-\zeta_{cr}}}}$$

non universal

Depends on the choice $V_<(r)$




Selected for a Viewpoint in *Physics*

PRL **112,** 190401 (2014) PHYSICAL REVIEW LETTERS week ending 16 MAY 2014

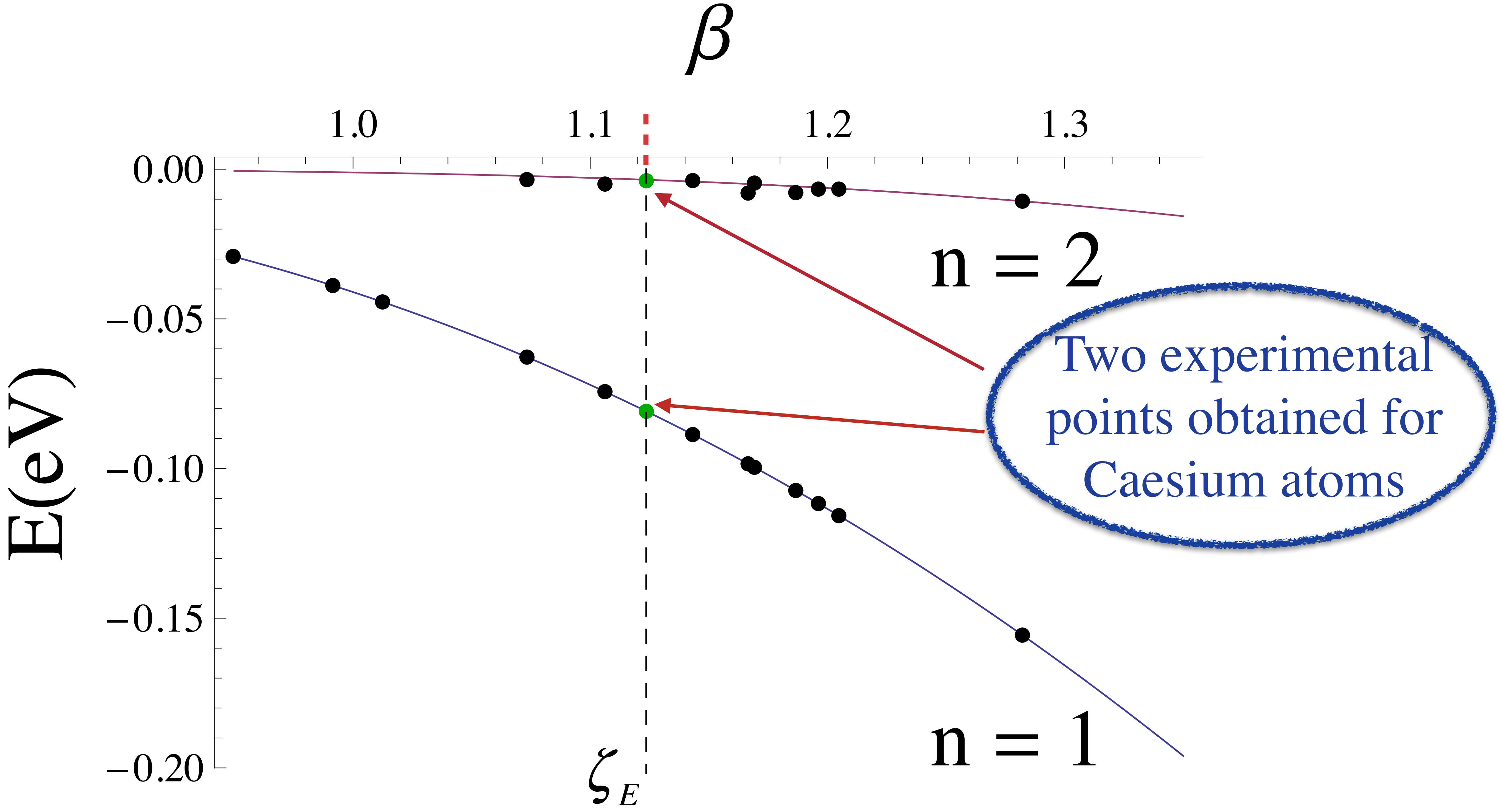
Observation of the Second Triatomic Resonance in Efimov's Scenario

Bo Huang (黄博),[1] Leonid A. Sidorenkov,[1,2] and Rudolf Grimm[1,2]
[1]*Institut für Experimentalphysik, Universität Innsbruck, 6020 Innsbruck, Austria*
[2]*Institut für Quantenoptik und Quanteninformation (IQOQI), Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria*

Jeremy M. Hutson
Joint Quantum Centre (JQC) Durham/Newcastle, Department of Chemistry,
Durham University, South Road, Durham DH1 3LE, United Kingdom
(Received 26 February 2014; published 12 May 2014)

Measurement of a second Efimov state : n=2

Universality



Two experimental points obtained for Caesium atoms

Not obvious at all ! Two very different physical phenomena share the same universal energy spectrum.

Summary-Further directions

- Breaking of continuous scale invariance (CSI) into discrete scale invariance (DSI) on two examples.
- Observed this quantum phase transition on graphene. It raises more questions than it solved.
- Efimov physics belongs to this universality class. It does not allow observing the transition.

- Other problems can be described similarly as “conformality lost” (Kaplan et al., 2009) and emergence of limit cycles:
  - Kosterlitz-Thouless transition (deconfinement of vortices in the XY-model at a critical temp. above which the theory is conformal): mapping between the XY-model and the T=0 sine-Gordon in 1+1 dim.

$$\boxed{L = \frac{T}{2}\left(\partial_\mu \phi\right)^2 - 2z\cos\phi}$$

Thank you for your attention.

Continuous *vs*. discrete scale symmetry

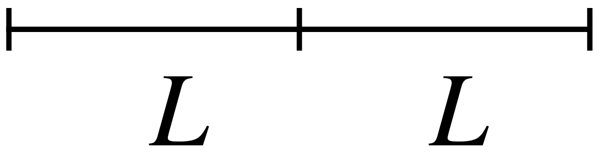
Homogeneous string (uniform mass per unit length)

$d=1$

$m(L)$

Expect : $m(L)\propto L$

How to obtain this result ?



$\boxed{m(2L)=2\ m(L)}$

or more generally, $m(aL)=b\,m(L)$ $\forall\, a\in\mathbb{R}$

Continuous scale invariance (CSI)

Scaling relation: $f(a\,x) = b\,f(x)$

If this relation is **satisfied for all** a and $b(a)$, the system has a continuous scale invariance (CSI).

Continuous scale invariance (CSI)

Scaling relation: $f(ax) = b\,f(x)$

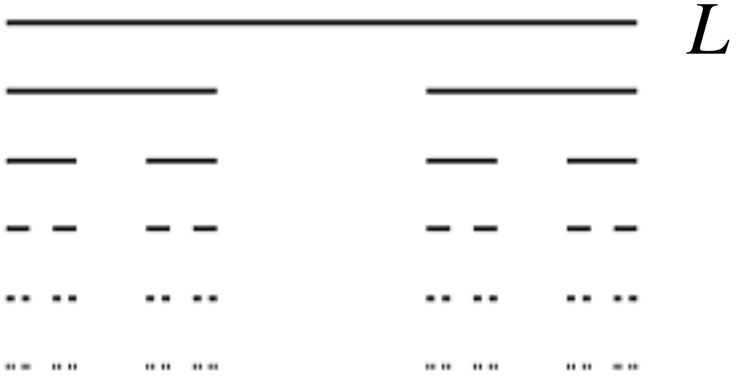
If this relation is **satisfied for all** a and $b(a)$, the system has a continuous scale invariance (CSI).

Discrete scale invariance (DSI)

discrete scale invariance is a weaker version of scale invariance, *i.e.*,

$$f(ax) = b\,f(x),\quad \boxed{\text{with fixed}\,(a,b)}$$

Cantor set



$$M_n = 2^n M$$

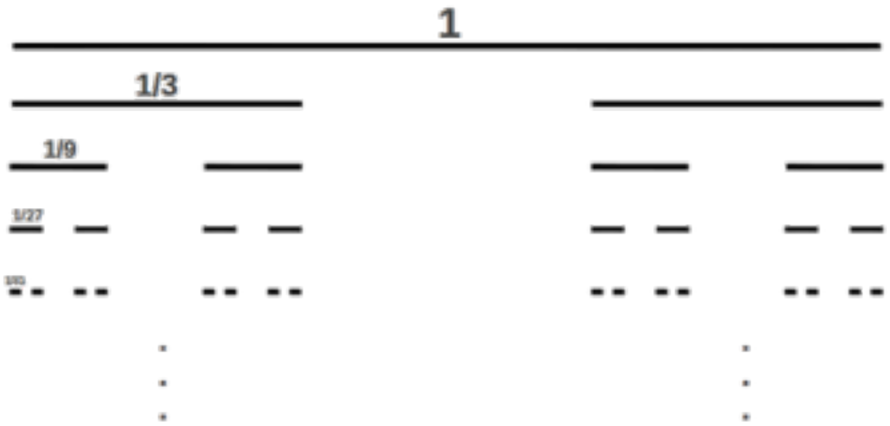
$$L_n = 3^n L$$

$$\frac{\ln M_n}{\ln L_n} \xrightarrow[n\to\infty]{} d_h = \frac{\ln 2}{\ln 3}$$

Alternatively, define the mass density $m\left(L\right)$ of the Cantor set

$$2m\left(L\right) = m\left(3L\right)$$

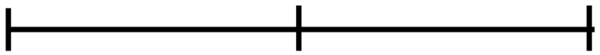
Relation between the different cases :



$$m(3L) = 2m(L)$$

$$(a,b) = (3,2)$$

Cantor set



$$d = 1$$

$$m(2L) = 2\ m(L)$$

$$\forall b(a) \in \mathbb{R}$$

Euclidean lattice

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f\left(a\,x\right)=b\,f\left(x\right)$$

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f(a\,x) = b\,f(x)$$

If satisfied $\forall\, b(a) \in \mathbb{R}$ (CSI),

General solution :

$$f(x) = C\,x^{\alpha}$$

with $\alpha = \dfrac{\ln b}{\ln a}$

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

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If satisfied *with fixed* (a,b) (DSI),

General solution:

$$f(x) = x^{\alpha}\,G\left(\frac{\ln x}{\ln a}\right)$$

where $G(u+1) = G(u)$ is a periodic function of period unity

Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f\left(a\,x\right)=b\,f\left(x\right)$$

If satisfied $\forall\, b\left(a\right)\in\mathbb{R}$ (CSI),

General solution :

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Break CSI into DSI ?

If satisfied *with fixed* $\left(a,b\right)$ (DSI),

General solution :

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Continuous *vs.* discrete scale invariance (CSI *vs.* DSI)

$$f\left(a\,x\right)=b\,f\left(x\right)$$

If satisfied $\forall\, b\left(a\right)\in\mathbb{R}$ (CSI),

General solution (by direct inspection)

$$f\left(x\right)=C\,x^{\alpha}$$

If satisfied *with fixed* $\left(a,b\right)$ (DSI),

General solution is

$$f\left(x\right)=x^{\alpha}\,G\left(\frac{\ln x}{\ln a}\right)$$

Break CSI into DSI ?

Claim : breaking of CSI into DSI occurs at the quantum level : quantum phase transition (scale anomaly)

A simple example of continuous scale invariance in quantum physics

Schr$\ddot{o}$dinger equation for a particle of mass μ in d-dimensions in an attractive potential :

$$V\left(r\right) = -\frac{\xi}{r^2}$$

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

Radial Schrödinger eq.

$$\psi''(r)+\frac{d-1}{r}\psi'(r)+\frac{\zeta}{r^2}\psi(r)=k^2\psi(r)$$

The only parameter $\zeta = 2\mu\xi - l(l+d-2)$ is dimensionless : no characteristic length (energy) scale.

Consequence: Schrödinger eq. displays continuous scale invariance : it is invariant under:

$$\begin{cases} r \to \lambda r \\ k \to \frac{1}{\lambda} k \end{cases} \qquad \forall \lambda \in \mathbb{R}$$

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To every normalisable wave function $\psi(r,k)$ corresponds a family of wave functions $\psi(\lambda r,\, k/\lambda)$ of energy $(\lambda k)^2$

Radial Schrödinger eq.

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To every normalisable wave function $\psi(r,k)$
corresponds a family of wave functions $\psi(\lambda r,\, k/\lambda)$ of energy $(\lambda k)^2$

The existence of one bound state implies those of a continuum of related bound states. No ground state. Problem !

It is a problem, but a well known (textbook) one.

It results essentially from :

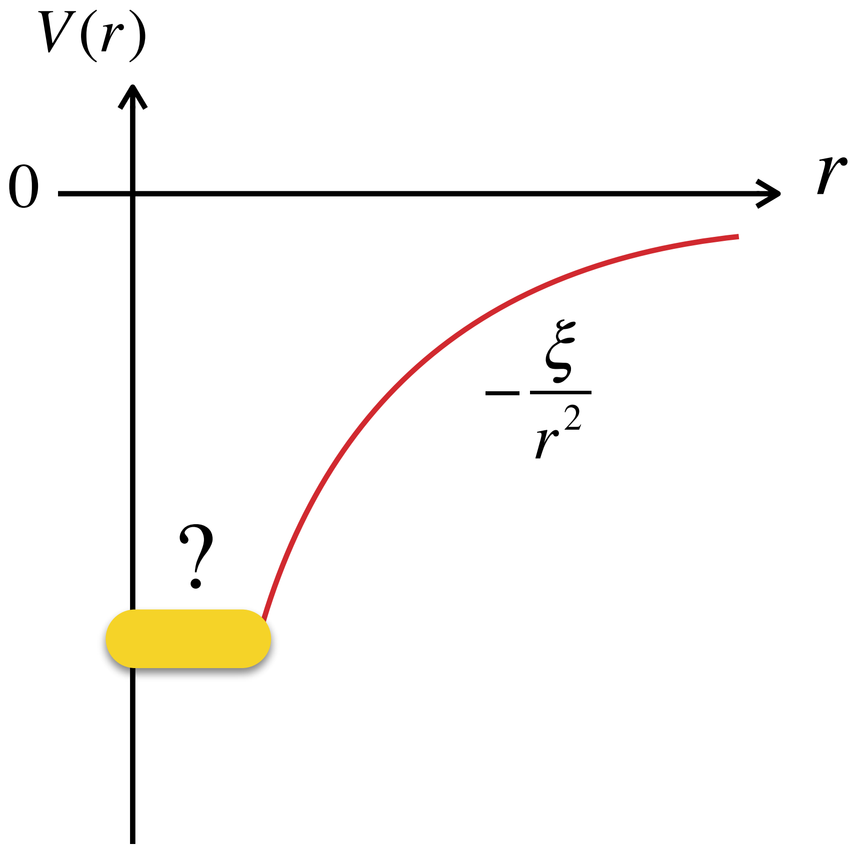
- the ill-defined behaviour of the potential $V(r) = -\frac{\xi}{r^2}$ for $r \to 0$
- the absence of characteristic length/energy.

Technically : non hermitian (self-adjoint) Hamiltonian.
To cure it : need to properly define boundary conditions (somewhere)

$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$ is scale invariant (CSI) : $r \rightarrow \lambda r \Rightarrow \hat{H} \rightarrow \frac{1}{\lambda^2}\hat{H}$

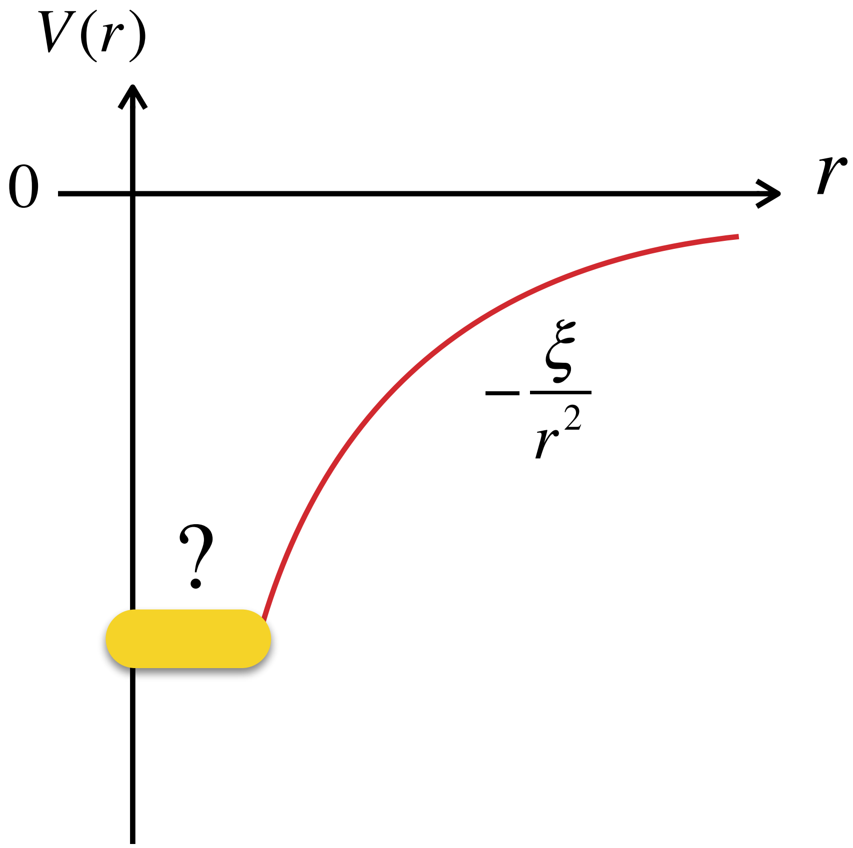
Any type of boundary conditions needed to find a well defined hermitian Hamiltonian break CSI.

Outline of the main results

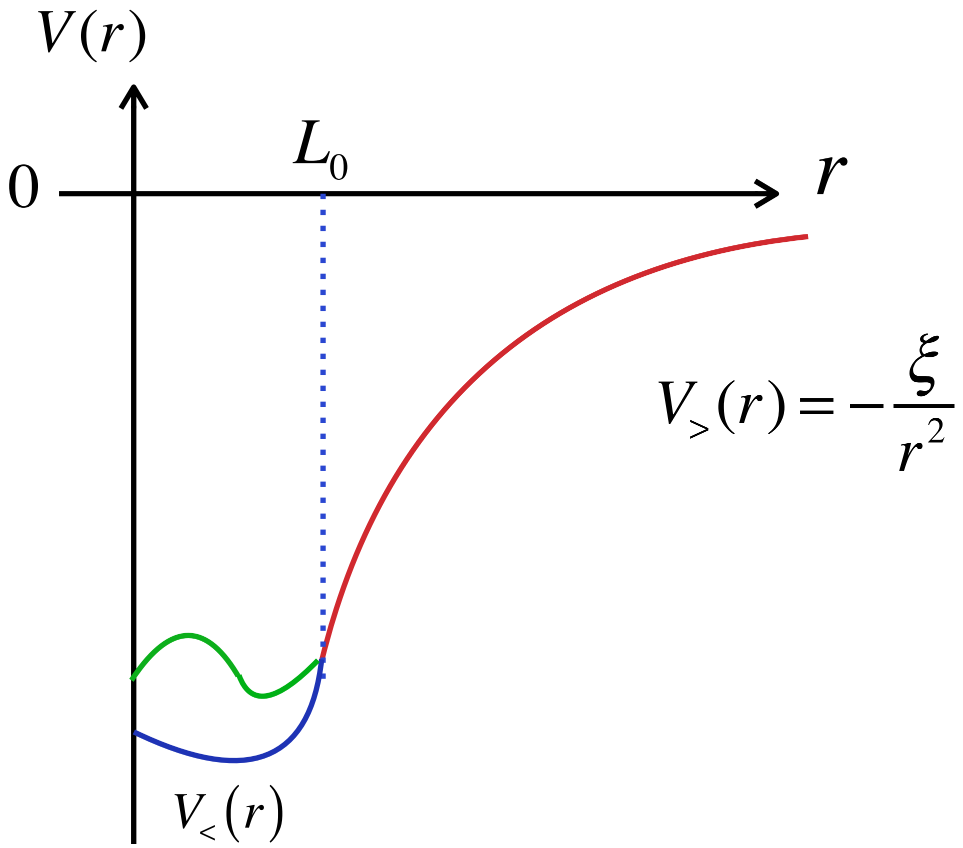


No characteristic scale

Outline of the main results



short-range regularisation

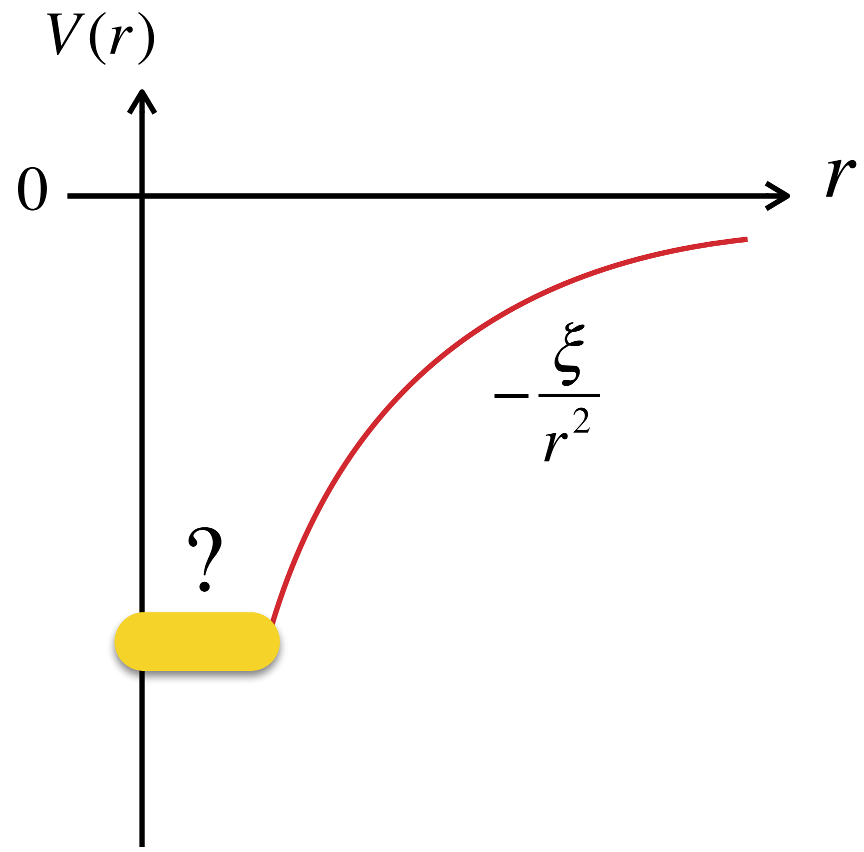


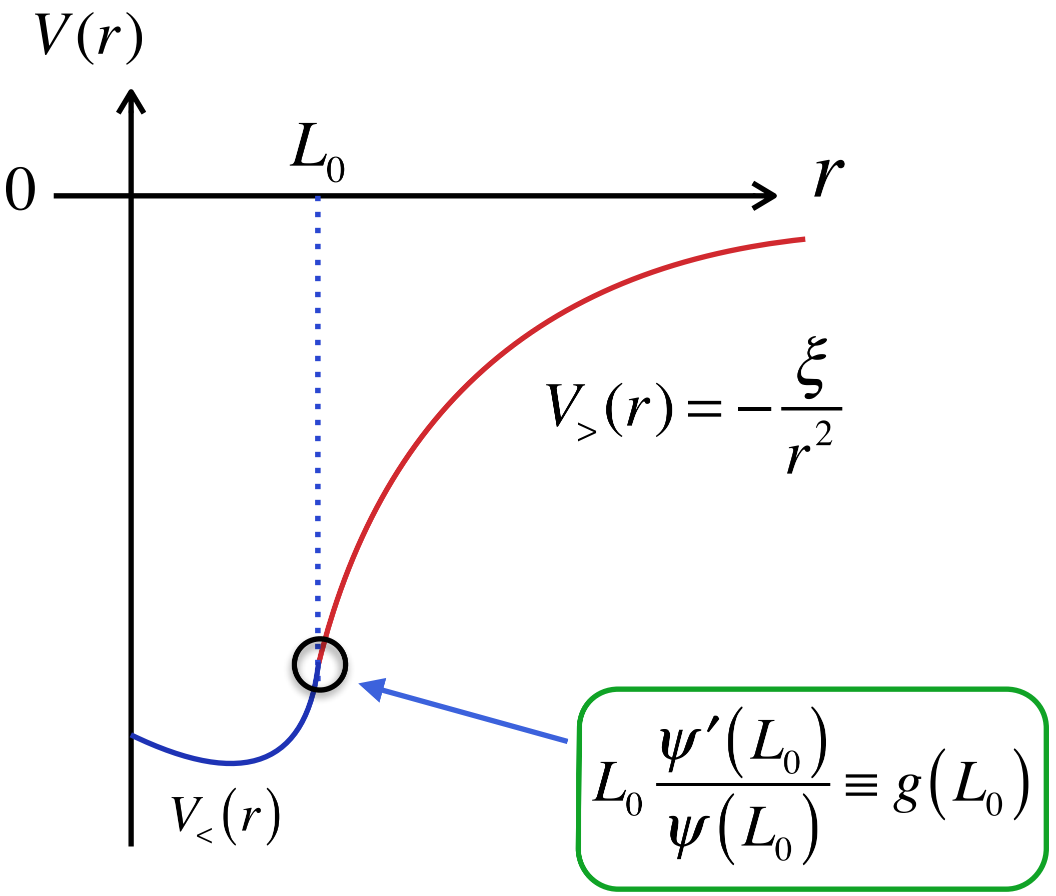
No characteristic scale

Some potential $V_<(r)$: accounts for “real” short-range physics.

Exact expression is not important .

Outline of the main results





Problem becomes well-defined :

- characteristic length L_0
- continuity of ψ and ψ' at L_0 (boundary condition)

$\Rightarrow$ energy spectrum

How the energy spectrum looks like ?

At low enough energies $\left(E \simeq 0\right)$,the spectrum has a "universal" behaviour.

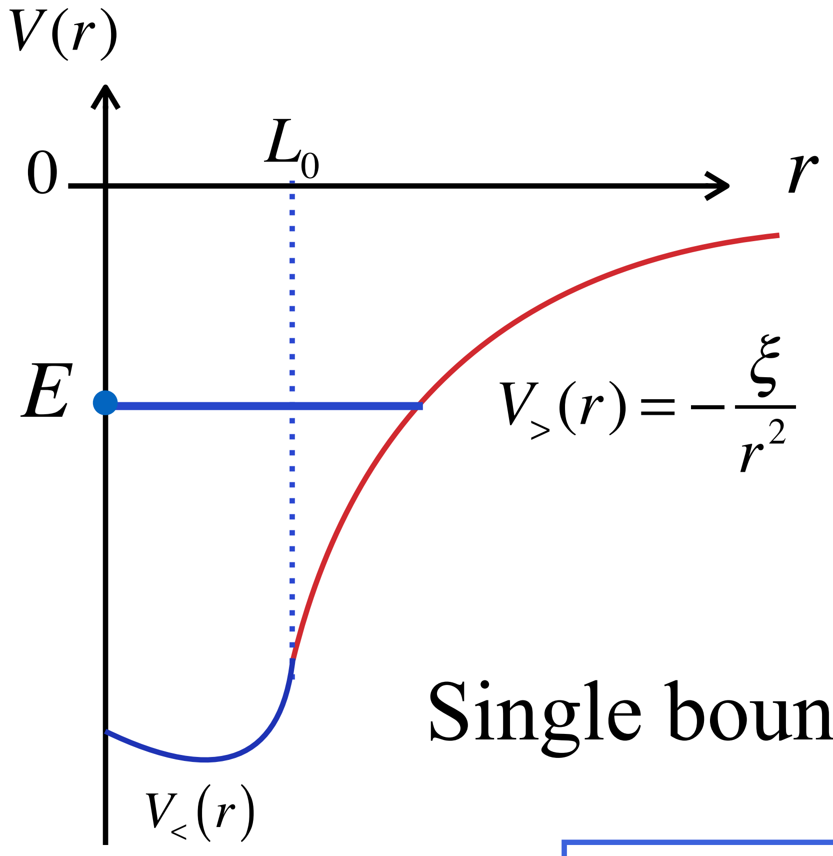
- It depends on the parameter $\zeta = 2\mu\xi - l\left(l + d - 2\right)$
- It exists a singular value $\boxed{\zeta_{cr} = \frac{\left(d-2\right)^2}{4}}$

Universal part of the energy spectrum

It depends on the parameter $\zeta = 2\mu\xi - l(l+d-2)$

It exists a singular value $\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$

$\zeta < \zeta_{cr}$



Single bound state

$$\boxed{E = -\frac{1}{L_0^2} f(g)}$$

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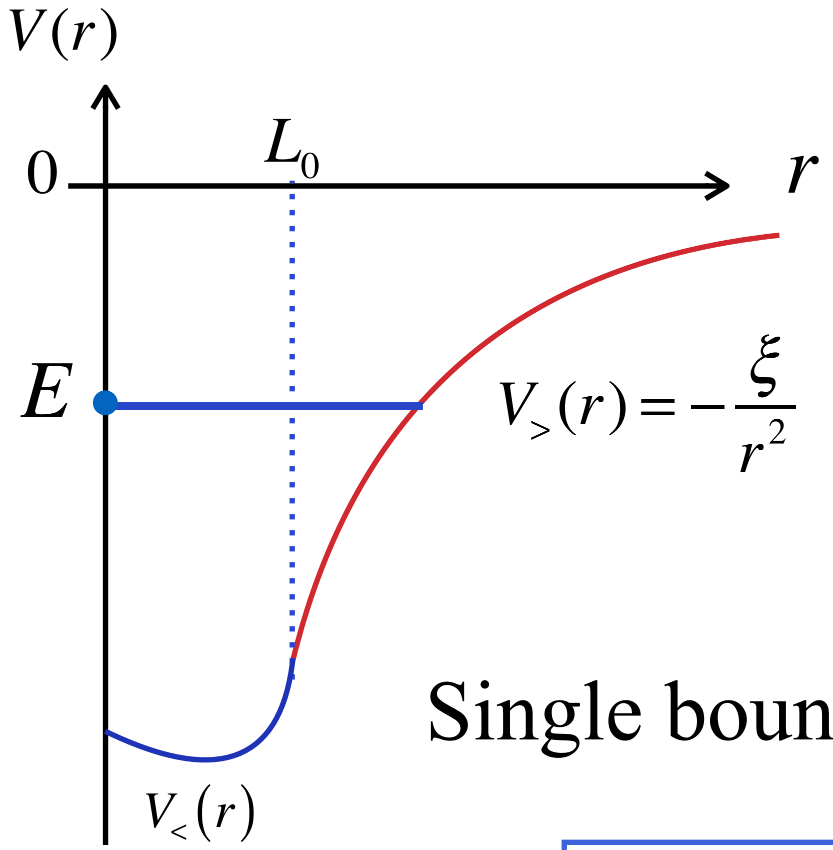

Single bound state

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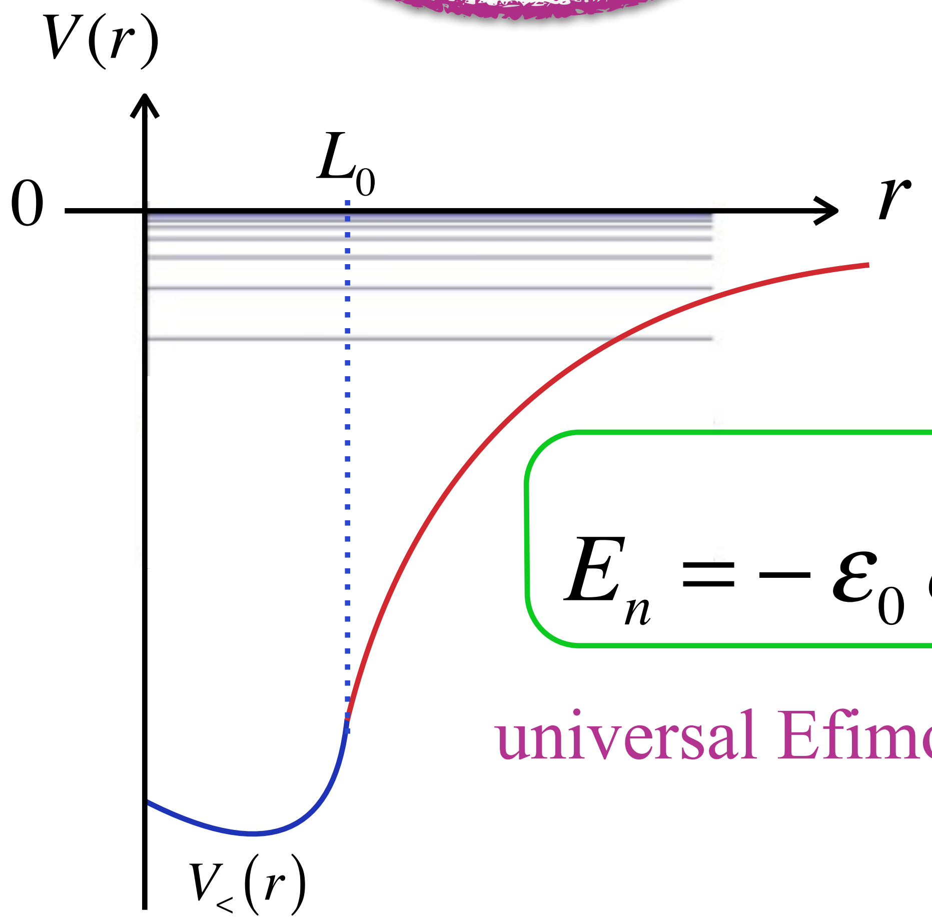

$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

universal Efimov spectrum

Universal part of the energy spectrum

It depends on the parameter $\zeta = 2\mu\xi - l(l+d-2)$

It exists a singular value $\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$

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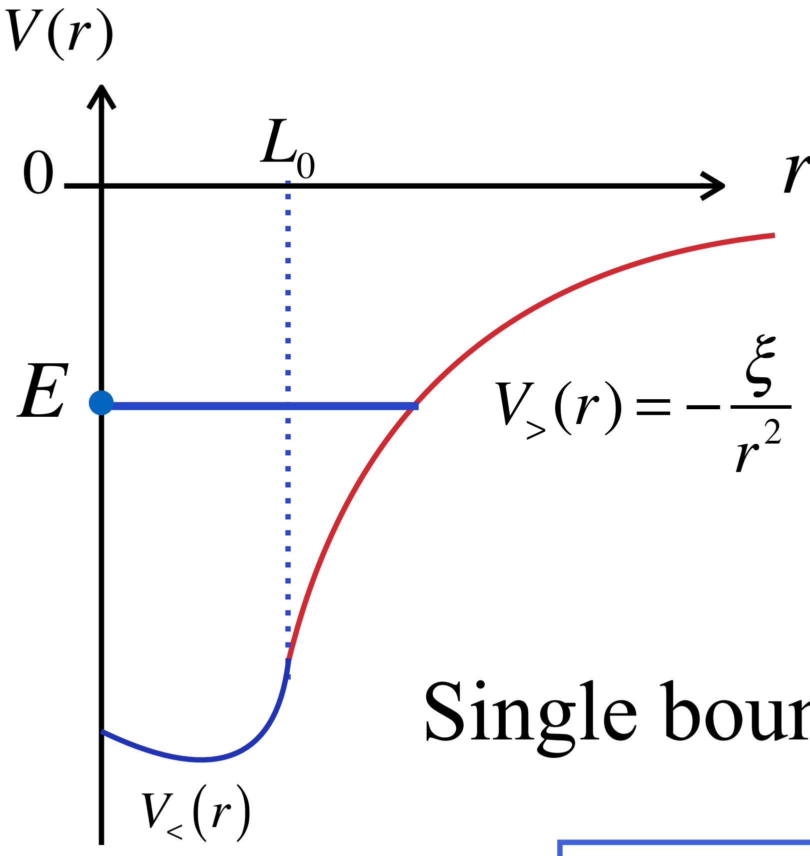
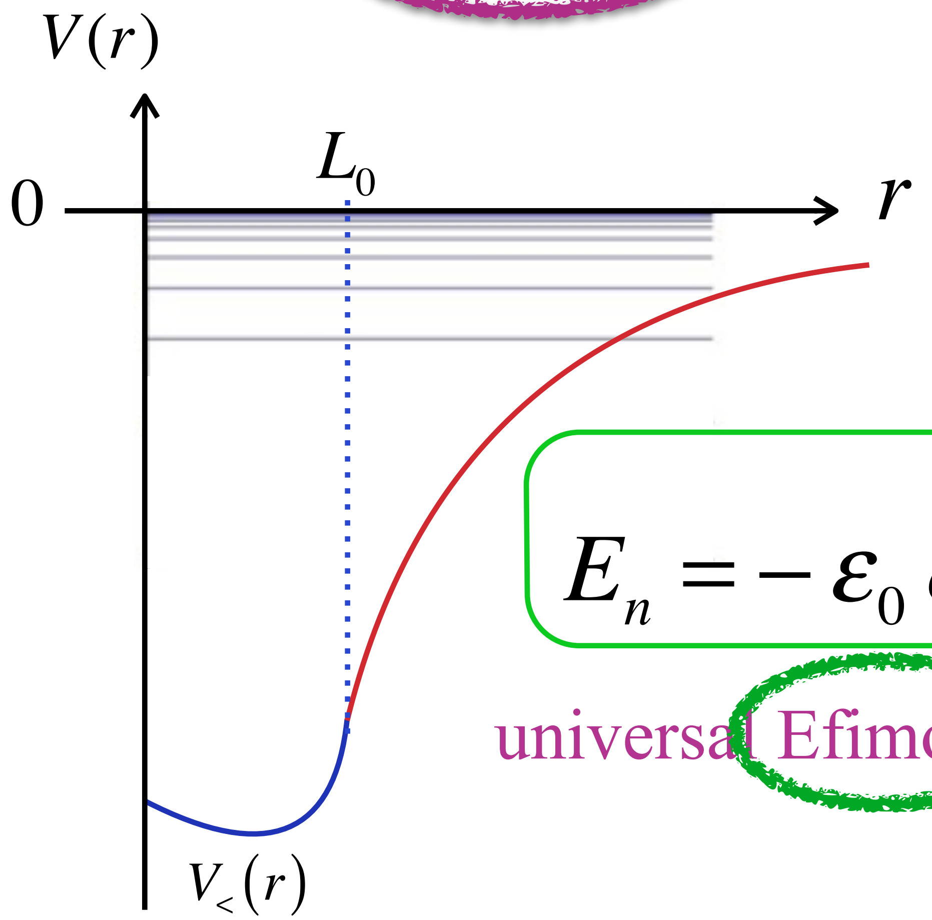


Single bound state

$$\boxed{E = -\frac{1}{L_0^2} f(g)}$$

$\zeta > \zeta_{cr}$



$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

universal Efimov spectrum

Just a name for the moment

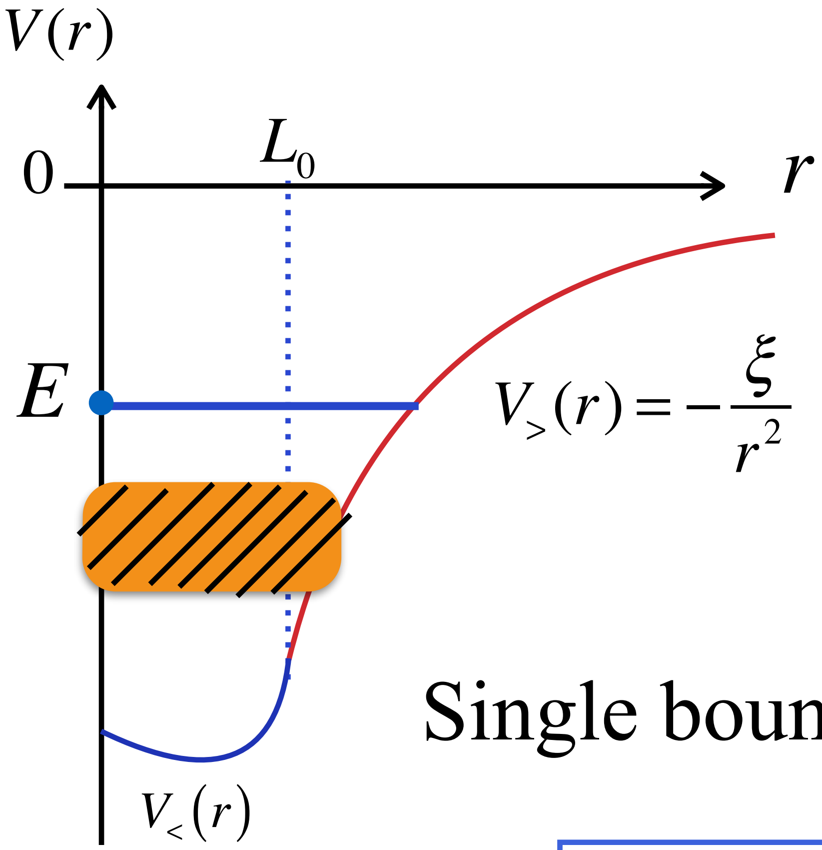
Universal part of the energy spectrum

It depends on the parameter $\zeta = 2\mu\xi - l(l+d-2)$

It exists a singular value

$$\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$$

$\zeta < \zeta_{cr}$

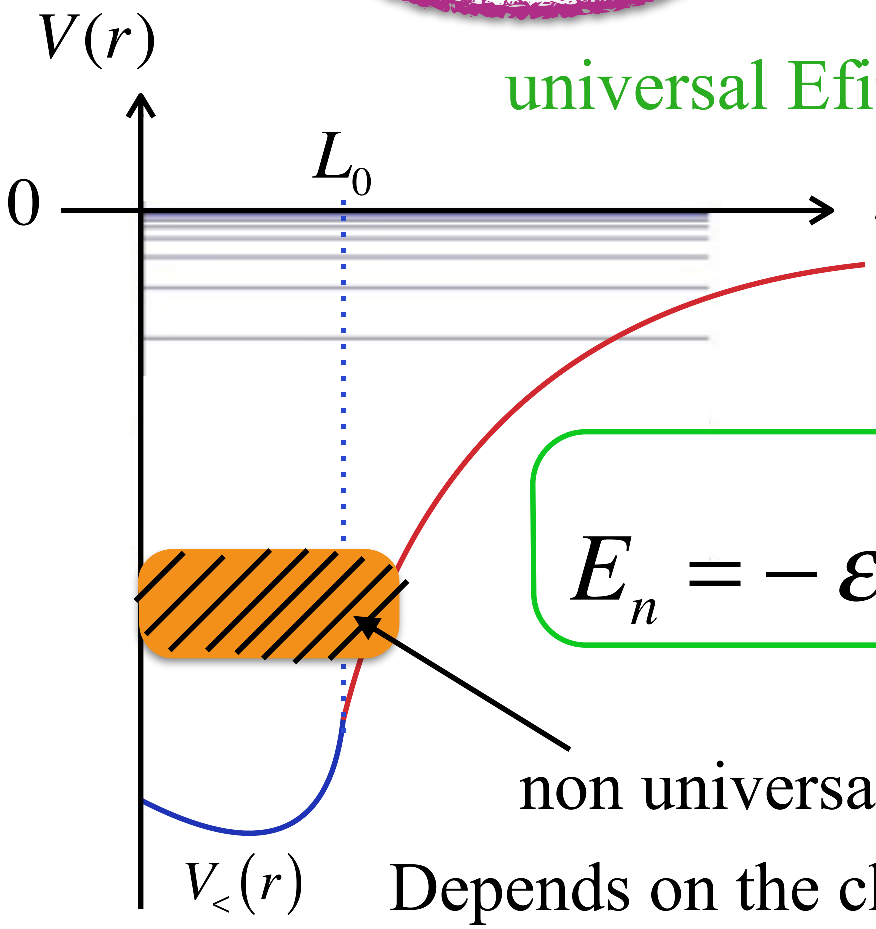


Single bound state

$$\boxed{E = -\frac{1}{L_0^2} f(g)}$$

$\zeta > \zeta_{cr}$

universal Efimov spectrum



$$\boxed{E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\zeta-\zeta_{cr}}}}}$$

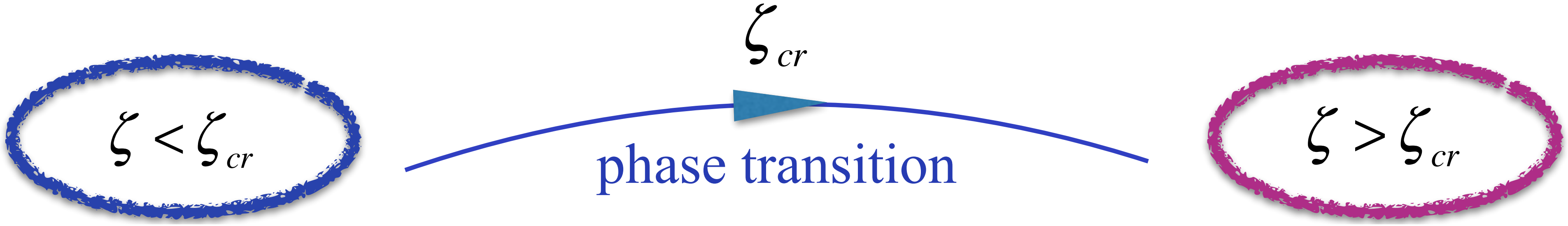
non universal part

Depends on the choice of $V_<(r)$

A quantum phase transition

It exists a singular value $\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$

Take the limit $L_0 \to \infty$
with EL_0^2 fixed



ζ_{cr}

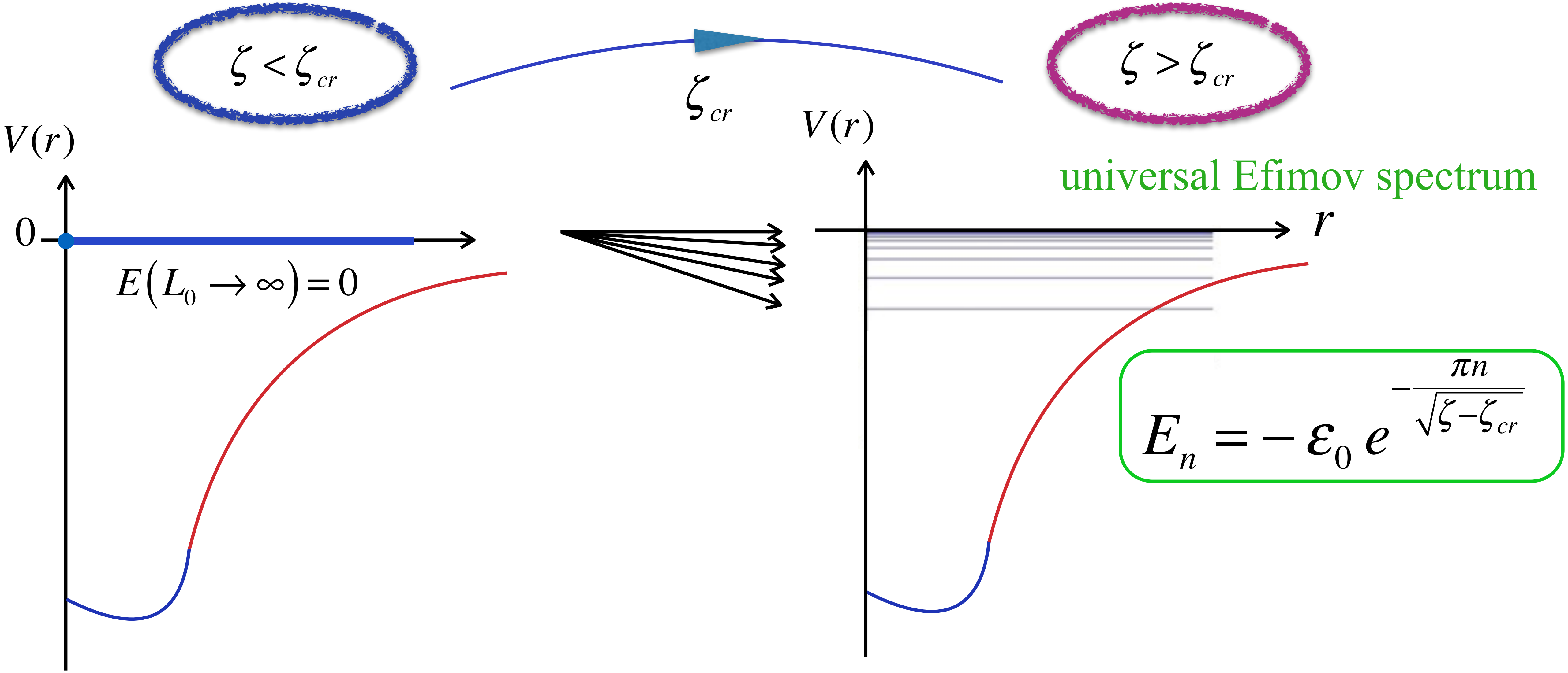
$\zeta < \zeta_{cr}$

phase transition

$\zeta > \zeta_{cr}$

A quantum phase transition

It exists a singular value $\boxed{\zeta_{cr} = \frac{(d-2)^2}{4}}$ Take the limit $L_0 \to \infty$



universal Efimov spectrum

$$\boxed{E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

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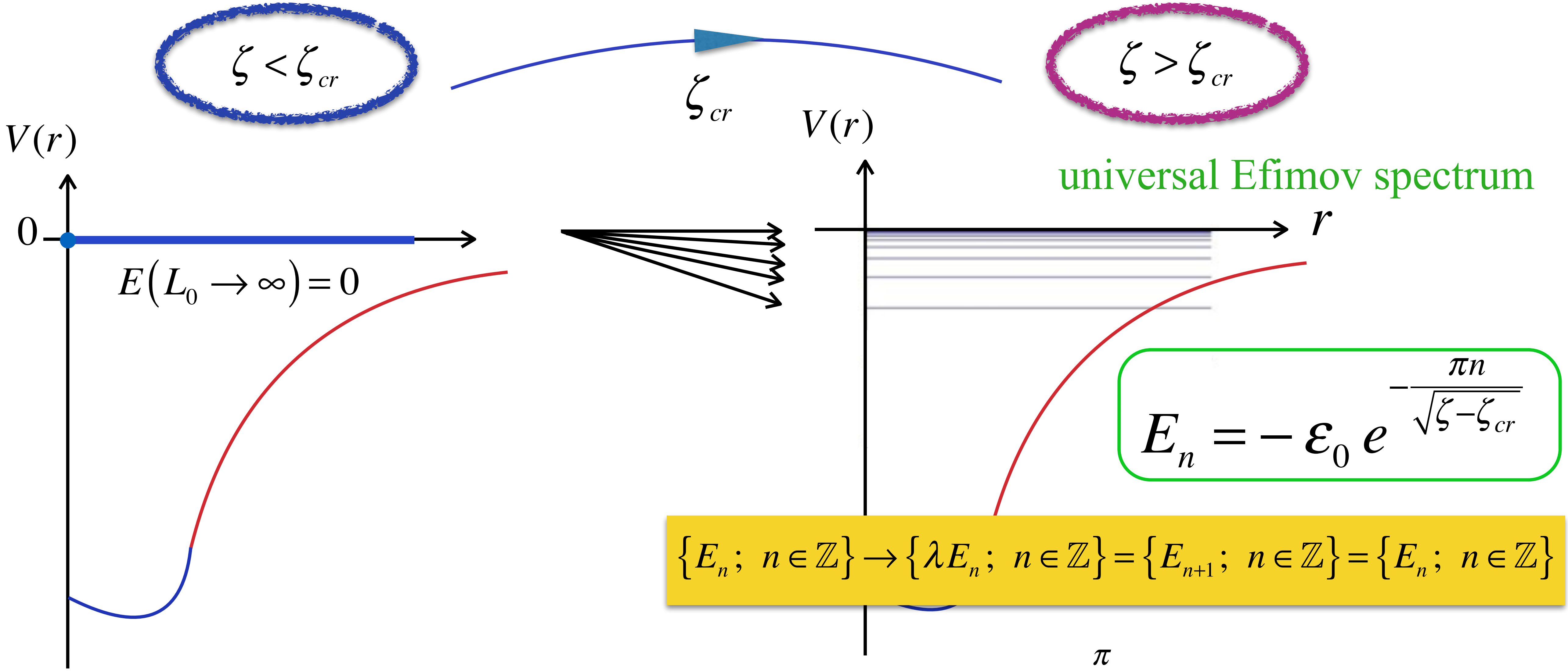

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$$\boxed{E_n = -\varepsilon_0\, e^{-\frac{\pi n}{\sqrt{\zeta - \zeta_{cr}}}}}$$

$$\{E_n;\ n \in \mathbb{Z}\} \to \{\lambda E_n;\ n \in \mathbb{Z}\} = \{E_{n+1};\ n \in \mathbb{Z}\} = \{E_n;\ n \in \mathbb{Z}\}$$

continuous scale invariance (CSI)

but trivial : $\lambda E = 0 \quad \forall \lambda$

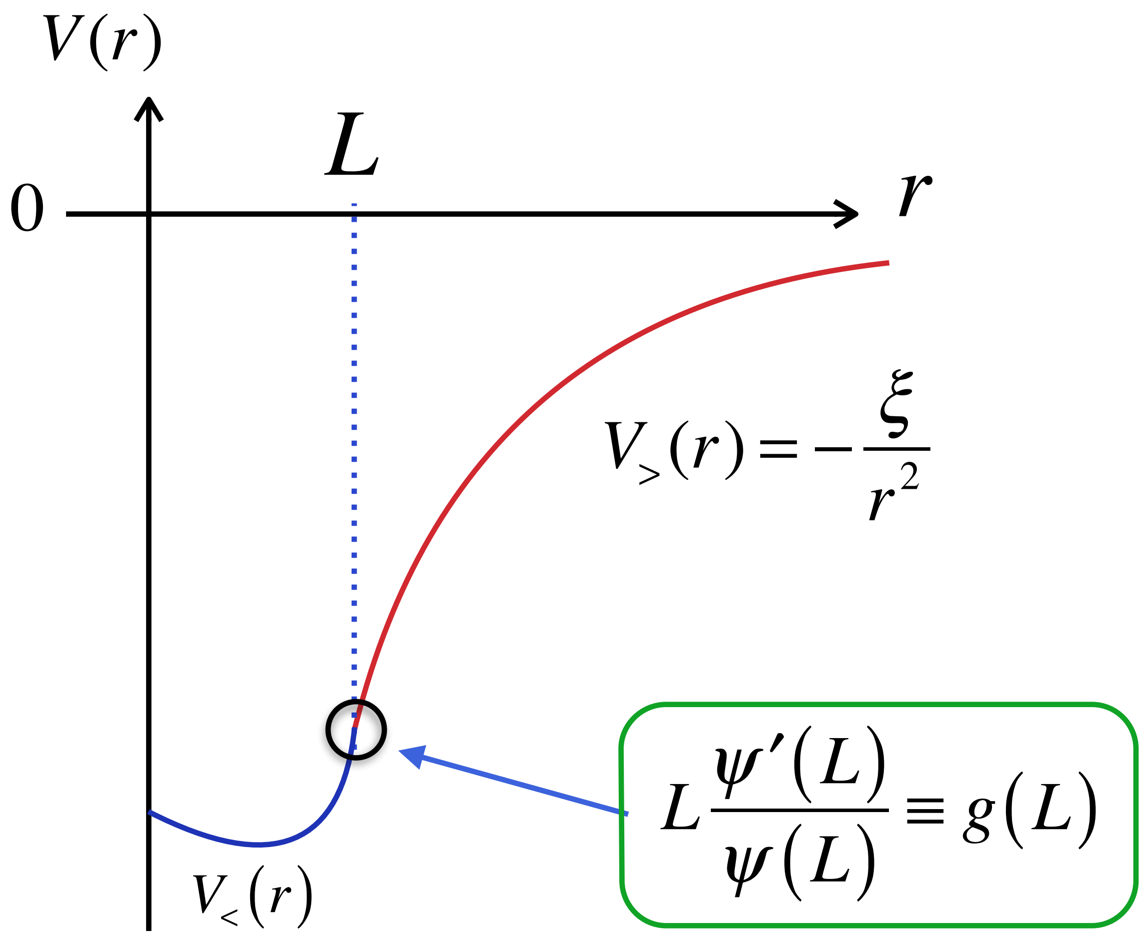
$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta - \zeta_{cr}}}}$ is fixed :

discrete scale invariance (DSI)

The same problem from another point of view

Renormalisation group (RG) and limit cycles

Is it possible to consistently change $\left(L,\xi,g\right)$ so that the energy spectrum remains unchanged ?



Problem becomes well-defined :

- characteristic length L
- continuity of ψ and ψ' at L

$\Rightarrow$ energy spectrum

ξ is a dimensionless number. To make it change with L we take

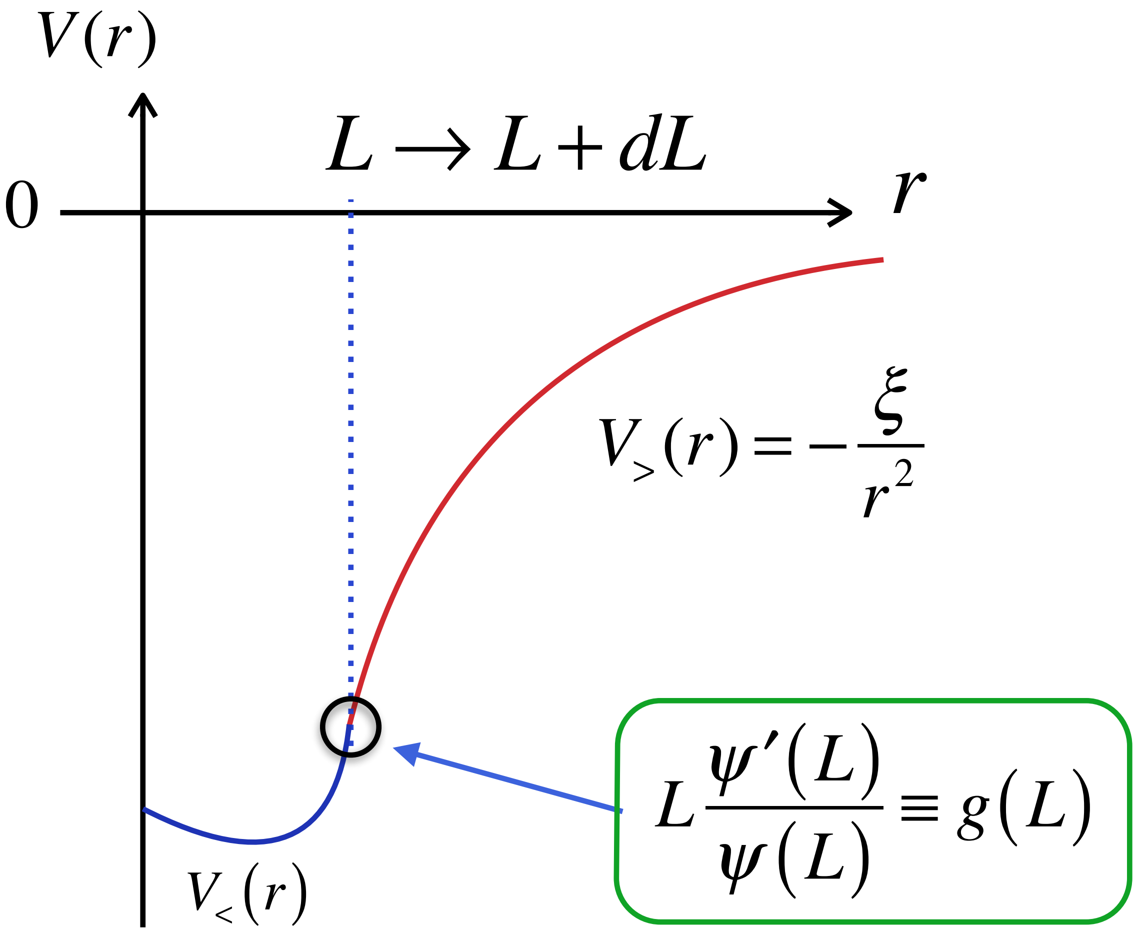
Note

$$\begin{cases} V_{>}(r) = -\xi / r^{s} & for\ r > L \\ V_{<}(r) & for\ \ r < L \end{cases}$$

eventually, $s \rightarrow 2$

so that now, $(L, \xi(L), g(L))$

Perform a RG transformation : change the cutoff distance $L \rightarrow L + dL$



$$L \frac{\psi'(L)}{\psi(L)} \equiv g(L)$$

leaves the energy spectrum unchanged provided :

- coupling strength changes as

$$\boxed{L \frac{d\xi}{dL} = (2 - s)\xi}$$

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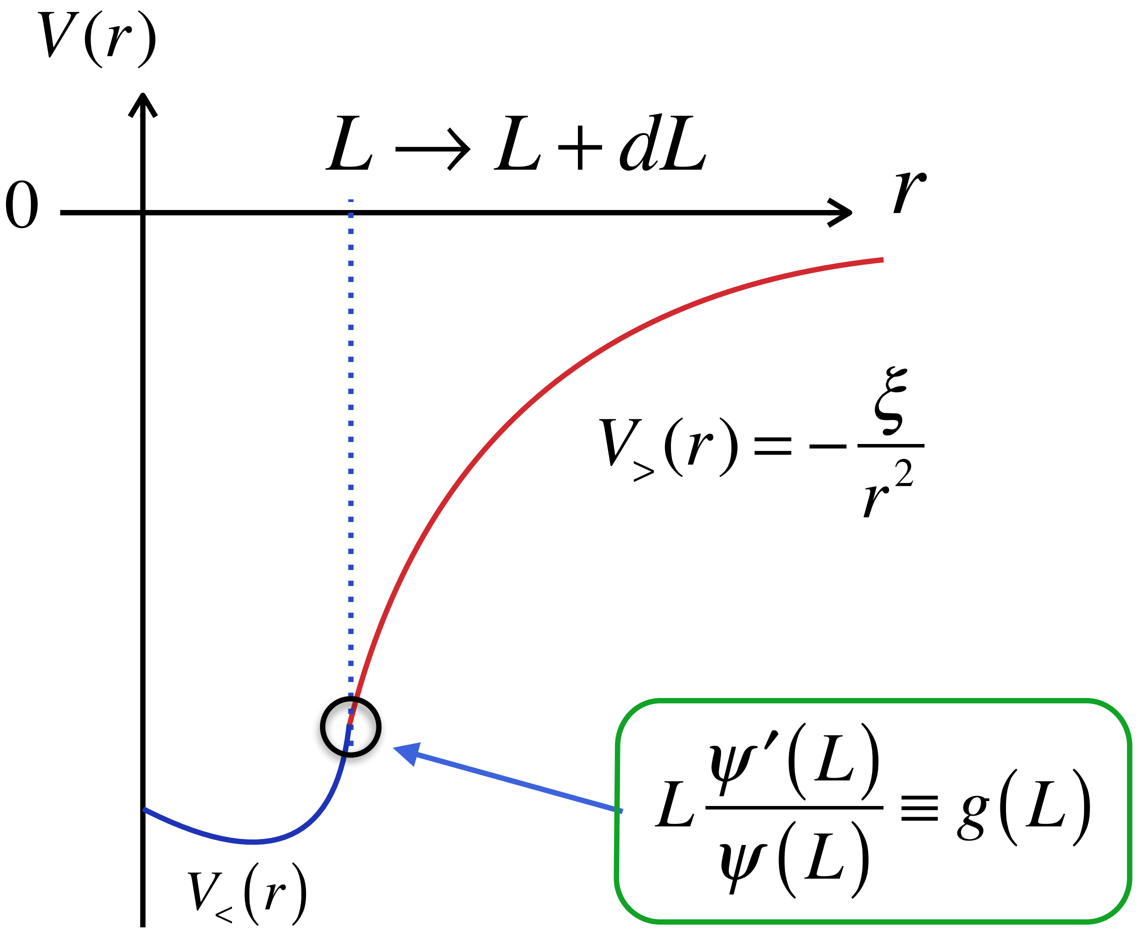

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- boundary condition parameter $g(L)$ changes according to

$$\boxed{L\frac{dg}{dL} = (2-d)g - g^2 - \zeta}$$

(for low enough energies, i.e. for $L \to \infty$)

Those are the renormalisation group (RG) equations.

- coupling strength changes as

$$\boxed{L\frac{d\xi}{dL}=(2-s)\xi}$$

- boundary condition parameter $g(L)$ changes according to

$$\boxed{L\frac{dg}{dL}=(2-d)g-g^2-\zeta}$$

(for low enough energies, i.e.
for $L\to\infty$)

Evolution of the coupling $g(L)$ - quantum phase transition

$$\beta(g) = \frac{\partial g}{\partial \ln L} = (2-d)g - g^2 - \zeta$$

Evolution of the coupling $g(L)$ - quantum phase transition

$$\beta(g)=\frac{\partial g}{\partial \ln L}=(2-d)g-g^2-\zeta=-(g-g_+)(g-g_-)$$

$$\boxed{g_\pm=\frac{2-d}{2}\pm\sqrt{\zeta_{cr}-\zeta}}$$

$$\boxed{\zeta_{cr}=\frac{(d-2)^2}{4}}$$

Evolution of the coupling $g(L)$ - quantum phase transition

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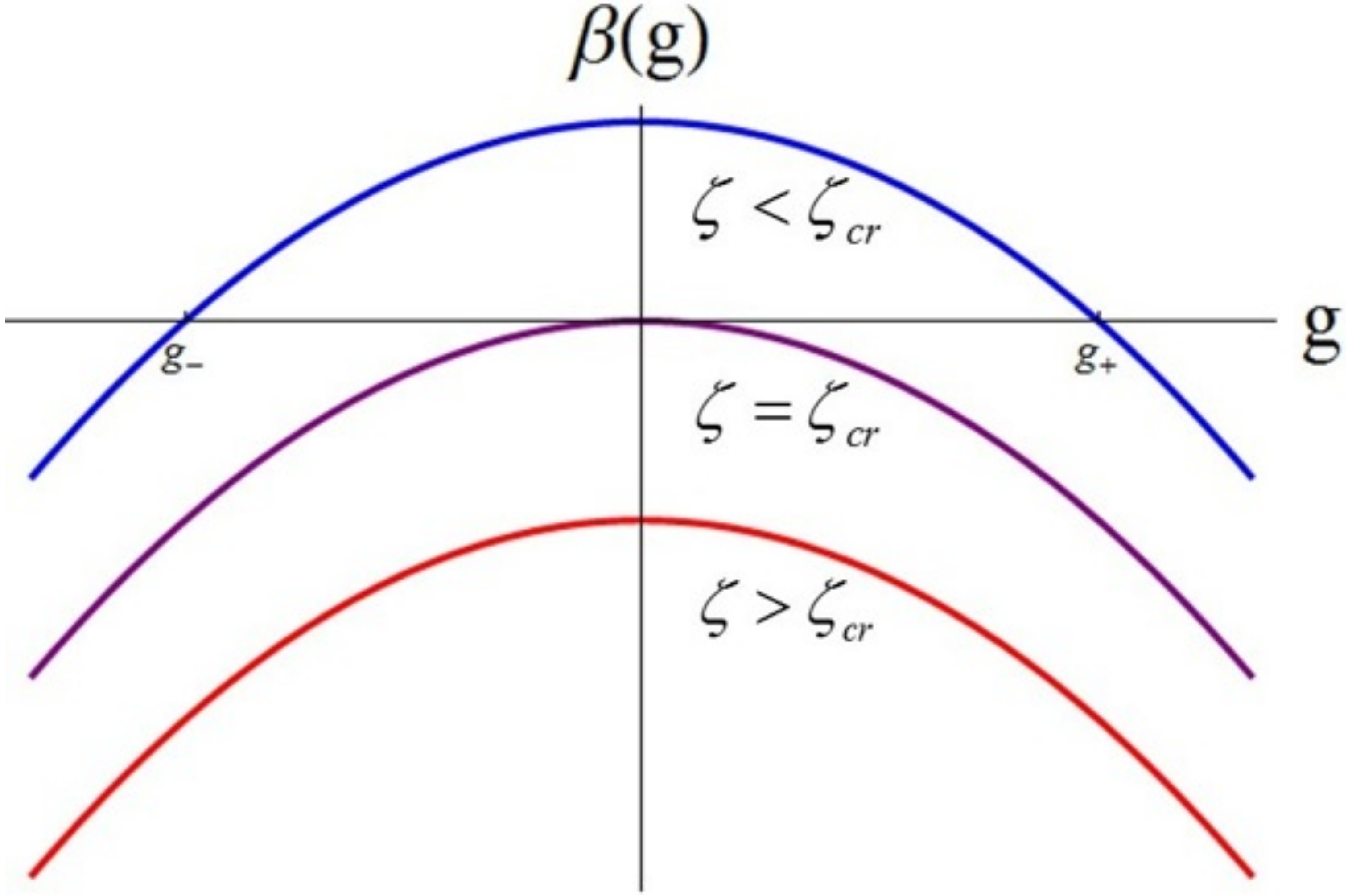


$$g_\pm = \frac{2-d}{2} \pm \sqrt{\zeta_{cr} - \zeta}$$

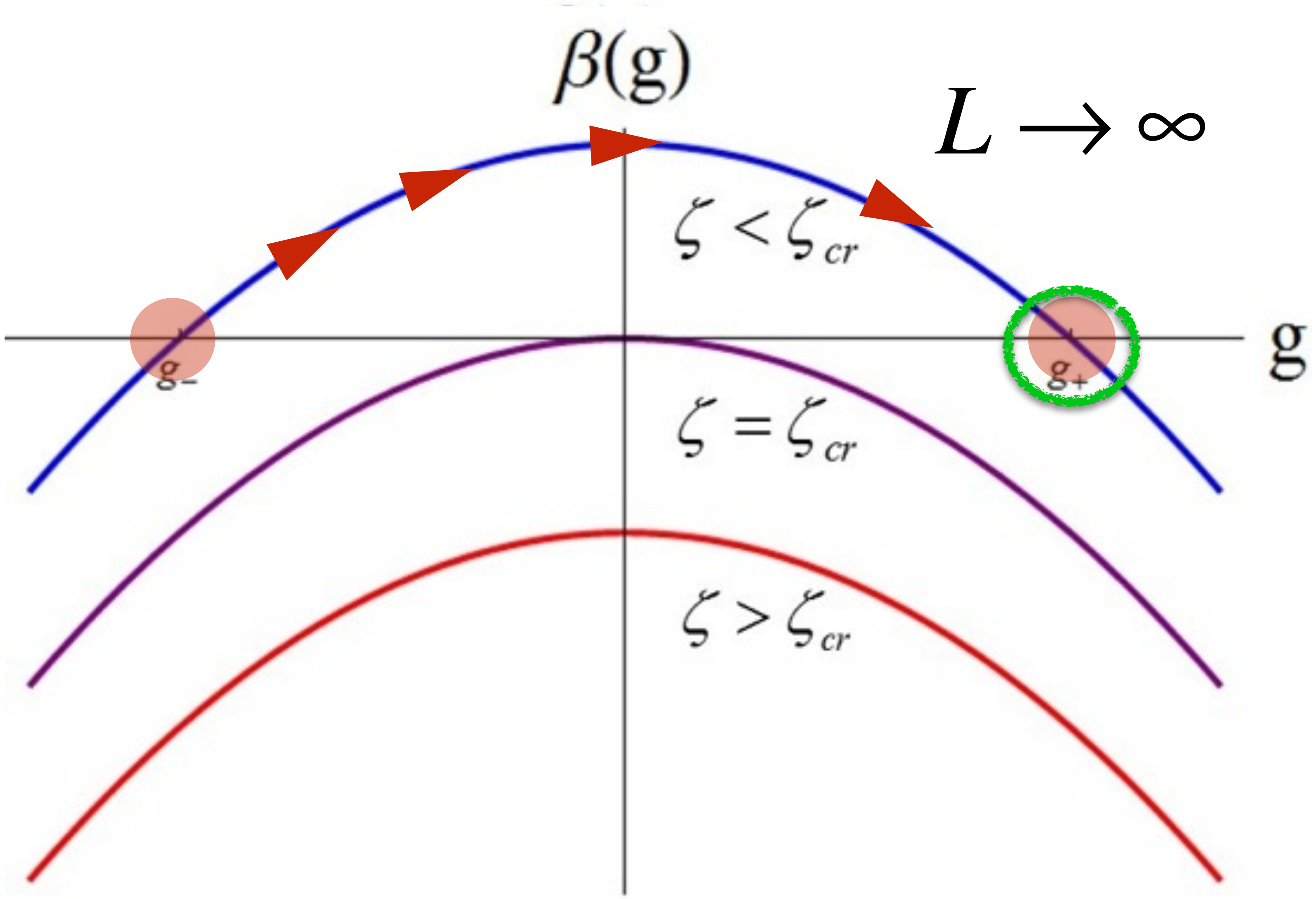
$$\zeta_{cr} = \frac{(d-2)^2}{4}$$

Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta < \zeta_{cr}$ two real fixed points (g_+ , g_-)

g_+ → stable

g_- → unstable

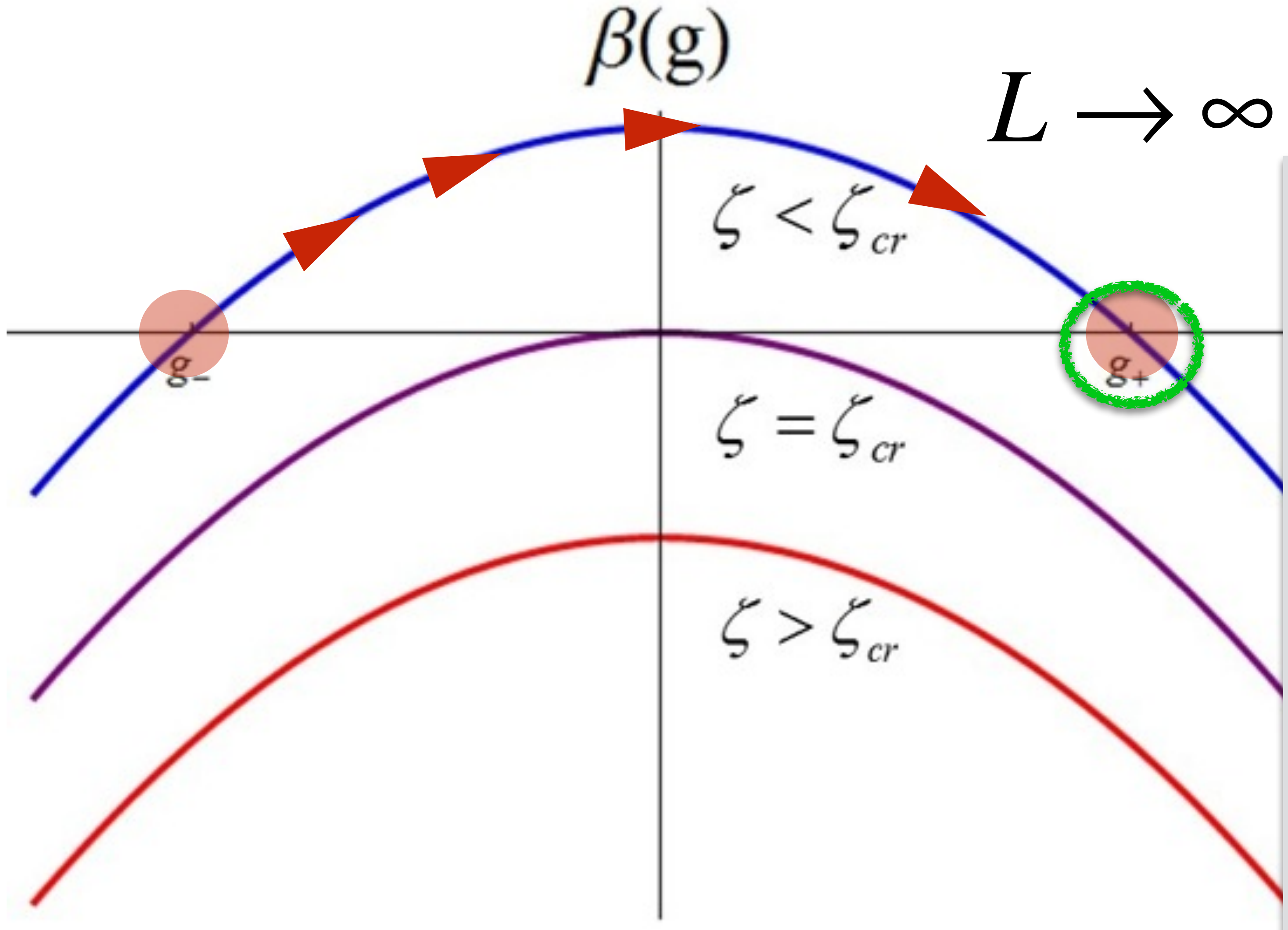


Universal behaviour of the energy spectrum

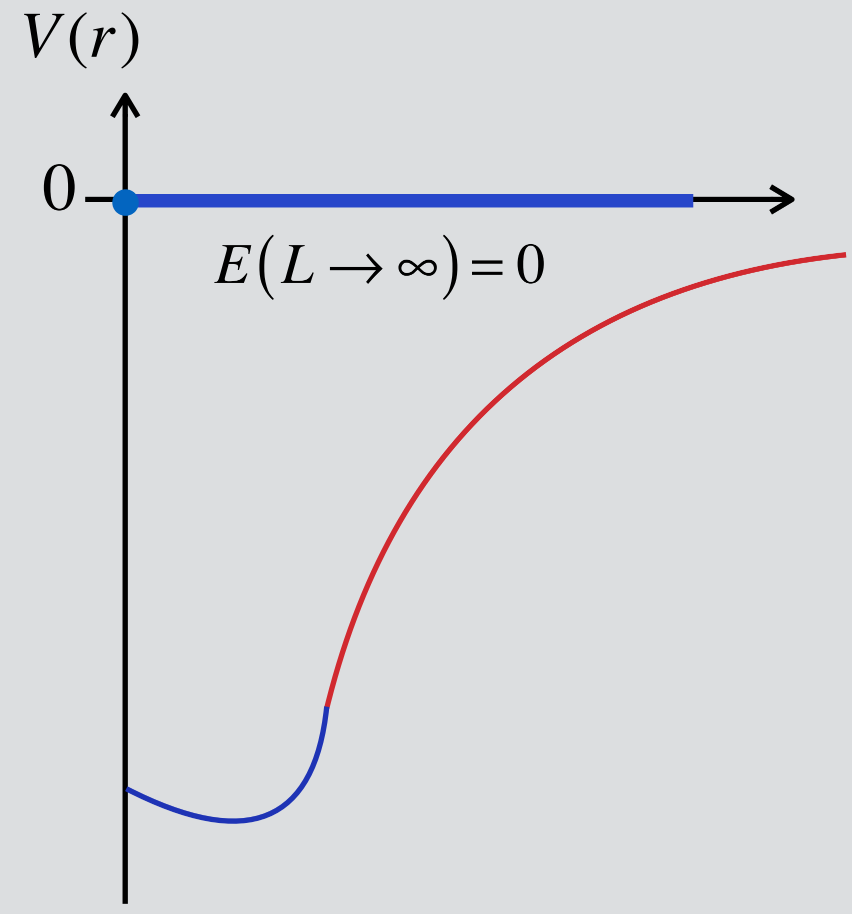
Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta < \zeta_{cr}$ two real fixed points (g_+ , g_-)

stable unstable



$L \rightarrow \infty$



continuous scale invariance (CSI)

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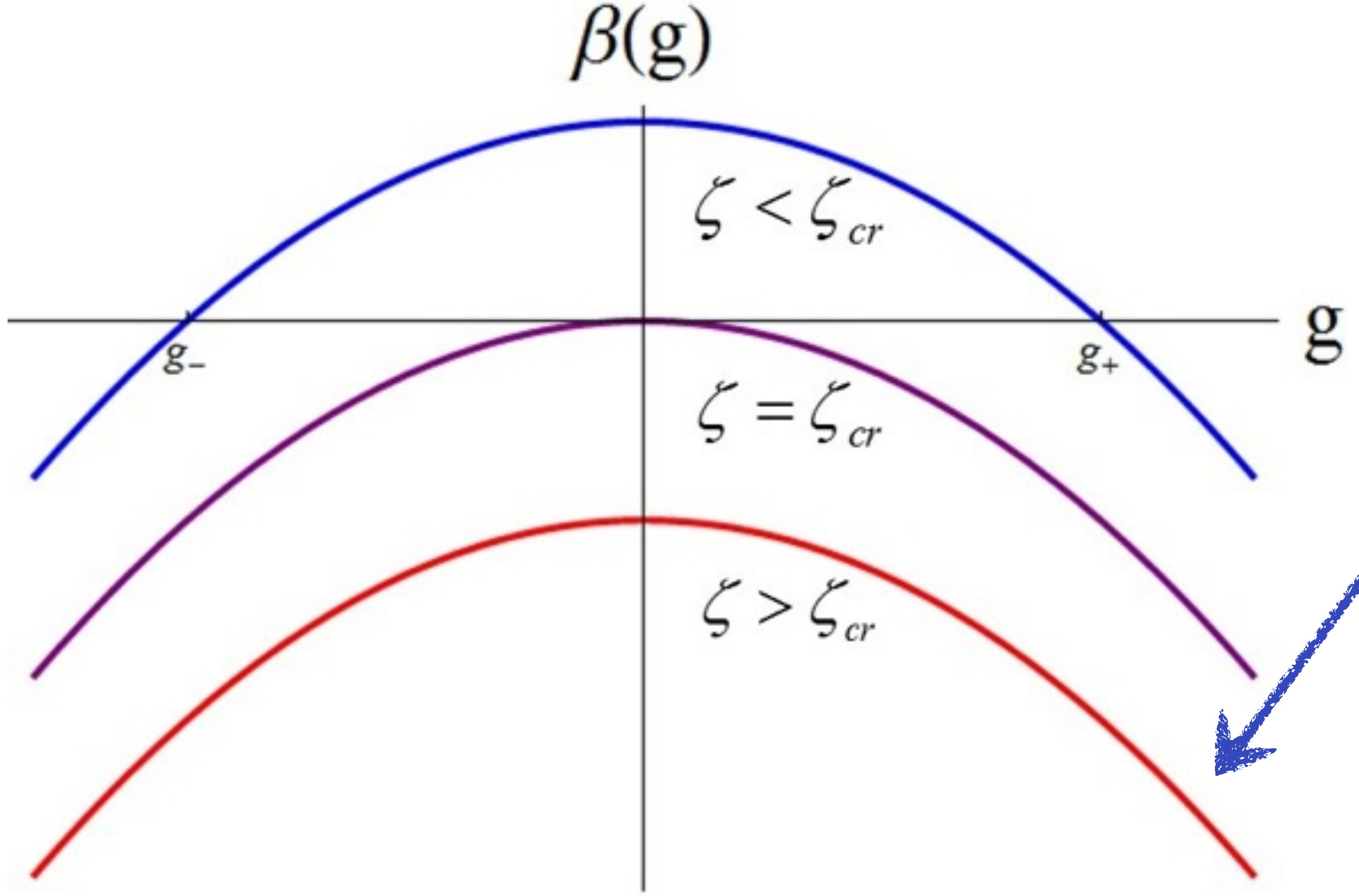
Universal behaviour of the e

Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta > \zeta_{cr}$

$$\beta(g) = \frac{\partial g}{\partial \ln L} = -(g - g_+)(g - g_-)$$



$$g_\pm = \frac{2-d}{2} \pm \sqrt{\zeta_{cr} - \zeta}$$

Two complex valued solutions

No fixed point

The solution for $g(L)$ is a limit cycle.

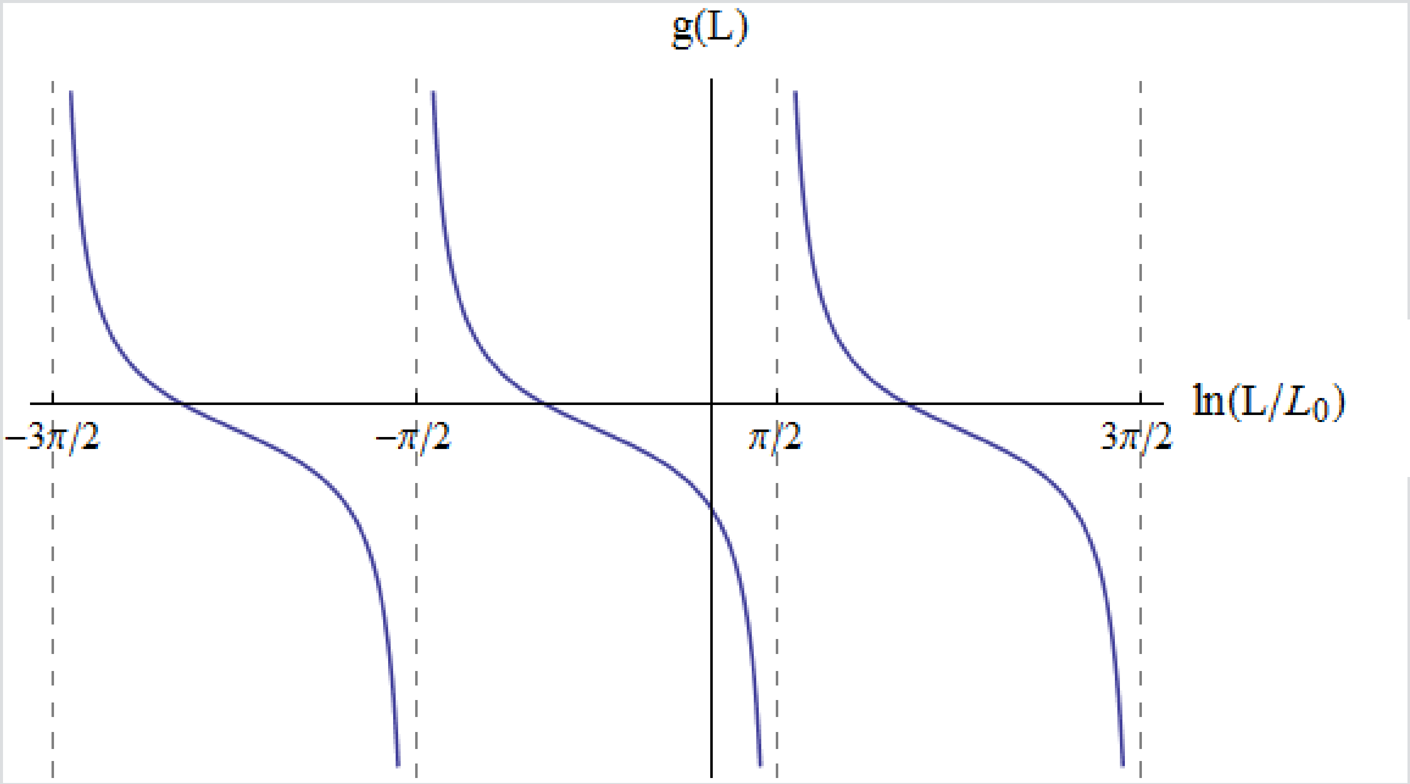
Evolution of the coupling $g(L)$ - quantum phase transition

For $\zeta > \zeta_{cr}$

$$\beta(g) = \frac{\partial g}{\partial \ln L} = -(g-g_+)(g-g_-)$$

The solution for $g(L)$ is a limit cycle.

The cycle completes a period for every $L \to e^{\frac{-\pi}{\sqrt{\zeta-\zeta_{cr}}}} L$



Evolution of the coupling $g(L)$ - quantum phase transition

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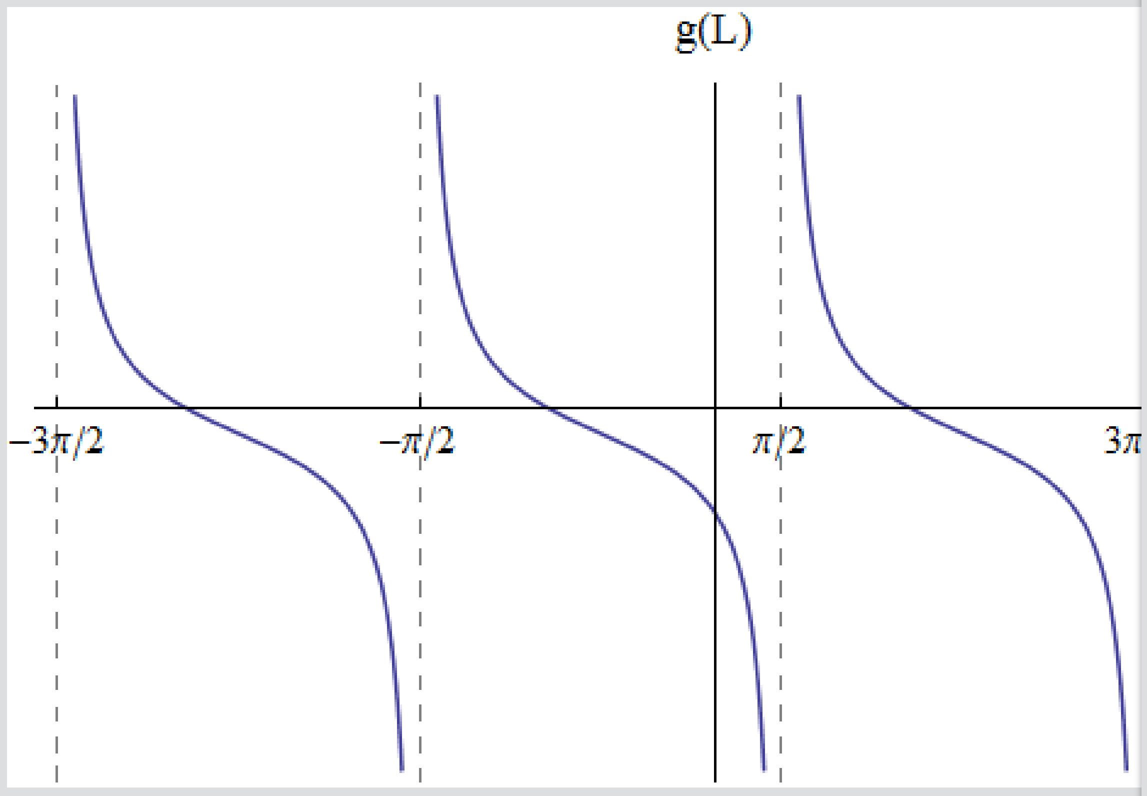
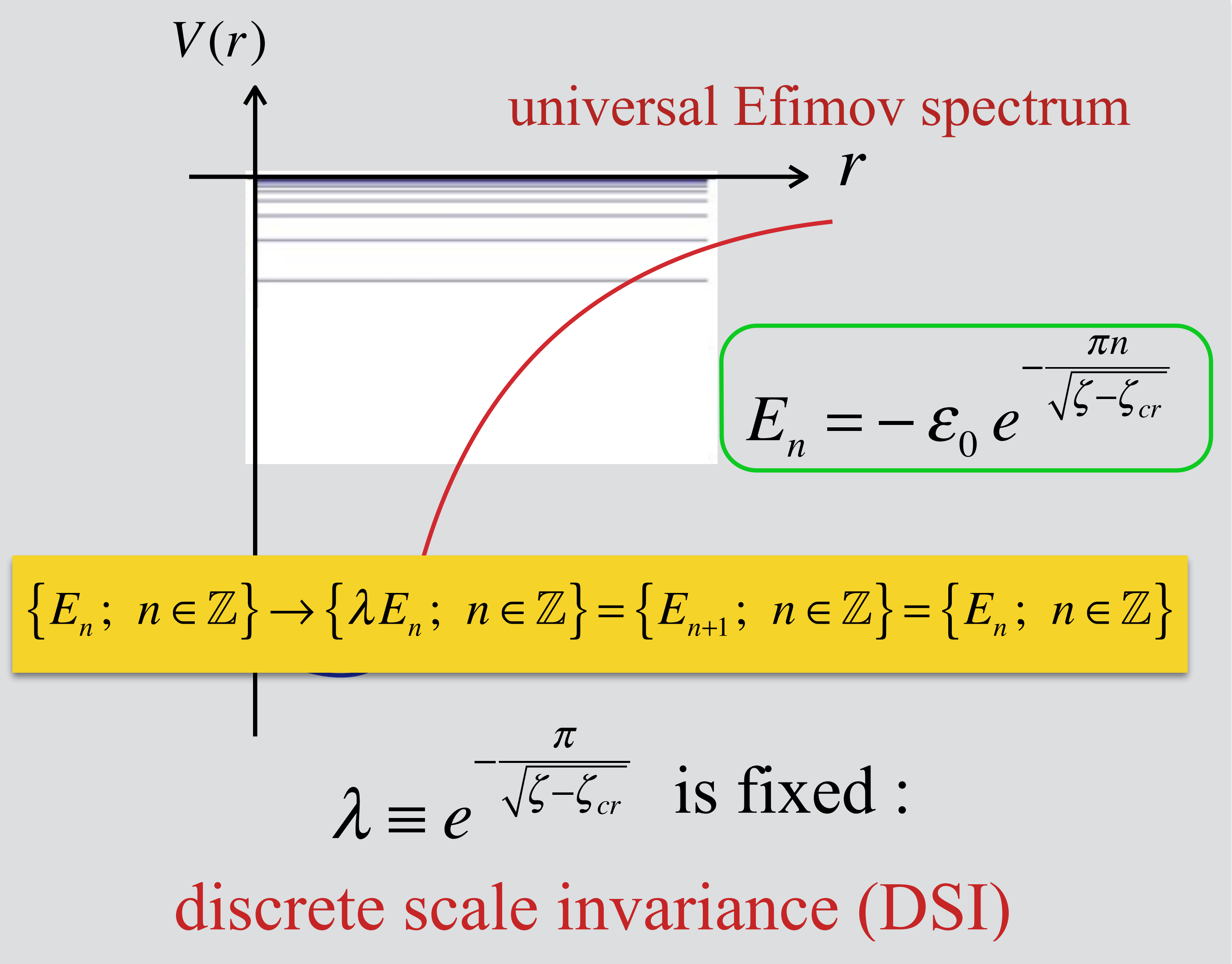




universal Efimov spectrum

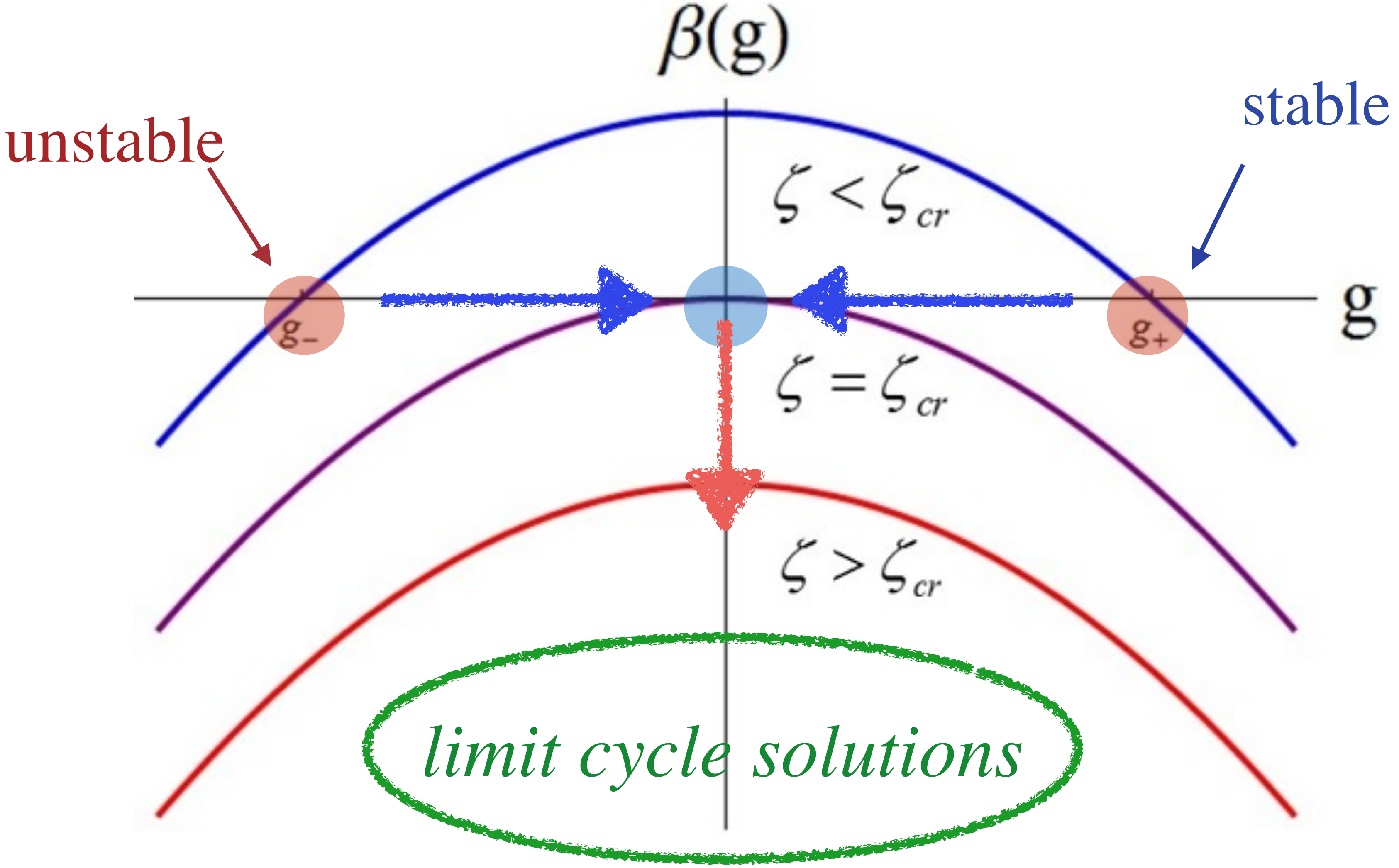
$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta-\zeta_{cr}}}}$$

$$\{E_n;\ n\in\mathbb{Z}\} \to \{\lambda E_n;\ n\in\mathbb{Z}\} = \{E_{n+1};\ n\in\mathbb{Z}\} = \{E_n;\ n\in\mathbb{Z}\}$$

$\lambda \equiv e^{-\frac{\pi}{\sqrt{\zeta-\zeta_{cr}}}}$ is fixed :

discrete scale invariance (DSI)

Breaking of CSI into DSI is now interpreted as a transition of the RG flow from a stable fixed point into the emergence of limit cycle solutions.



Dirac equation + Coulomb : The graphene approach

Dirac equation + Coulomb potential

Continuous scale invariance (CSI) of the Hamiltonian :

$$\hat{H} = -\frac{\hbar^2}{2\mu}\Delta - \frac{\xi}{r^2}$$

A immediate question : What about the Dirac eq. with a Coulomb potential ?

Dirac eq. $\boxed{i\sum_{\mu=0}^{d}\gamma^{\mu}\left(\partial_{\mu}+ieA_{\mu}\right)\Psi\left(x^{\nu}\right)=0}$ is linear with momentum and

Coulomb potential

$$\begin{aligned} &eA_0 = V(r) = -\frac{\xi}{r}, \quad \xi \equiv Z\alpha \\ &A_i = 0, \quad i = 1,\ldots,d \end{aligned}$$

fine structure constant

These two problems share the same continuous scale invariance (CSI).

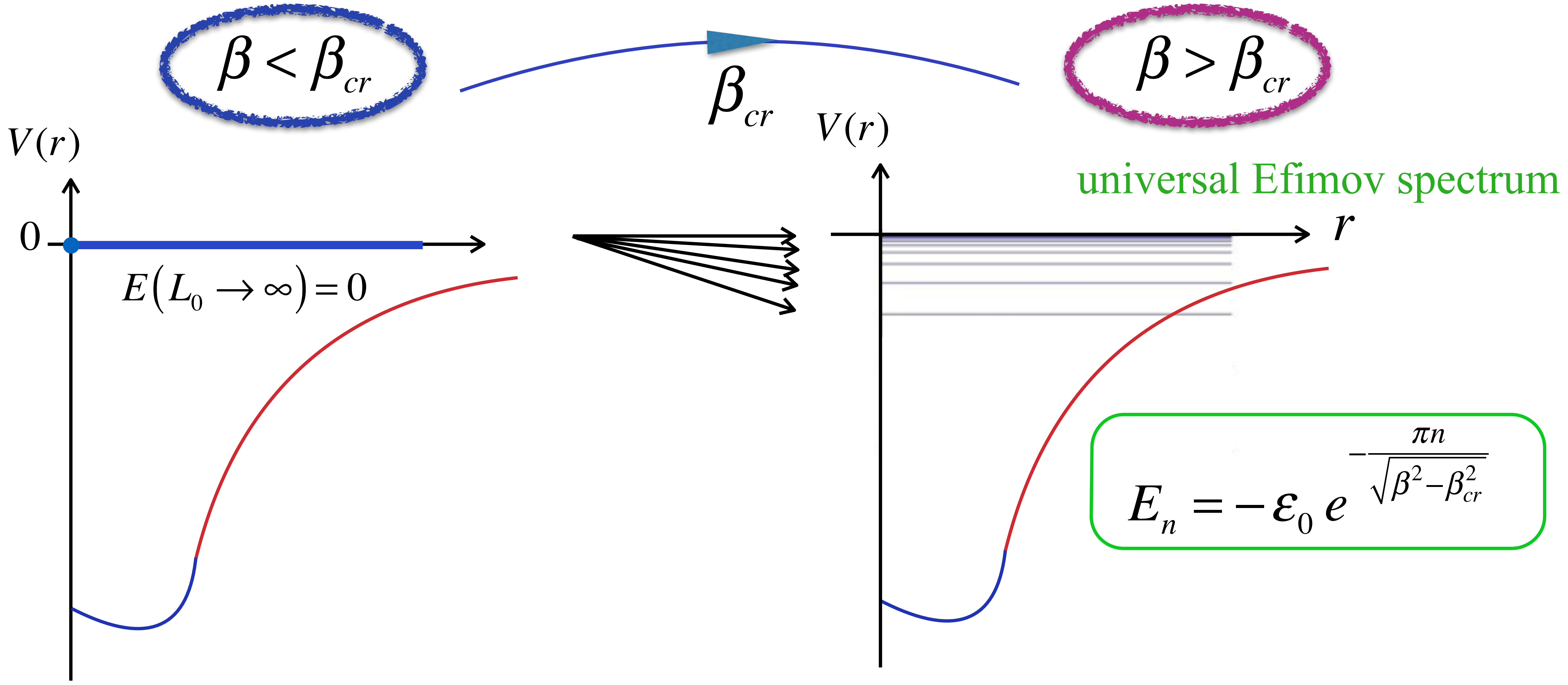
The instability in the Dirac + Coulomb problem is an example of the breaking of CSI into DSI.

Efimov spectrum for the massless Dirac problem obtained using the RG picture.

Dirac quantum phase transition

Dimensionless coupling $\beta \equiv Z\alpha$

Singular value $\boxed{\beta_{cr} = \frac{d-1}{2} = \frac{1}{2}}$



$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\beta^2 - \beta_{cr}^2}}}}$$

Continuous scale invariance (CSI)

Discrete scale invariance (DSI)

Problem : to observe this instability, we need $Z \geq {}^{1}\!/_{\alpha} \simeq 137$

No such stable nuclei have been created.

Problem : to observe this instability, we need $Z \geq 1/\alpha \simeq 137$

No such stable nuclei have been created.

Idea: consider analogous condensed matter systems with a "much larger effective fine structure constant".

Graphene : Effective massless Dirac excitations with a Fermi velocity $v_F \simeq 10^6\, m/s$ so that $\boxed{\alpha_G = e^2/\hbar v_F \approx 2.5}$

and $Z_c \geq 1/\alpha_G \simeq 0.4$

($Z_c \simeq 1$ with screening effects)

Graphene : Effective massless Dirac excitations with a Fermi velocity $v_F \simeq 10^6 \, m/s$ so that

$$\boxed{\alpha_G = e^2/\hbar v_F \approx 2.5}$$

and $Z_c \geq 1/\alpha_G \simeq 0.4$

$(Z_c \simeq 1$ with screening effects)

- Charged impurities in graphene (Coulomb potential)

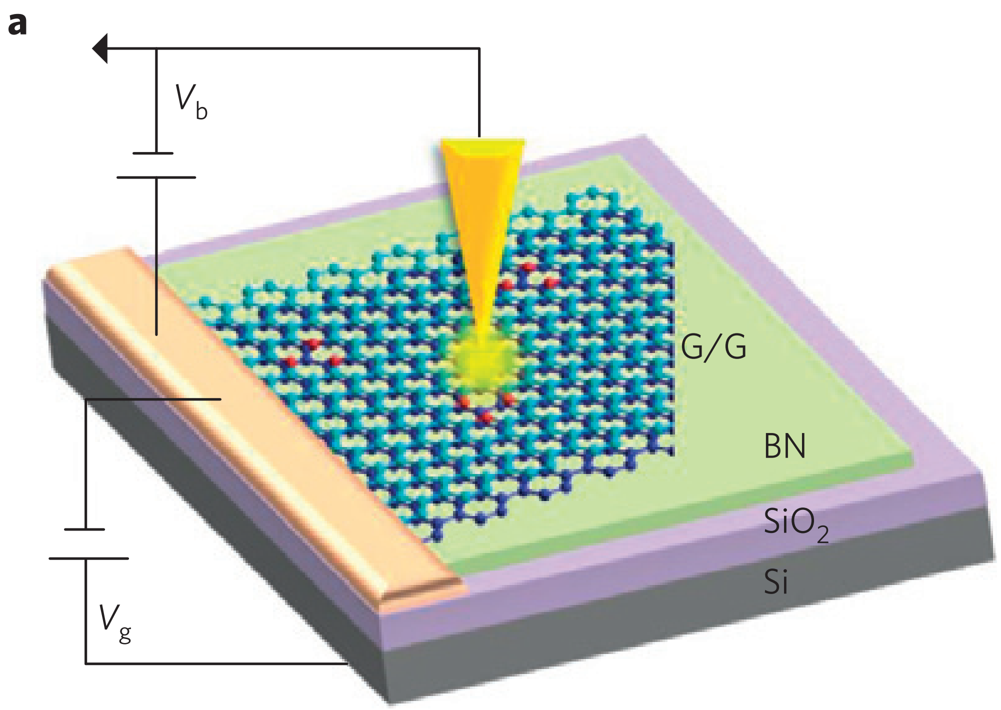
$\Rightarrow$ scattering of quasi- bound states

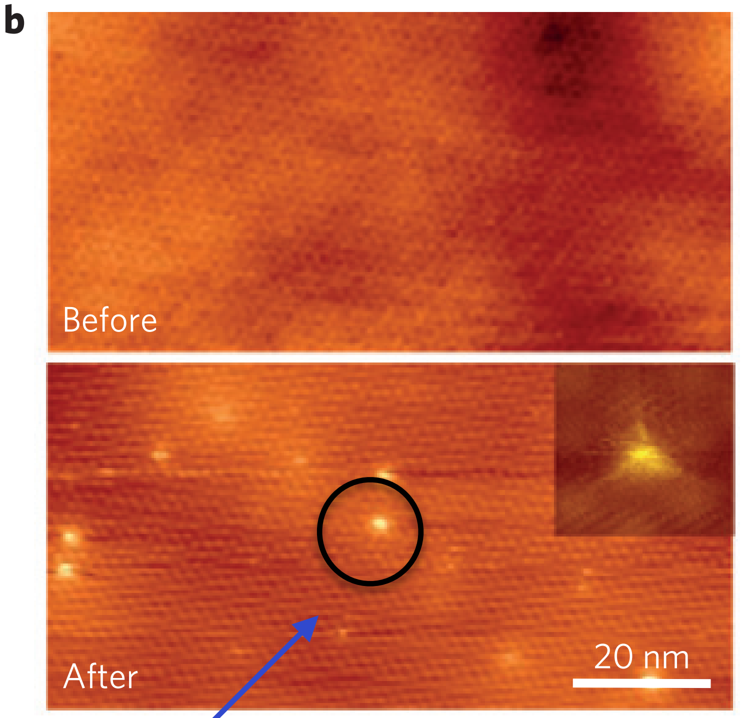
$\Rightarrow$ singular behaviour of the total phase shift

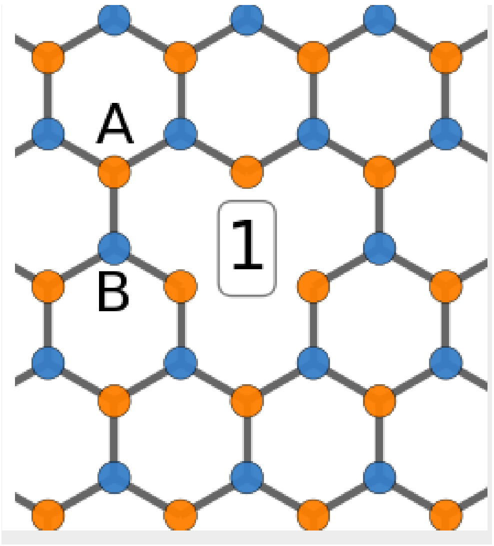
Dirac equation + Coulomb : The experiment

Jinhai Mao, Eva Andrei et al. Nature Physics (2016)

Building an artificial atom in graphene







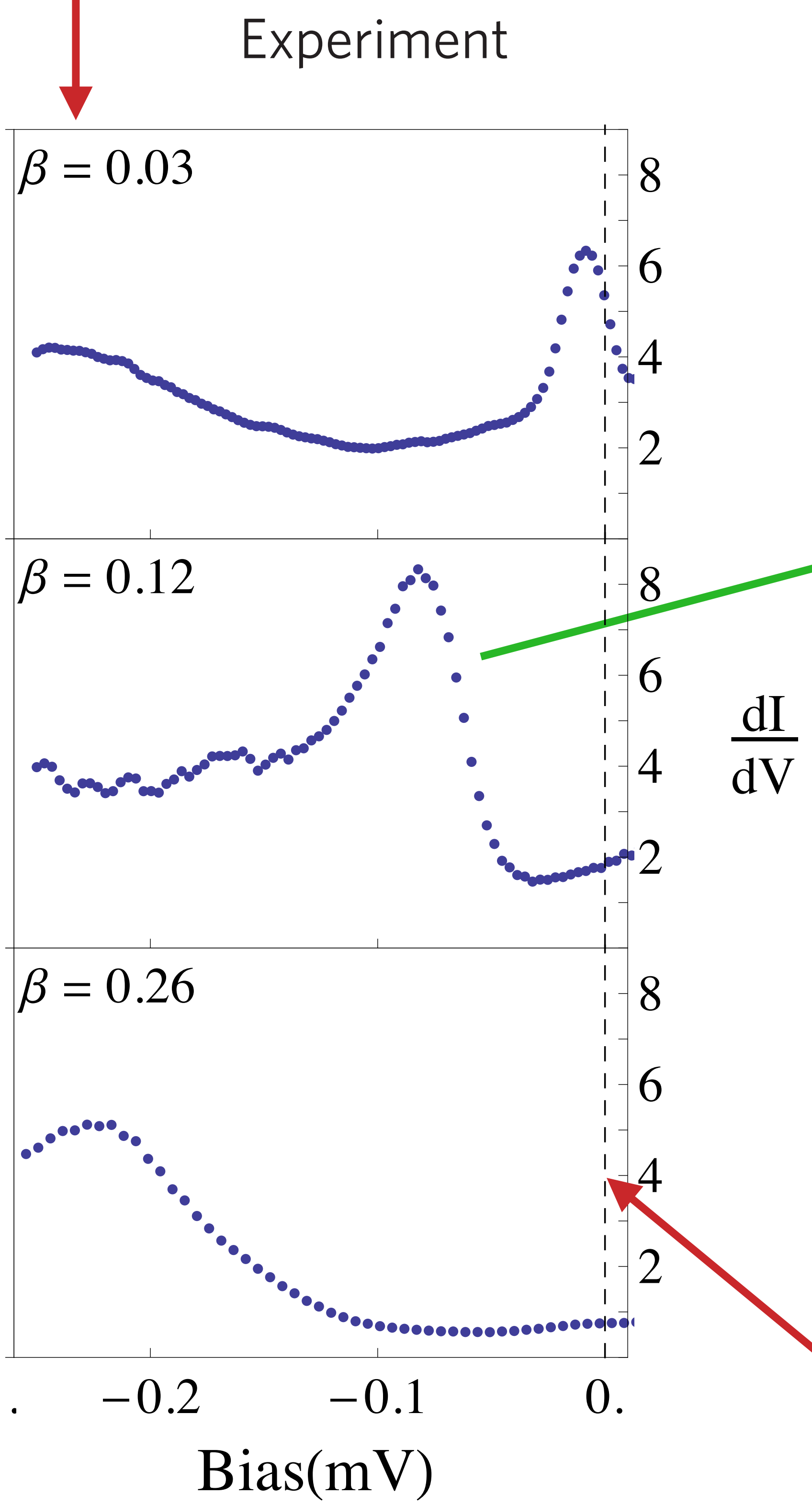
Local vacancy. Local charge is changed by applying voltage pulses with the tip of an STM

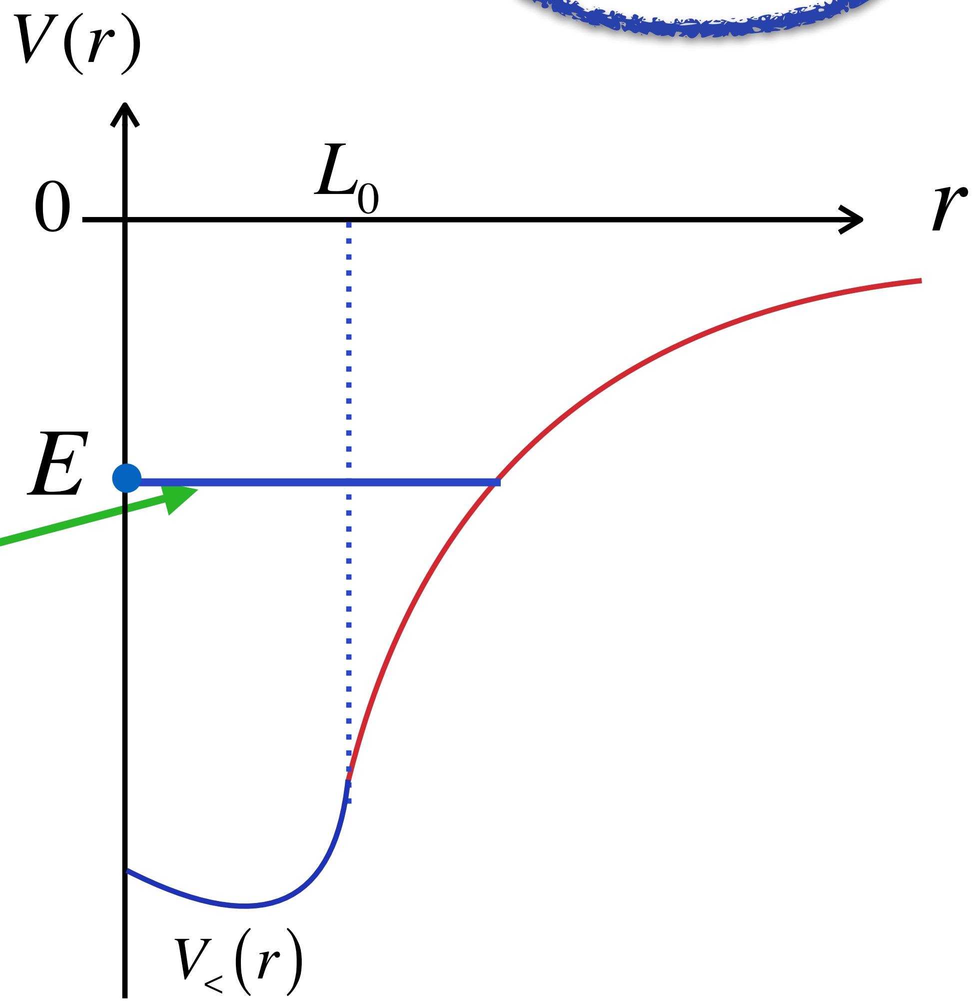
Jinhai Mao, Eva Andrei et al. Nature Physics (2016)

Increasing the charge, quasi-bound states are trapped.

$$\beta < \beta_{cr} \equiv \frac{1}{2}$$

$$\boxed{\beta < \beta_{cr}}$$

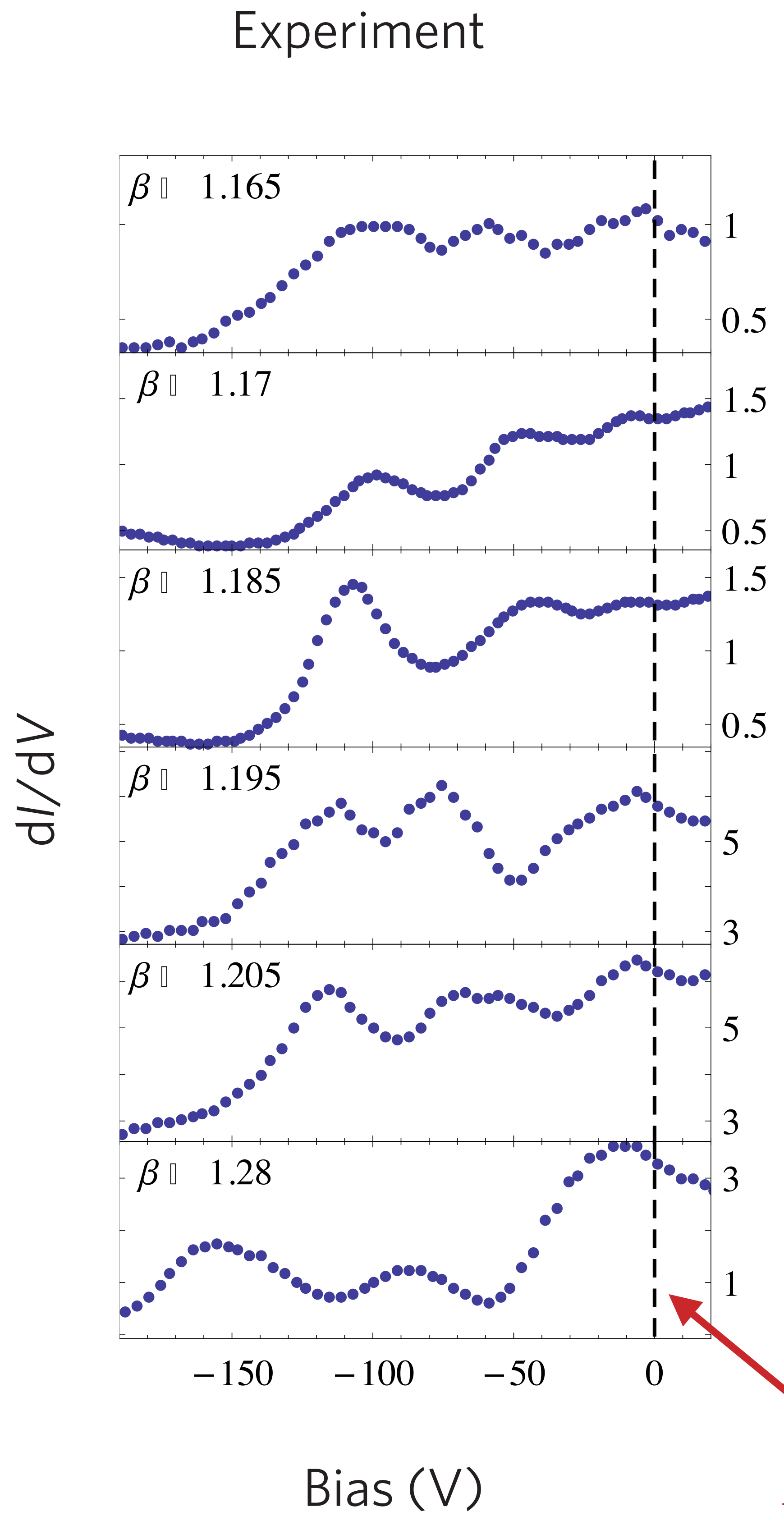


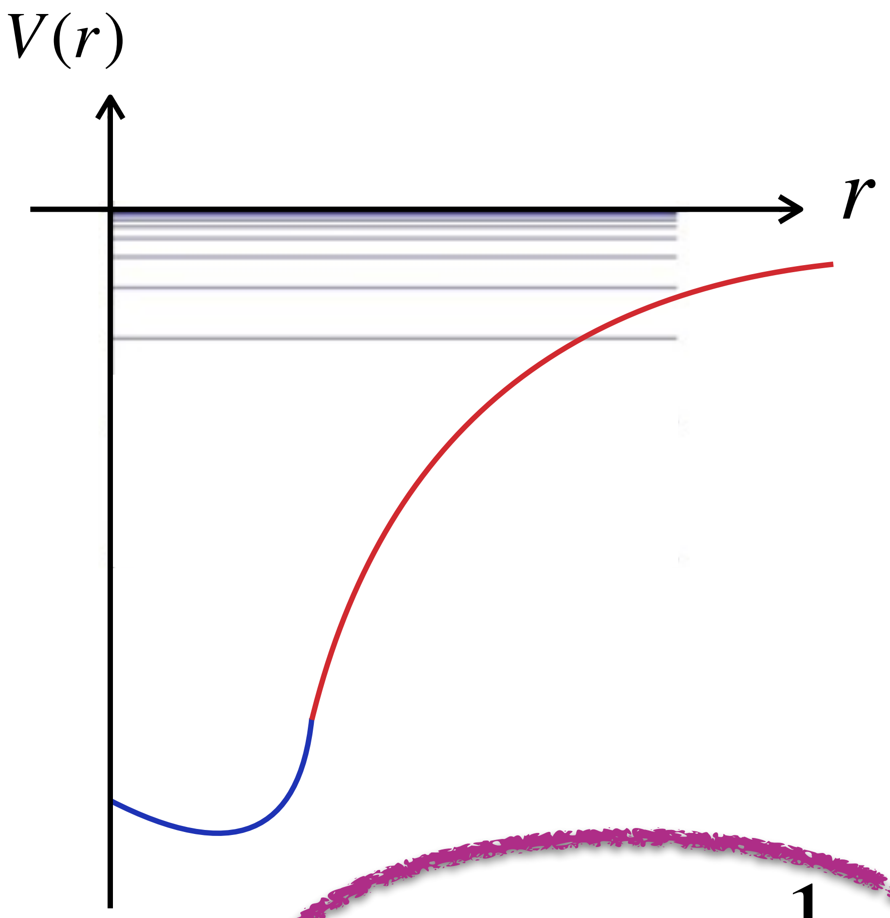


Measure the local density of states from the tunnelling conductance of the STM

Dirac point

Increasing the charge, quasi-bound states are trapped. For a large enough coupling, a discrete set of Efimov states shows up.

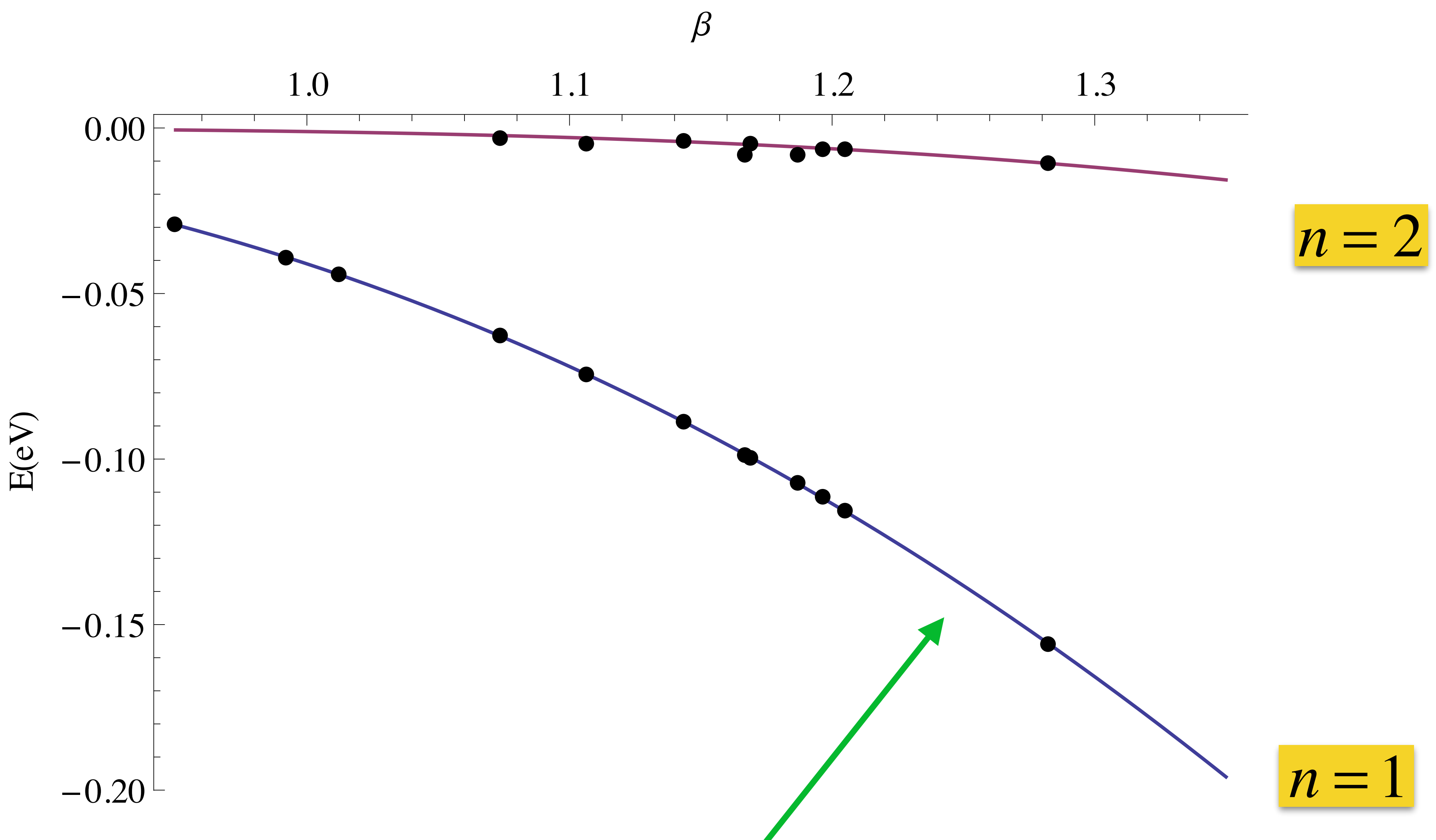




$$\beta > \beta_{cr} \equiv \frac{1}{2}$$

Dirac point

The Efimov universal spectrum



$$\beta > \beta_{cr} \equiv \frac{1}{2}$$

$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\beta^2 - \beta_{cr}^2}}}$$

Omrie Ovdat, J. Mao, Eva Andrei, E.A (2016)

What is Efimov physics ?

Universality in cold atomic gases

DSI in the non relativistic quantum 3-body problem

Universality in cold atomic gases

non relativistic quantum 3-body problem

3-body (nucleon) system interacting through zero-range interactions (r_0)
Existence of universal physics at low energies, $E \ll \hbar^2 / m r_0^2$

When the scattering length a of the 2-body interaction becomes $a \gg r_0$
there is a sequence of 3-body bound states whose binding energies are
spaced geometrically in the interval between $\hbar^2 / m a^2$ and $\hbar^2 / m r_0^2$

As $|a|$ increases, new bound states appear according to

$$\boxed{E_n = -\varepsilon_0 \, e^{-\frac{2\pi n}{s_0}}}$$

Efimov spectrum

where $s_0 \approx 1.00624$ is a universal number

The corresponding 3-body problem reduces to an <u>effective</u> Schrödinger equation with the attractive potential :

$$V(r) = -\frac{s_0^2 + 1/4}{r^2}$$

Efimov physics is always super-critical :

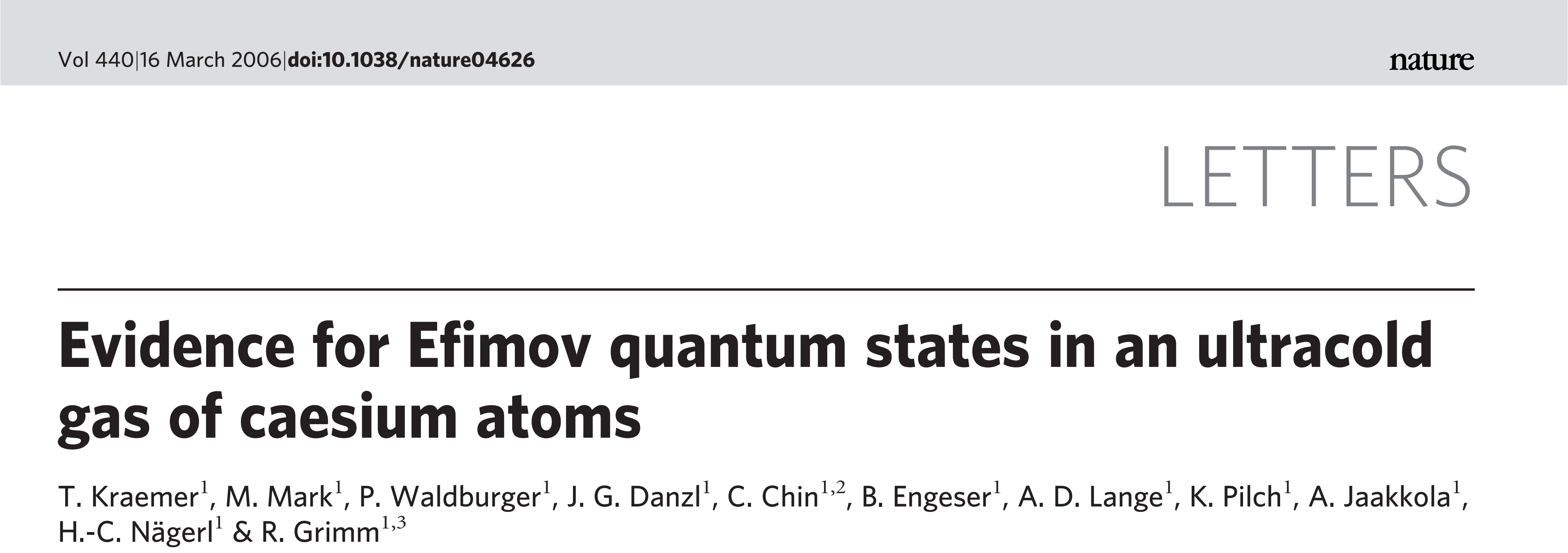
Schrodinger equation with an effective attractive potential $(d=3)$:

$$\boxed{V(r) = -\frac{s_0^2 + 1/4}{r^2}} \qquad s_0 \approx 1.00624$$

$$\zeta_{cr} = \frac{(d-2)^2}{4} = \frac{1}{4} \qquad \Longrightarrow$$ Efimov physics occurs at :

$$\boxed{\zeta_E = s_0^2 + 1/4 = 1.26251 > \zeta_{cr}}$$

ζ_E is <u>fixed</u> in Efimov physics. It cannot be changed !



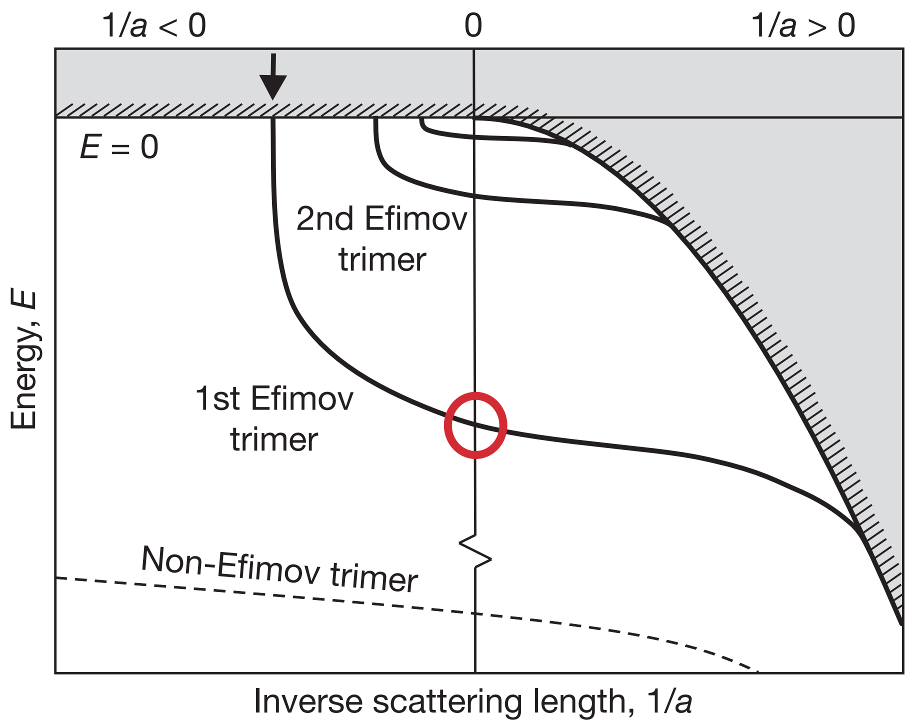
Vol 440|16 March 2006|**doi:10.1038/nature04626** nature

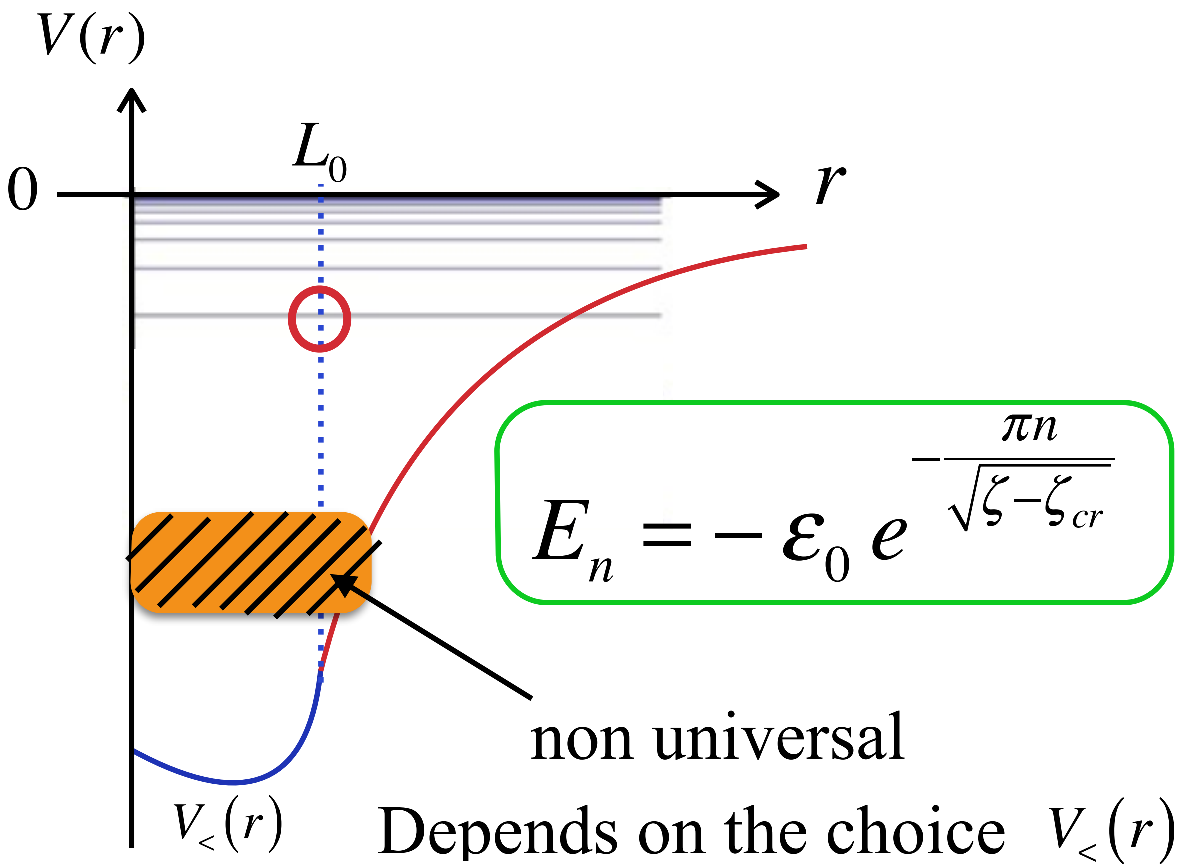
LETTERS

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer[1], M. Mark[1], P. Waldburger[1], J. G. Danzl[1], C. Chin[1,2], B. Engeser[1], A. D. Lange[1], K. Pilch[1], A. Jaakkola[1], H.-C. Nägerl[1] & R. Grimm[1,3]

Measurement of a single Efimov state : n=1

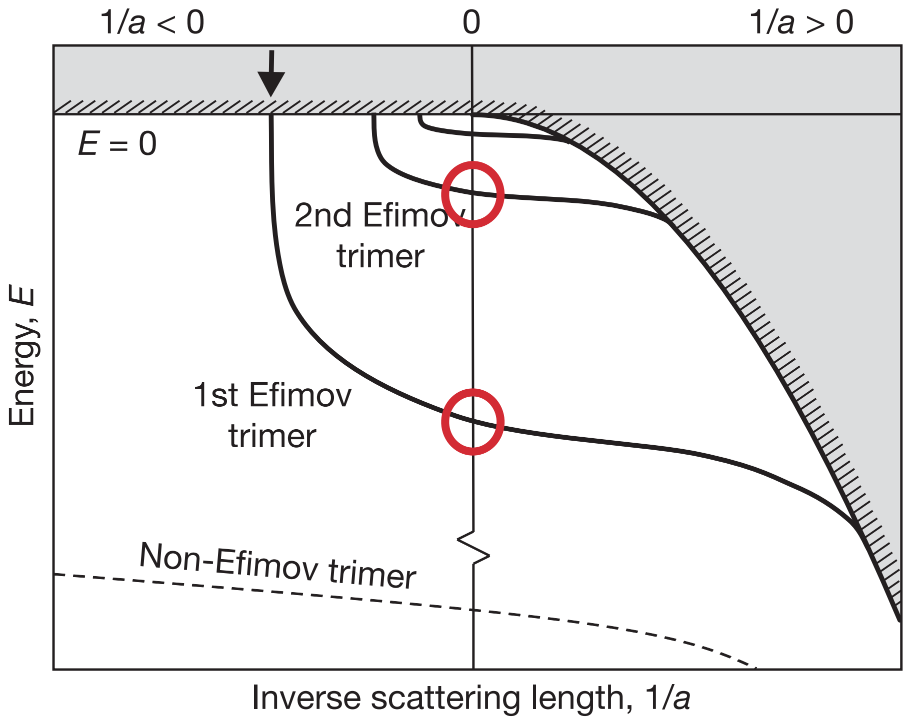


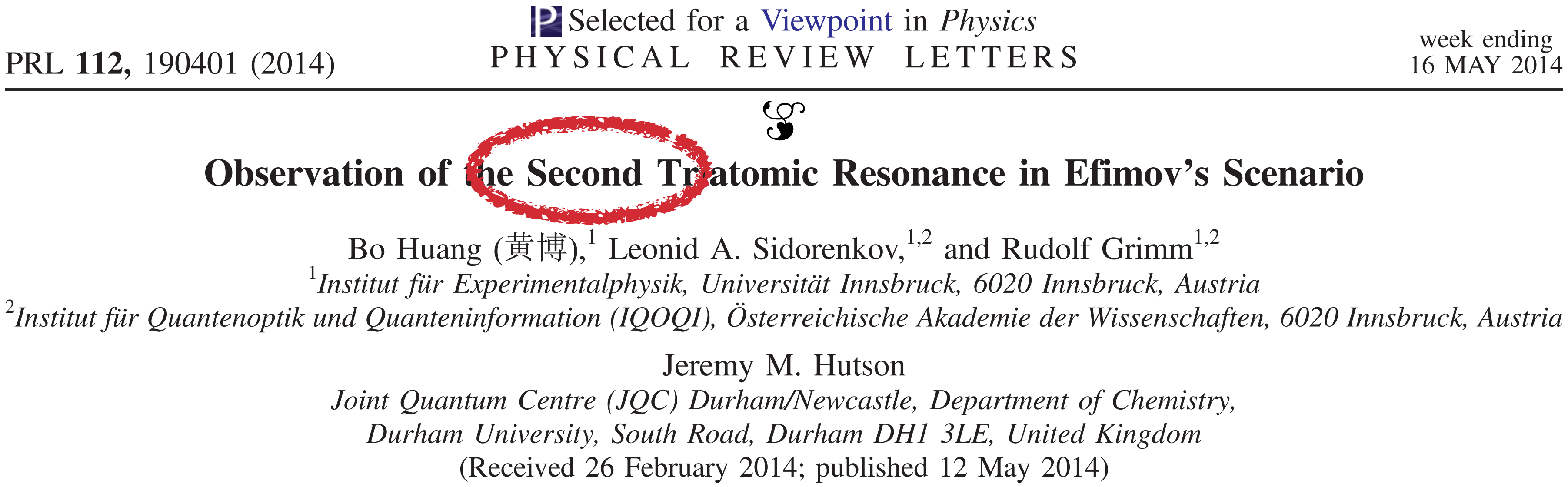


$$E_n = -\varepsilon_0 \, e^{-\frac{\pi n}{\sqrt{\zeta-\zeta_{cr}}}}$$

non universal

Depends on the choice $V_<(r)$





Selected for a Viewpoint in *Physics*

PRL **112,** 190401 (2014) PHYSICAL REVIEW LETTERS week ending 16 MAY 2014

Observation of the Second Triatomic Resonance in Efimov's Scenario

Bo Huang (黄博),[1] Leonid A. Sidorenkov,[1,2] and Rudolf Grimm[1,2]
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[2]*Institut für Quantenoptik und Quanteninformation (IQOQI), Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria*
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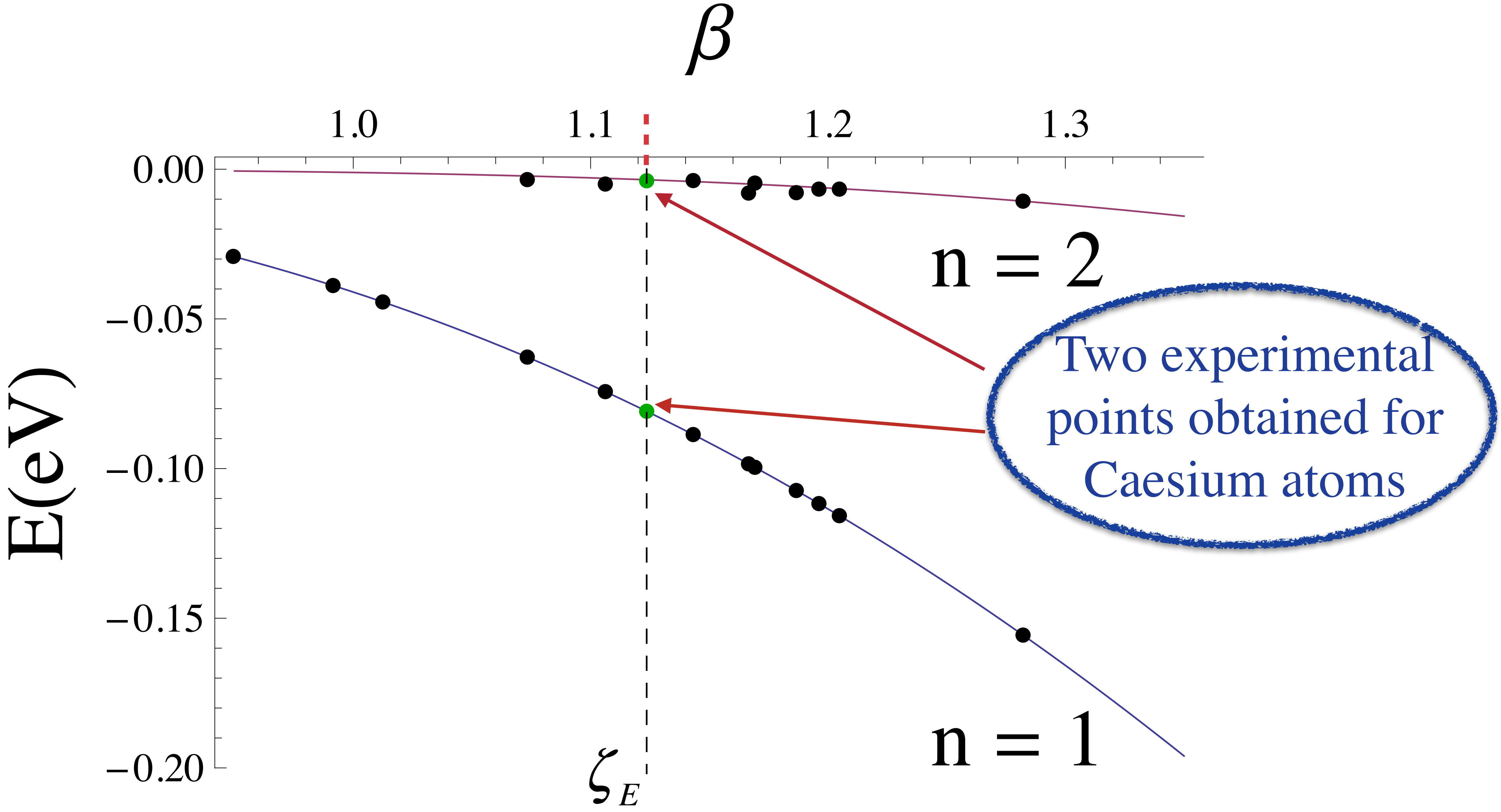
Durham University, South Road, Durham DH1 3LE, United Kingdom

(Received 26 February 2014; published 12 May 2014)

Measurement of a second Efimov state : n=2

Universality



Two experimental points obtained for Caesium atoms

Not obvious at all ! Two very different physical phenomena share the same universal energy spectrum.

Summary-Further directions

- Breaking of continuous scale invariance (CSI) into discrete scale invariance (DSI) on two examples.
- Observed this quantum phase transition on graphene. It raises more questions than it solved.
- Efimov physics belongs to this universality class. It does not allow observing the transition.

- Other problems can be described similarly as “conformality lost” (Kaplan et al., 2009) and emergence of limit cycles:
  - Kosterlitz-Thouless transition (deconfinement of vortices in the XY-model at a critical temp. above which the theory is conformal): mapping between the XY-model and the T=0 sine-Gordon in 1+1 dim.

$$\boxed{L = \frac{T}{2}\left(\partial_\mu \phi\right)^2 - 2z\cos\phi}$$

Thank you for your attention.