Fractal calculus from fractal arithmetic

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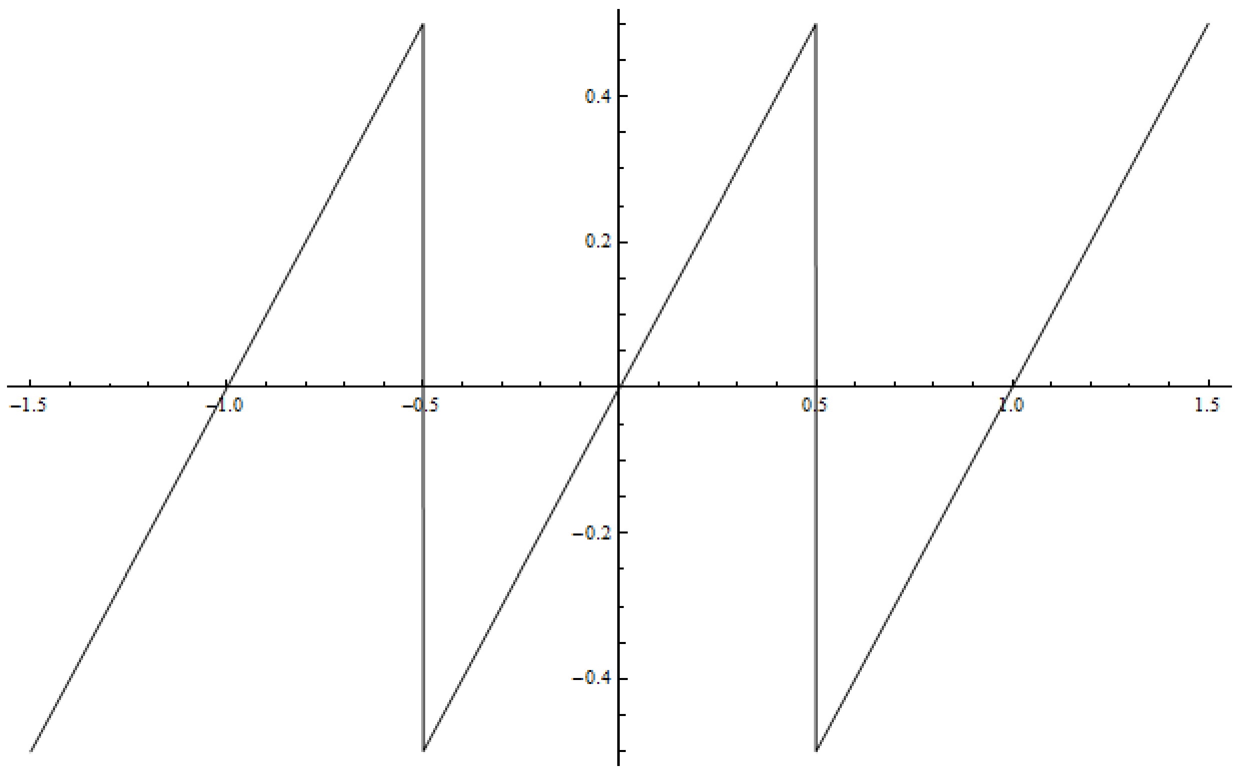
POLITECHNIKA GDAŃSKA



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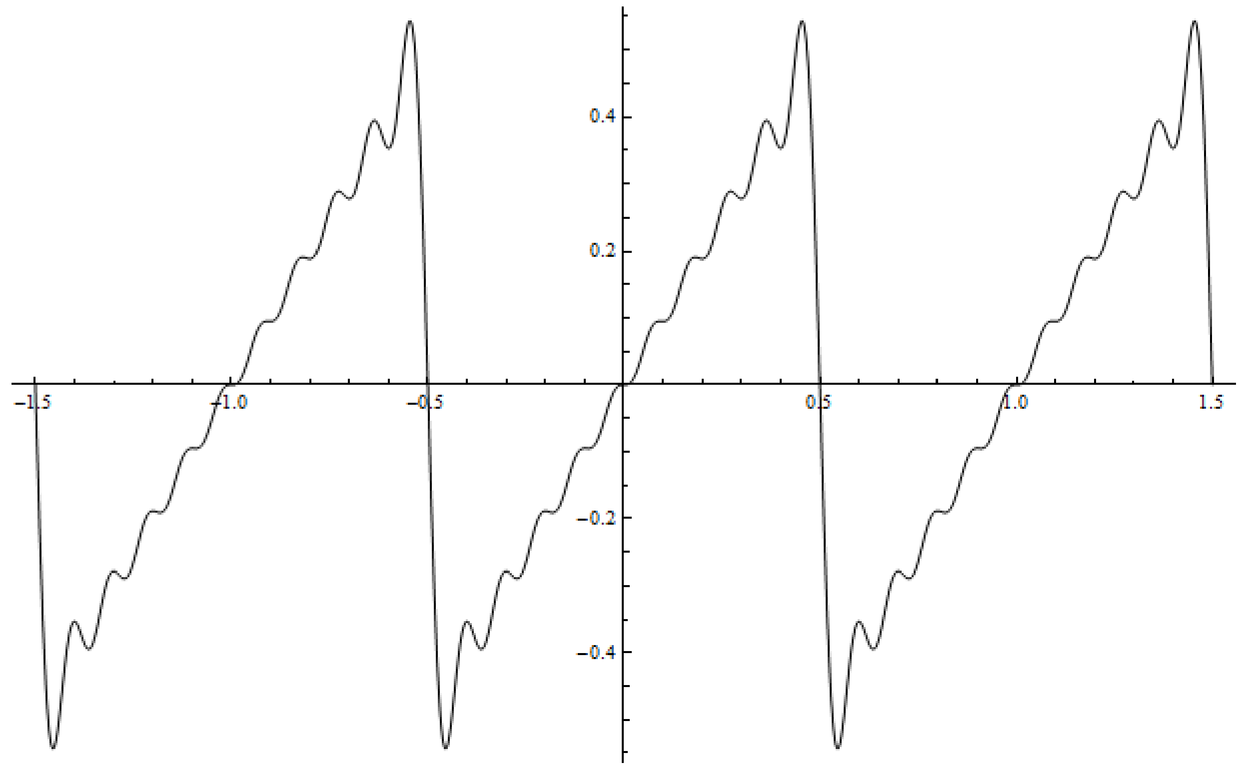

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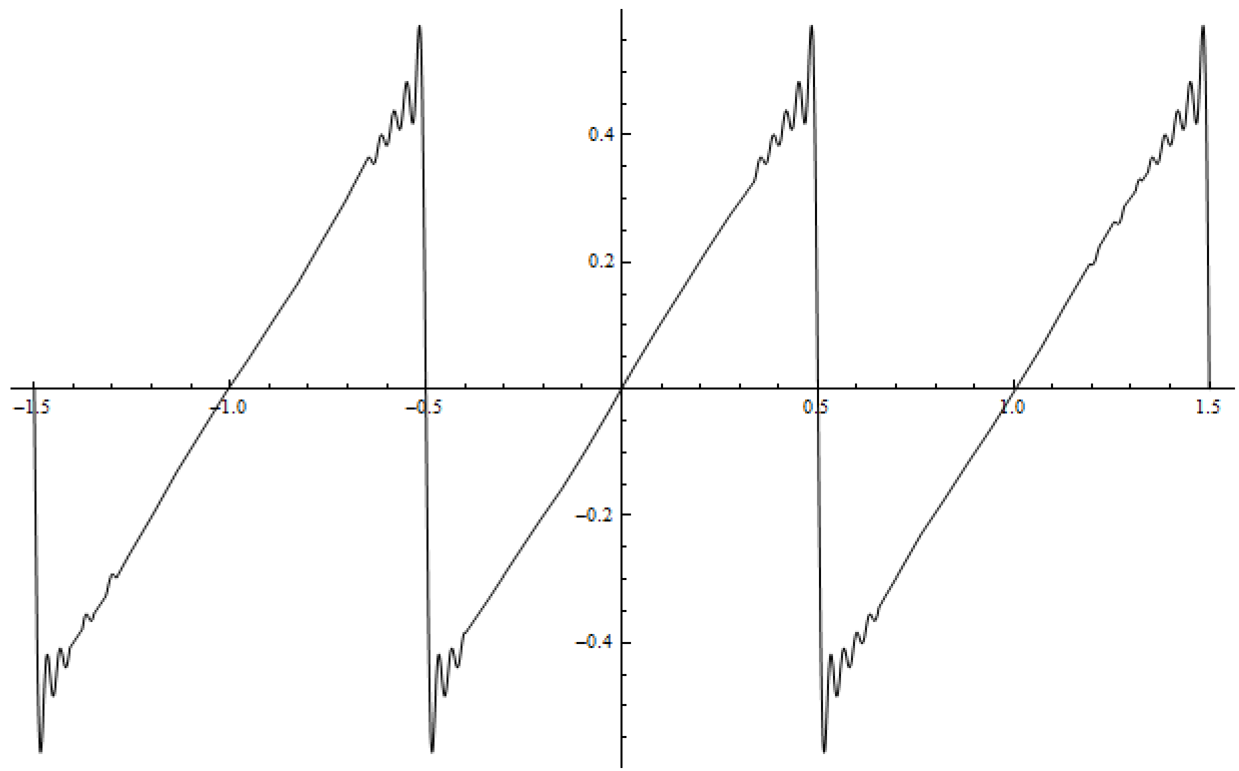

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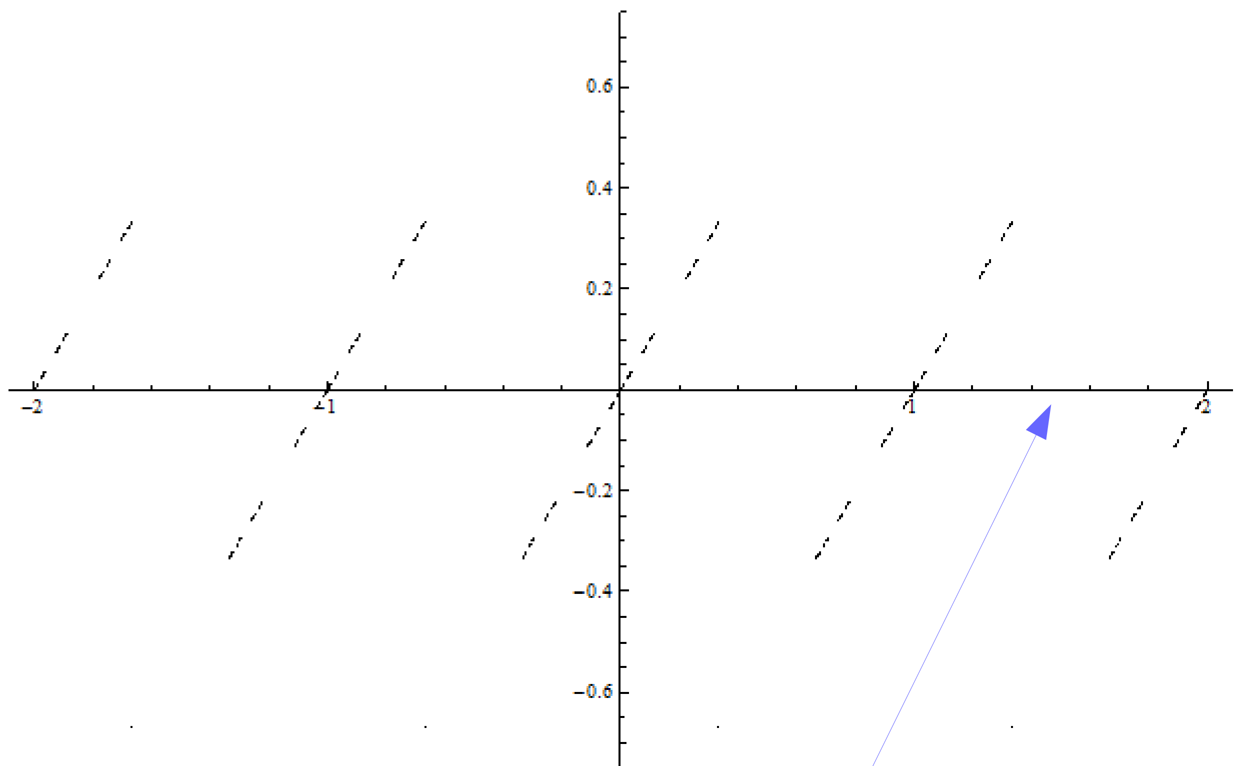
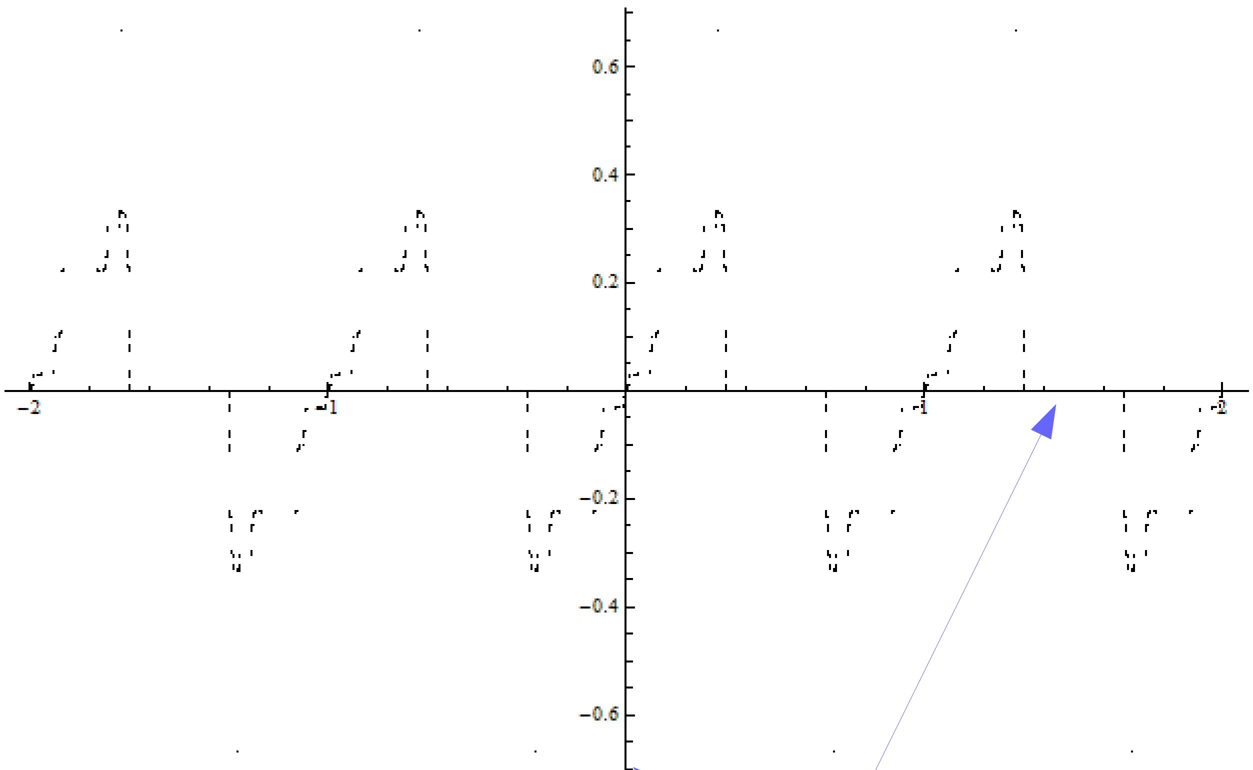


$$\mathbb{X} = \cup_{k\in\mathbb{Z}} C_{-}(k, k+1)$$

Sum of translated middle-third Cantor sets



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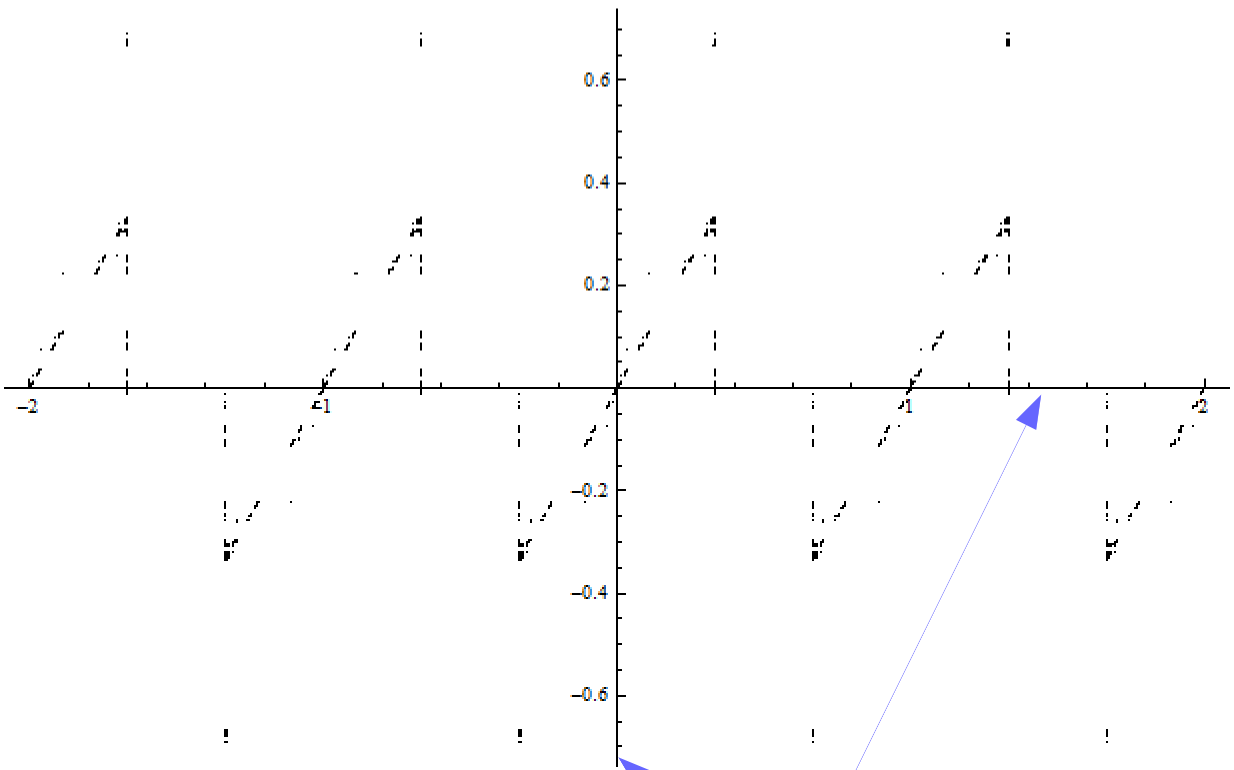

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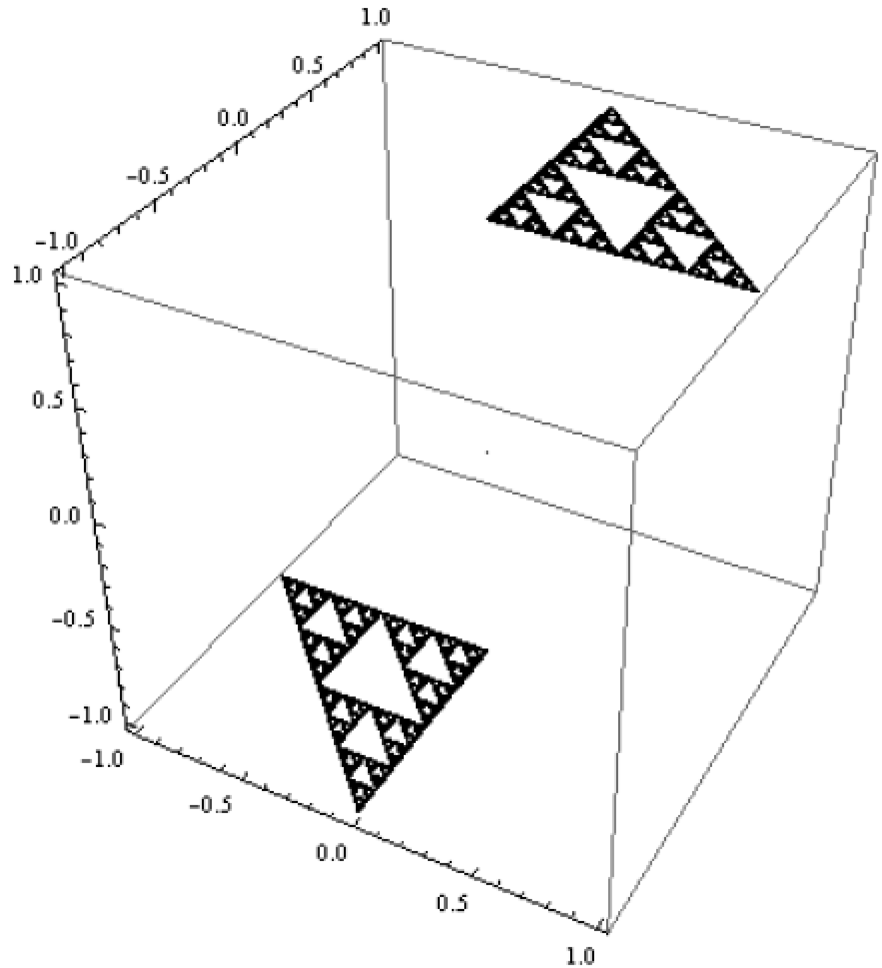
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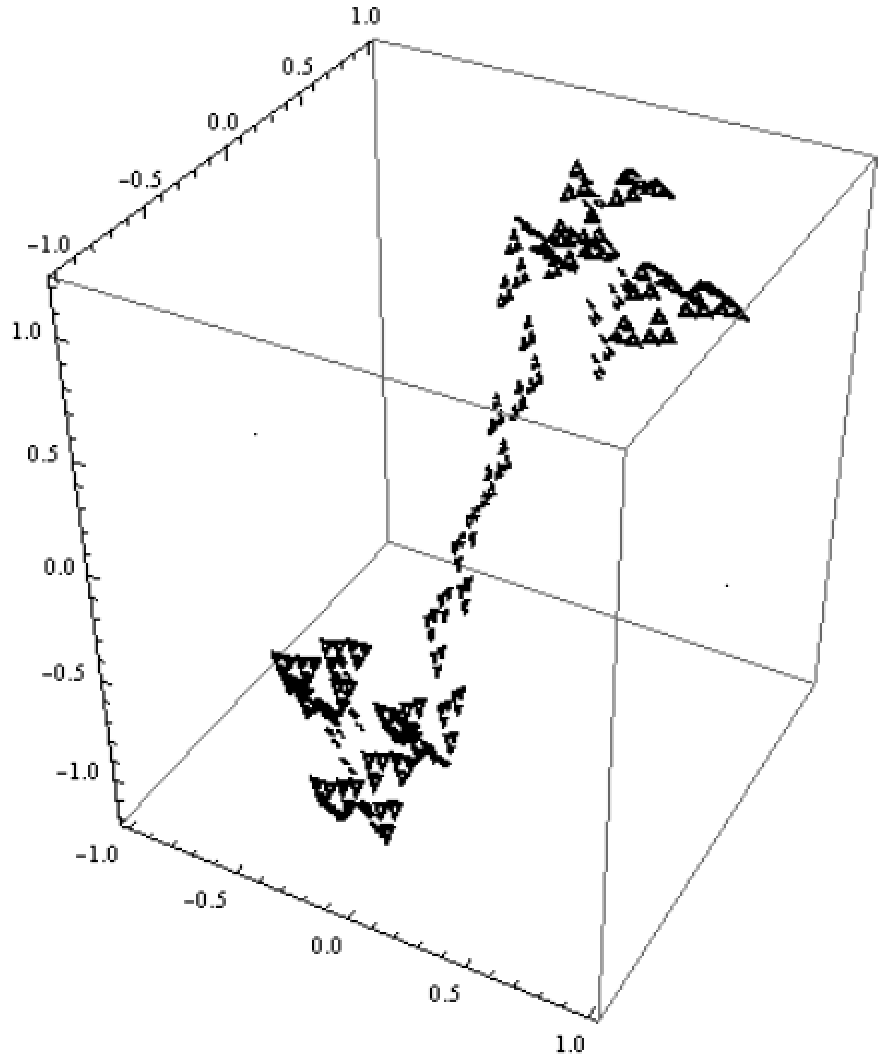

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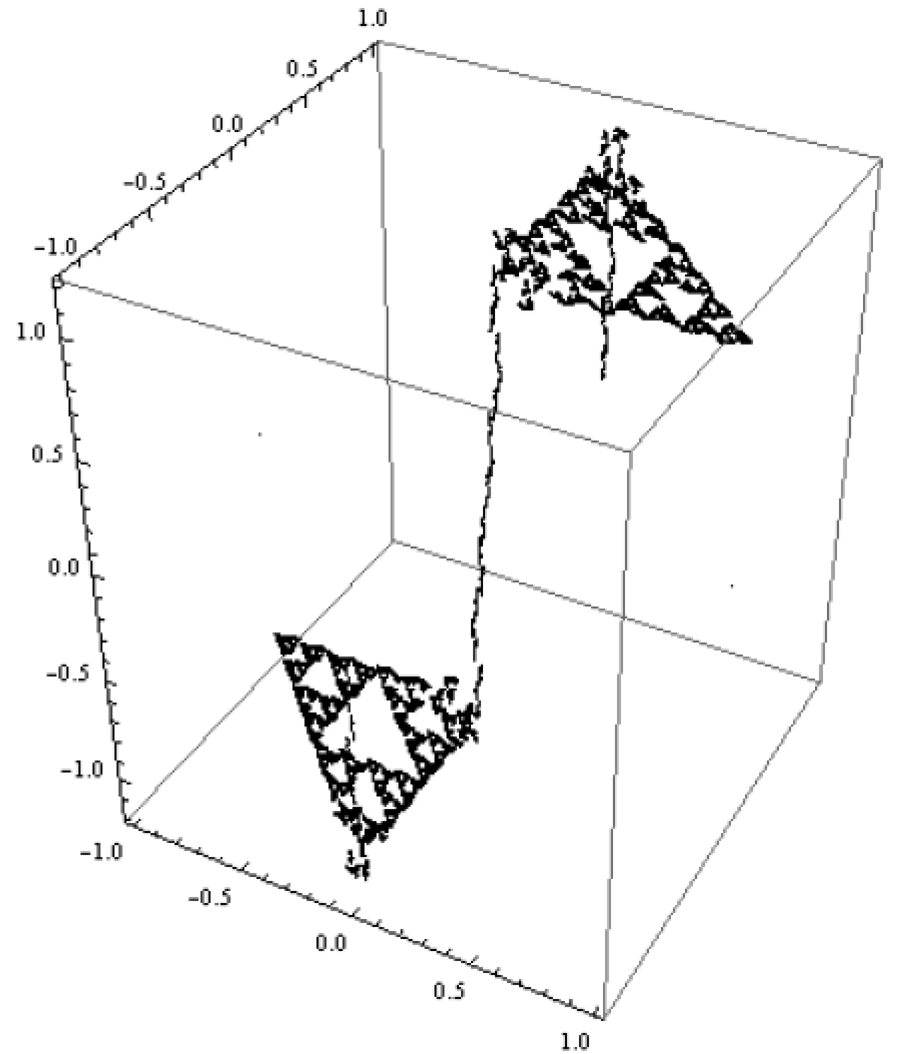
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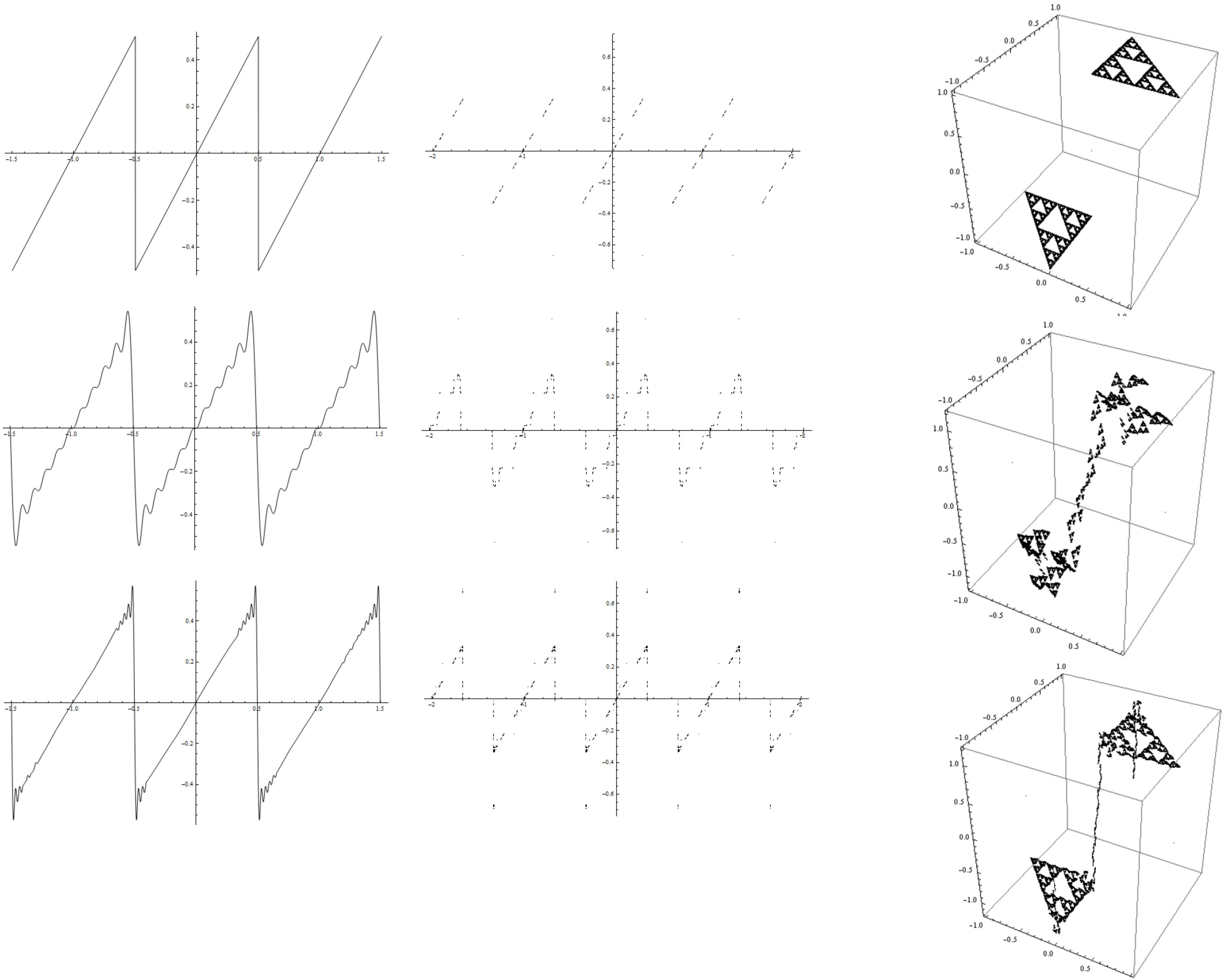


Assumptions

$\mathbb{X}$ any set whose cardinality is continuum

$f : \mathbb{X} \to \mathbb{R}$ a bijection

Continuity at 0 of the inverse map

$$\lim_{x \to 0_-} f^{-1}(x) = \lim_{x \to 0_+} f^{-1}(x)$$

Arithmetic in $\mathbb{X}$ (field isomorphism)

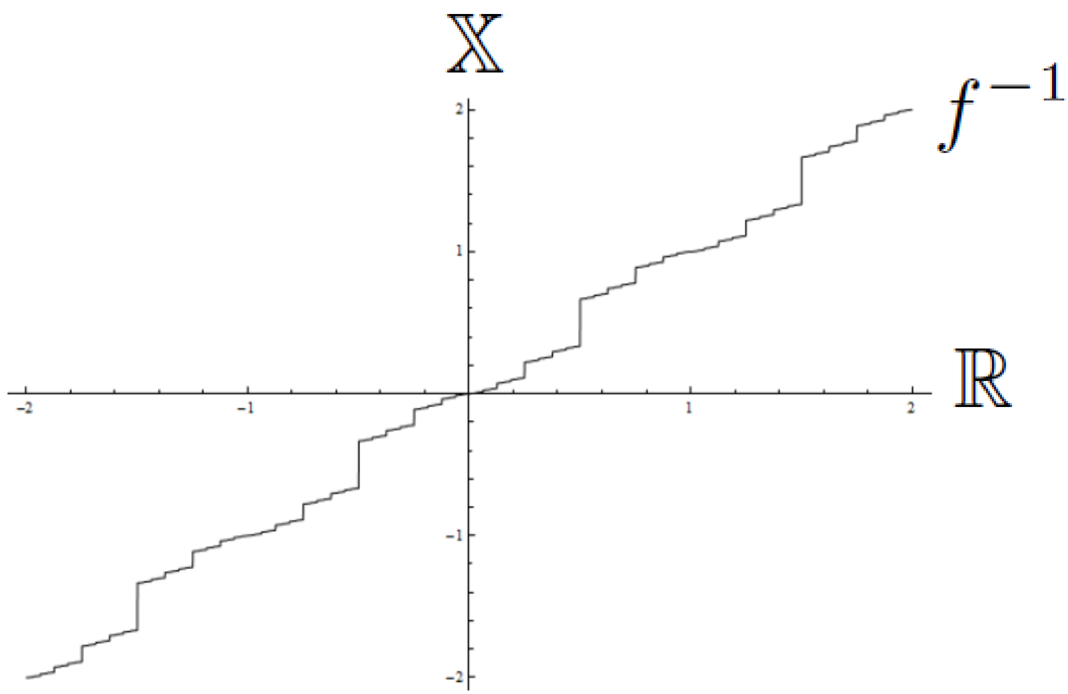
$$
\begin{aligned}
x \oplus y &= f^{-1}\big(f(x) + f(y)\big) \\
x \ominus y &= f^{-1}\big(f(x) - f(y)\big) \\
x \odot y &= f^{-1}\big(f(x) f(y)\big) \\
x \oslash y &= f^{-1}\big(f(x) / f(y)\big)
\end{aligned}
$$

One verifies the standard properties: (1) associativity $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, $(x \odot y) \odot z = x \odot (y \odot z)$, (2) commutativity $x \oplus y = y \oplus x$, $x \odot y = y \odot x$, (3) distributivity $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$. Elements $0', 1' \in \mathbb{X}$ are defined by $0' \oplus x = x$, $1' \odot x = x$, which implies $f(0') = 0$, $f(1') = 1$. One further finds $x \ominus x = 0'$, $x \oslash x = 1'$, as expected. A negative of $x \in \mathbb{X}$ is defined as $\ominus x = 0' \ominus x = f^{-1}\big(-f(x)\big)$, i.e. $f(\ominus x) = -f(x)$ and $f(\ominus 1') = -f(1') = -1$, i.e. $\ominus 1' = f^{-1}(-1)$. Notice that

$$
(\ominus 1') \odot (\ominus 1') \;=\; f^{-1}\big(f(\ominus 1')^2\big) = f^{-1}(1) = 1'.
$$

Example: Triadic middle-third Cantor set (details later)

$$
\begin{aligned}
n' &= f^{-1}(n), \quad n \in \mathbb{N}, \\
1' &= 1, \\
0' &= 0, \\
1' \oslash 2' &= 1/3 = f^{-1}(1/2), \\
(1/3) \oplus (1/3) &= (1' \oslash 2') \oplus (1' \oslash 2') = 2' \odot (1' \oslash 2') = 1' = 1
\end{aligned}
$$



Multiplication can be regarded as repeated addition in the following sense. Let $n \in \mathbb{N}$ and $n' = f^{-1}(n) \in \mathbb{X}$. Then

$$\begin{aligned} n' \oplus m' &= (n+m)', && (2)\\ n' \odot m' &= (nm)' && (3)\\ &= \underbrace{m' \oplus \cdots \oplus m'}_{n\text{times}}. && (4) \end{aligned}$$

In particular $n' = 1' \oplus \cdots \oplus 1'$ (n times).

A power function $A(x) = x \odot \cdots \odot x$ (n times) will be denoted by $x^{n'}$. Such a notation is consistent in the sense that

$$x^{n'} \odot x^{m'} = x^{(n+m)'} = x^{n' \oplus m'}. \tag{5}$$

The derivative of a function $A : \mathbb{X} \to \mathbb{X}$

$$\frac{DA(X)}{DX} = \lim_{H \to 0'} \Big(A(X \oplus H) \ominus A(X) \Big) \oslash H$$
$$= \lim_{h \to 0} \Big(A\big(X \oplus f^{-1}(h)\big) \ominus A(X) \Big) \oslash f^{-1}(h)$$

Examples:

$$\mathrm{Cos}\, X = f^{-1}\big(\cos f(X)\big)$$
$$\mathrm{Sin}\, X = f^{-1}\big(\sin f(X)\big)$$

$$\frac{D\,\mathrm{Sin}\,(K \odot X)}{DX} = K \odot \mathrm{Cos}\,(K \odot X)$$
$$\frac{D\,\mathrm{Cos}\,(K \odot X)}{DX} = \ominus K \odot \mathrm{Sin}\,(K \odot X)$$

$$\frac{DX^{N'}}{DX} = f^{-1}\big(N f(X)^{N-1}\big)$$
$$= f^{-1}\big(f(N') f(X)^{N-1}\big)$$
$$= N' \odot X^{(N-1)'} = N' \odot X^{N' \ominus 1'}$$

Example: $\mathbb{X} = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3 \qquad f^{-1}(x) = \sqrt[3]{x}$$

$$\begin{aligned}
x \odot y &= \sqrt[3]{x^3 y^3} = xy \\
x \oslash y &= \sqrt[3]{x^3 / y^3} = x/y \\
x \oplus y &= \sqrt[3]{x^3 + y^3} \\
x \ominus y &= \sqrt[3]{x^3 - y^3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cos}\ x &= \sqrt[3]{\cos(x^3)} \\
\mathrm{Sin}\ x &= \sqrt[3]{\sin(x^3)}
\end{aligned}$$

$$\begin{aligned}
\frac{D}{Dx}\mathrm{Sin}\ x &= \sqrt[3]{\cos(x^3)} = \mathrm{Cos}\ x \\
\frac{d}{dx}\mathrm{Sin}\ x &= \frac{x^2 \cos(x^3)}{\sin^{2/3}(x^3)} \\
\frac{D}{Dx} e^{x^3/3} &= e^{x^3/3}
\end{aligned}$$

Integral of a function $A : \mathbb{X} \to \mathbb{X}$

$$\int_X^Y A(X')DX' = f^{-1}\left(\int_{f(X)}^{f(Y)} f \circ A \circ f^{-1}(x)dx\right)$$

satisfies

$$\begin{aligned} \frac{D}{DX}\int_Y^X A(X')DX' &= A(X) \\ \int_Y^X \frac{DA(X')}{DX'}DX' &= A(X) \ominus A(Y) \end{aligned}$$

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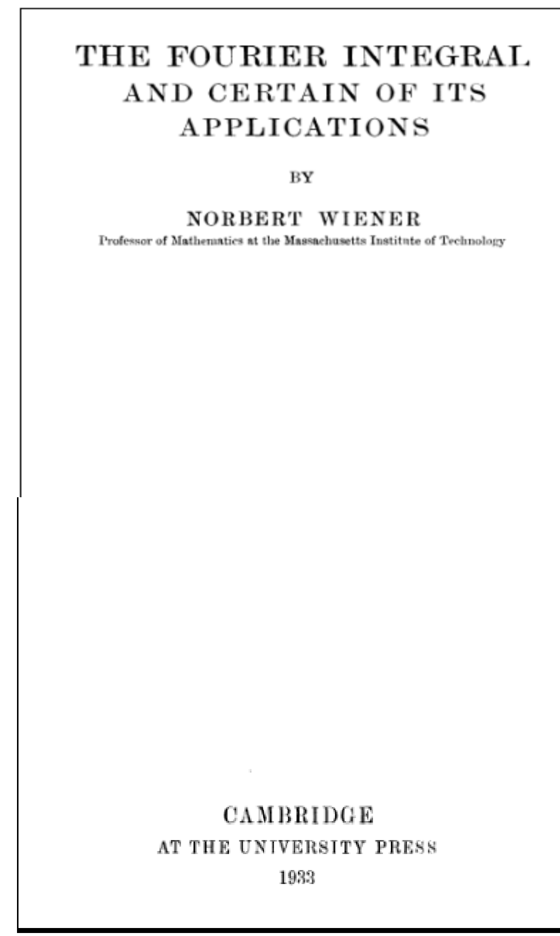


THE FOURIER INTEGRAL
AND CERTAIN OF ITS
APPLICATIONS

BY

NORBERT WIENER
Professor of Mathematics at the Massachusetts Institute of Technology

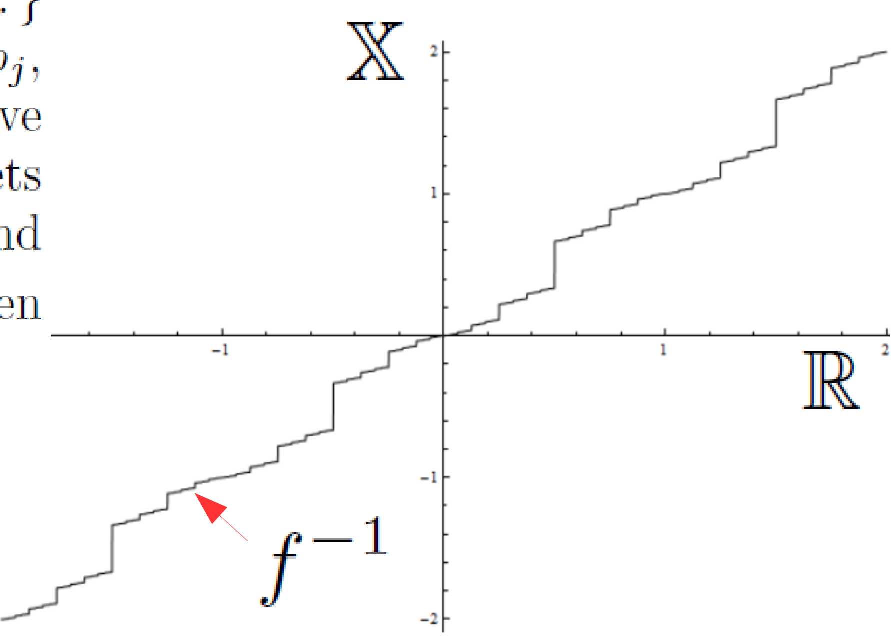
CAMBRIDGE
AT THE UNIVERSITY PRESS
1933

Application: Sine Fourier transform on a middle-third Cantor set

Step 1: $\mathbb{X}$ and $f : \mathbb{X} \to \mathbb{R}$

Let us start with the right-open interval $[0,1) \subset \mathbb{R}$, and let the (countable) set $\mathbb{Y}_2 \subset [0,1)$ consist of those numbers that have two different binary representations. Denote by $0.t_1 t_2 \dots$ a ternary representation of some $x \in [0,1)$. If $y \in \mathbb{Y}_1 = [0,1) \setminus \mathbb{Y}_2$ then y has a unique binary representation, say $y = 0.b_1 b_2 \dots$. One then sets $g_\pm(y) = 0.t_1 t_2 \dots$, $t_j = 2b_j$. The index $\pm$ appears for the following reason. Let $y = 0.b_1 b_2 \dots = 0.b'_1 b'_2 \dots$ be the two representations of $y \in \mathbb{Y}_2$. There are two options, so we define: $g_-(y) = \min\{0.t_1 t_2 \dots, 0.t'_1 t'_2 \dots\}$ and $g_+(y) = \max\{0.t_1 t_2 \dots, 0.t'_1 t'_2 \dots\}$, where $t_j = 2b_j$, $t'_j = 2b'_j$. We have therefore constructed two injective maps $g_\pm : [0,1) \to [0,1)$. The ternary Cantor-like sets are defined as the images $C_\pm(0,1) = g_\pm([0,1))$, and $f_\pm : C_\pm(0,1) \to [0,1)$, $f_\pm = g_\pm^{-1}$, is a bijection between $C_\pm(0,1)$ and the interval.

$$\mathbb{X} = \cup_{k \in \mathbb{Z}} C_-(k, k+1)$$



Application: Sine Fourier transform on a middle-third Cantor set

Step 2: Scalar product

$$\langle A|B\rangle = \int_{\ominus T\oslash 2'}^{T\oslash 2'} A(X)\odot B(X) DX$$

$$\begin{aligned}
\langle A|B\rangle &= \langle B|A\rangle \\
\langle A|B\oplus C\rangle &= \langle A|B\rangle \oplus \langle A|C\rangle \\
\langle A|\Lambda\odot B\rangle &= \Lambda\odot\langle A|B\rangle
\end{aligned}$$

Application: Sine Fourier transform on a middle-third Cantor set

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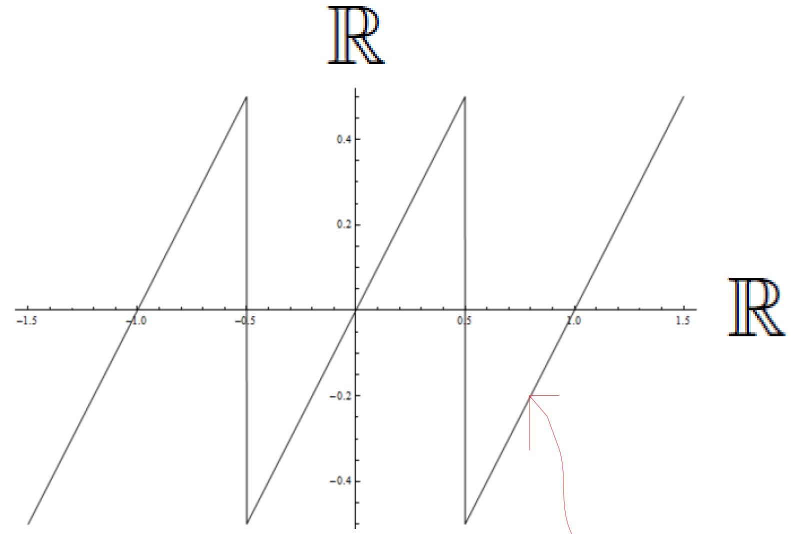
\end{aligned}$$

From now on it's just standard signal analysis...

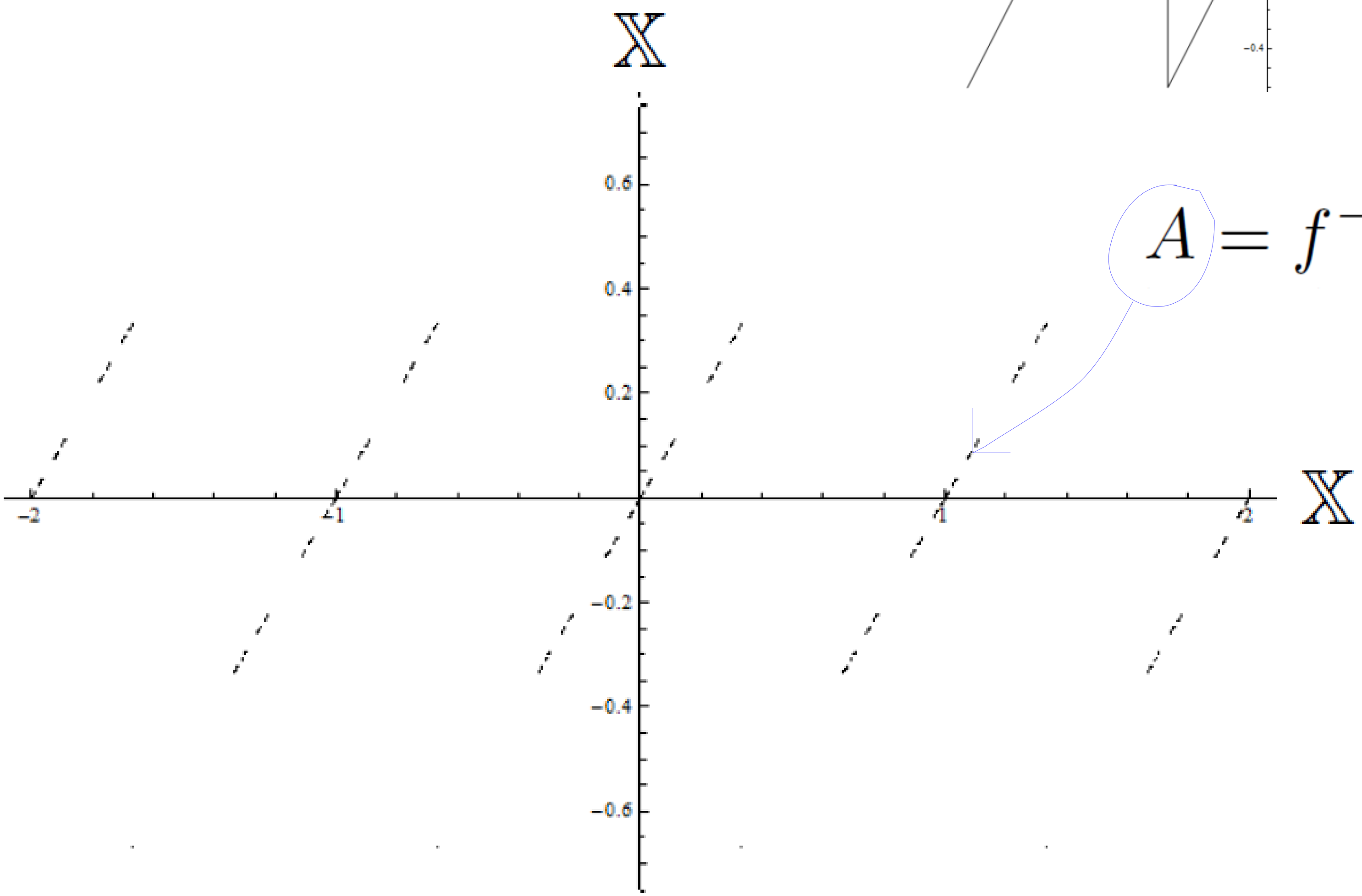
$$A(X) = \bigoplus_{n\geq 0}\Big(C_n(X)\odot\langle C_n|A\rangle \oplus S_n(X)\odot\langle S_n|A\rangle\Big)$$

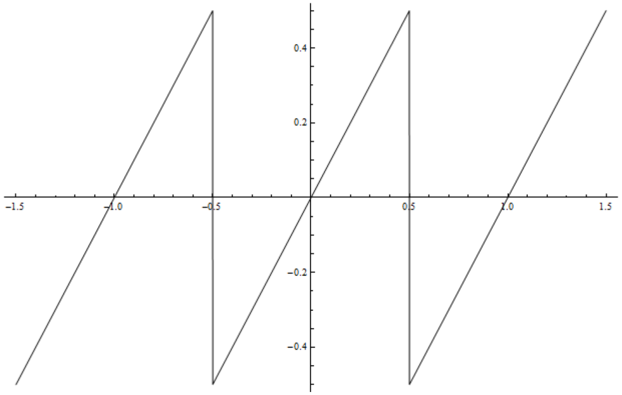
Application: Sine Fourier transform on a middle-third Cantor set

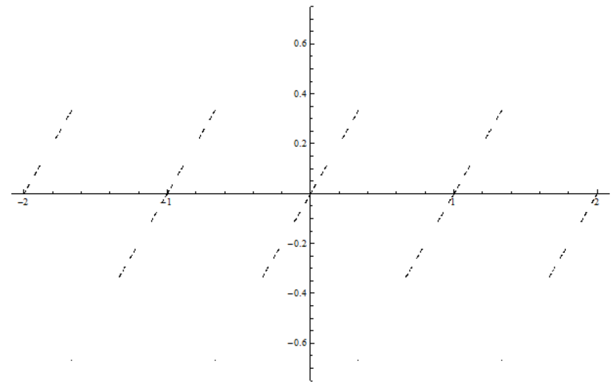
Example:



$$A = f^{-1} \circ a \circ f$$

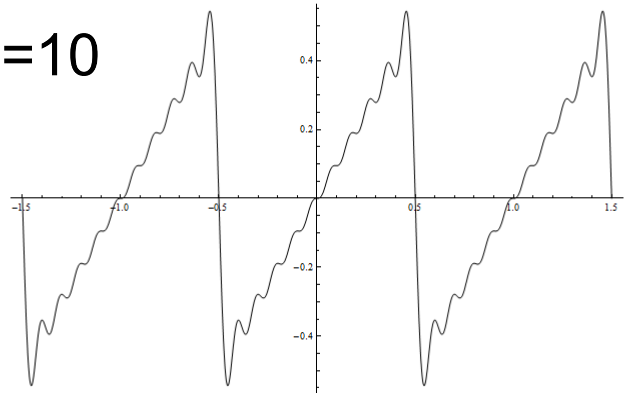




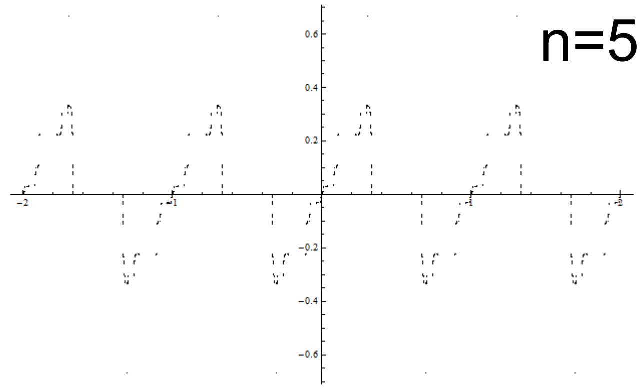


The original signals...

n=10

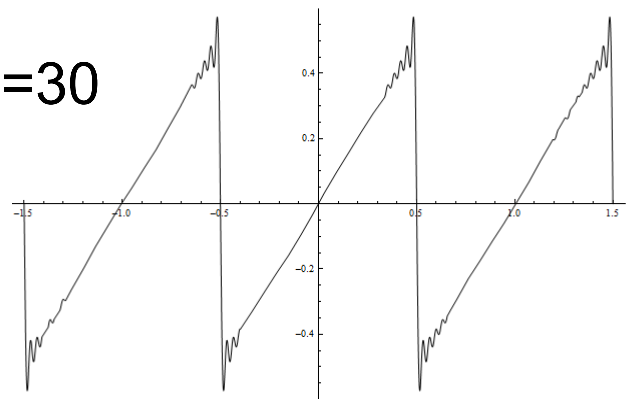


n=5



...and their finite-sum reconstructions

n=30



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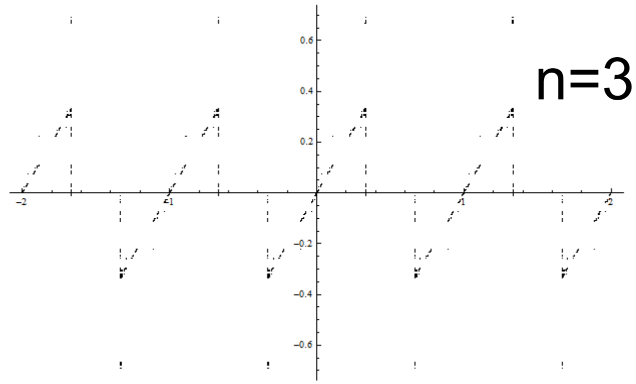
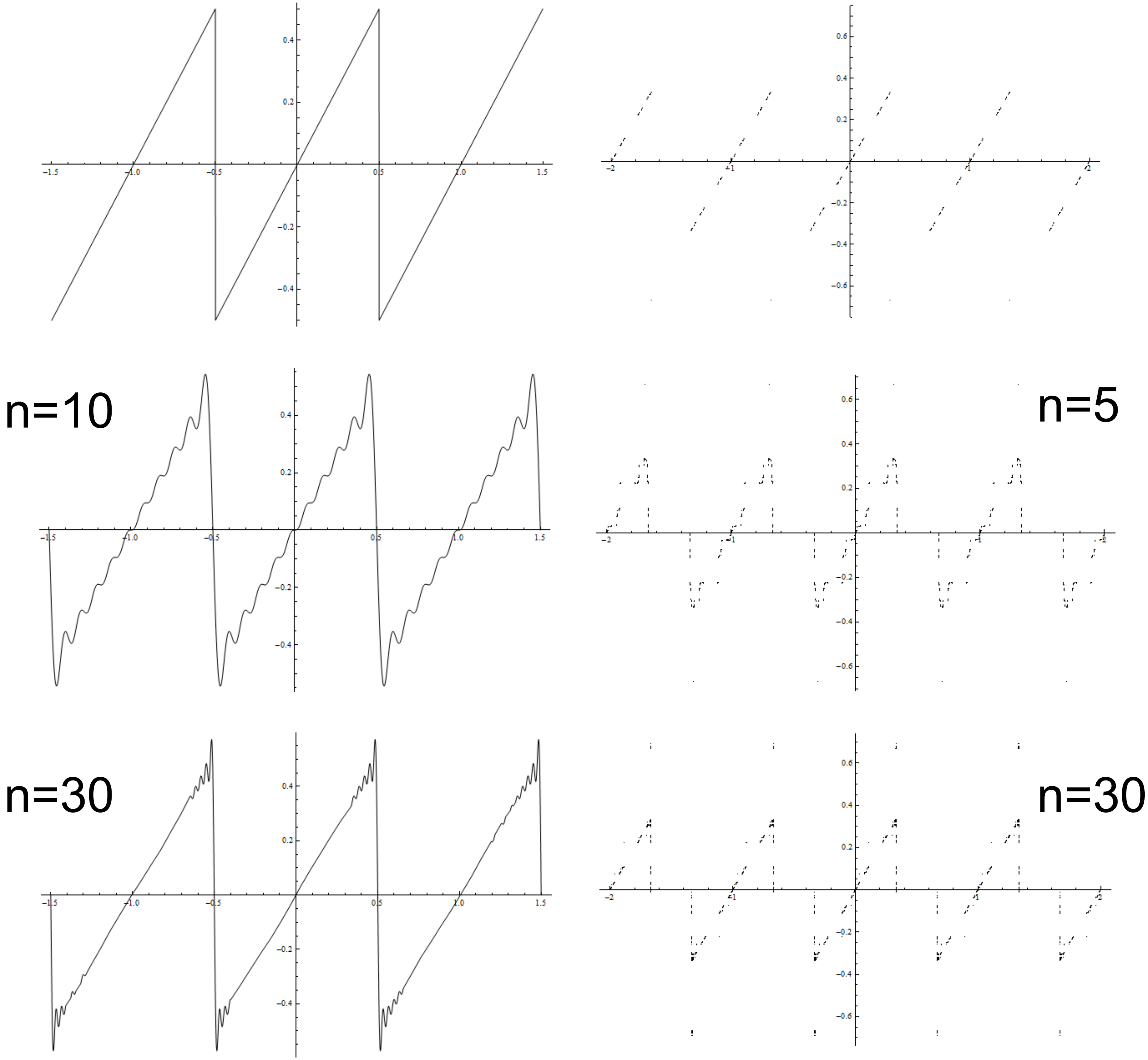


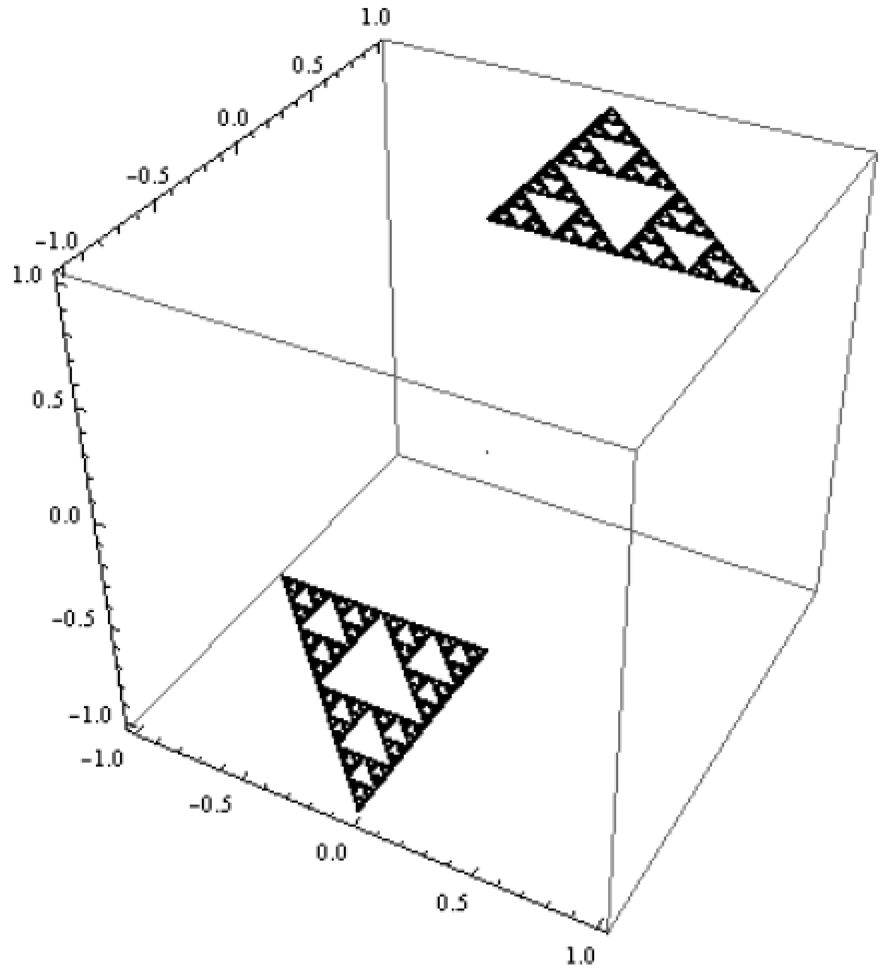


The original signals...

...and their finite-sum reconstructions

- The method works for all Cantor sets, even those that are not self-similar
- We circumvent limitations of the Jorgensen-Pedersen construction, based on self-similar measures

Example: Fourier analysis of



A cosmetic change in definitions

Consider two sets, $\mathbb{X}$ and $\mathbb{Y}$, equipped with bijections $f_{\mathbb{Y}} : \mathbb{Y} \to \mathbb{R}$ and $f_{\mathbb{X}} : \mathbb{X} \to \mathbb{R}$, and arithmetics $\{\oplus_{\mathbb{Y}}, \odot_{\mathbb{Y}} : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}\}$, $\{\oplus_{\mathbb{X}}, \odot_{\mathbb{X}} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}\}$, defined by $f_{\mathbb{Y}}$ and $f_{\mathbb{X}}$. The bijection $f = f_{\mathbb{Y}}^{-1} \circ f_{\mathbb{X}} : \mathbb{X} \to \mathbb{Y}$ makes it possible to consider derivatives of functions $A : \mathbb{X} \to \mathbb{Y}$. Let $0'_{\mathbb{X}} = f_{\mathbb{X}}^{-1}(0)$ be the neutral element of addition in $\mathbb{X}$.

$$\frac{DA(X)}{DX} = \lim_{H \to 0'_{\mathbb{X}}} \Big(A(X \oplus_{\mathbb{X}} H) \ominus_{\mathbb{Y}} A(X) \Big) \oslash_{\mathbb{Y}} f(H)$$

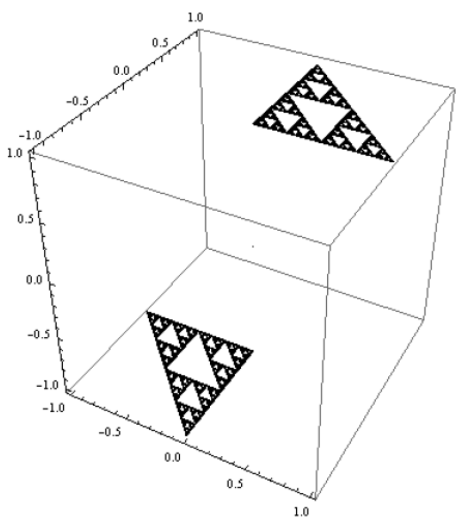
$$\int_Y^X A(X')DX' = f_{\mathbb{Y}}^{-1} \left(\int_{f_{\mathbb{X}}(Y)}^{f_{\mathbb{X}}(X)} f_{\mathbb{Y}} \circ A \circ f_{\mathbb{X}}^{-1}(x) dx \right)$$

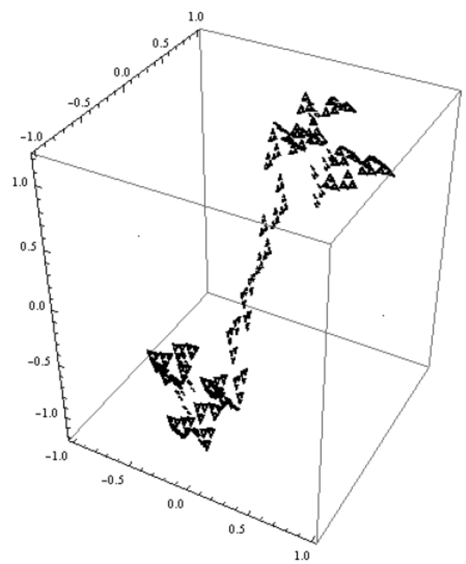
Bijection for the Sierpiński case

In the Cantor case we removed a coutable subset to have the bijection
In the Sierpiński case we add a countable subset to have the bijection

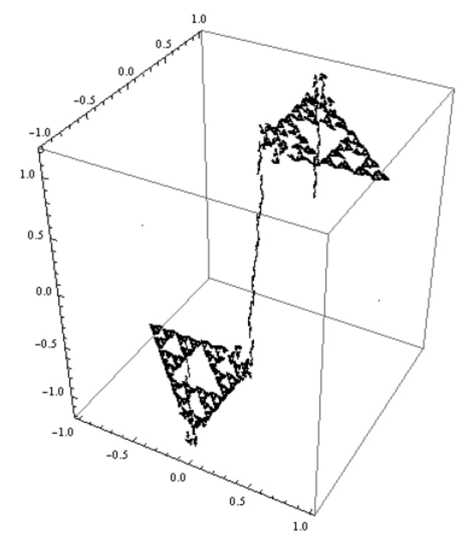
This is not needed in principle, but I'm not clever enough to find something more straightforward and yet easy to work with :(

I will describe the bijection since once we have it the rest is just standard signal analysis:





n=5



n=50

Algorithm

Step 1

Consider $x \in \mathbb{R}_+$ and its ternary representation $x = (t_n \dots t_0.t_{-1}t_{-2}\dots)_3$. If x has two different ternary representations, we choose the one that ends with infinitely many 2s. Keeping the digits unchanged let us change the base from 3 to 4, i.e.

$$x = (t_n \dots t_0.t_{-1}t_{-2}\dots)_3 \mapsto (t_n \dots t_0.t_{-1}t_{-2}\dots)_4 = y$$

Step 2

The quaternary representation of y is unique, and it does not involve the digit 3. Next, let us parametrize the quaternary digits in a binary way, but written in a column form: $0 = \begin{matrix}0\\0\end{matrix}$, $1 = \begin{matrix}0\\1\end{matrix}$, $2 = \begin{matrix}1\\0\end{matrix}$, $3 = \begin{matrix}1\\1\end{matrix}$. y has been converted into a pair of binary sequences,

$$(t_n \dots t_0.t_{-1}t_{-2}\dots)_4 \mapsto \begin{pmatrix} a_n \dots a_0.a_{-1}a_{-2}\dots \\ b_n \dots b_0.b_{-1}b_{-2}\dots \end{pmatrix}_2$$

$(a_j, b_j) \neq (1,1)$ for any j

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Until now the procedure is invertible...

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$$(t_n \ldots t_0.t_{-1}t_{-2}\ldots)_4 \mapsto \begin{pmatrix} a_n \ldots a_0.a_{-1}a_{-2}\ldots \\ b_n \ldots b_0.b_{-1}b_{-2}\ldots \end{pmatrix}_2 \in \mathbb{R}_+ \times \mathbb{R}_+$$

$(a_j, b_j) \neq (1,1)$ for any j

Until now the procedure is invertible and defines a Sierpiński set, but...

...once we identify the pair of binary sequences with a point in the plane, we no longer know how to return from the point to the sequences.

In principle there are 4 options, e.g. (1,1) could be either of

$$\begin{pmatrix} 1.(0) \\ 0.(1) \end{pmatrix}_2 \quad \begin{pmatrix} 0.(1) \\ 1.(0) \end{pmatrix}_2 \quad \begin{pmatrix} 0.(1) \\ 0.(1) \end{pmatrix}_2 \quad \begin{pmatrix} 1.(0) \\ 1.(0) \end{pmatrix}_2$$

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Can't occur in the algorithm as containing quaternary digit 3

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$$\begin{pmatrix} 1.(0) \\ 0.(1) \end{pmatrix}_2 \quad \begin{pmatrix} 0.(1) \\ 1.(0) \end{pmatrix}_2 \quad \cancel{\begin{pmatrix} 0.(1) \\ 0.(1) \end{pmatrix}}_2 \quad \cancel{\begin{pmatrix} 1.(0) \\ 1.(0) \end{pmatrix}}_2$$

Can't occur in the algorithm as containing quaternary digit 3

Can't occur in the algorithm as ending with infinitely 0s

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Can't occur in the algorithm as containing quaternary digit 3

Can't occur in the algorithm as ending with infinitely 0s

Only these two options count, and this turns out to be the only ambiguity of the inverse algorithm in general

Proof: The same mechanism eliminates all the remaining ambiguities:

(A) If a, b are both irrational, or a is irrational and b rational-periodic, their binary forms are unique.

(B) If a is irrational (or rational-periodic), but b rational non-periodic, then b cannot end with infinitely many 1s, as it would mean that a ends with infinitely many 0s. So these cases are again unique. Conclusions are unchanged if one interchanges a and b.

(C) The only ambiguity appears if a ends with infinitely many 0s, but b with infinitely many 1s (or the other way around). But this is the case we have started with.

Thus the bijection is not for a standard Sierpiński set, but for its double cover:

In cases (A) and (B) we identify $(a,b)_+ = (a,b)_- = (a,b)$. Only the (countable) case (C) requires a two-sided plane $(a,b)_+ \neq (a,b)_-$. The case (C) occurs for those $x \in \mathbb{R}$ whose ternary representation ends with $(2)_3$ or $(1)_3$. Only the latter numbers are mapped into $(a,b)_-$.

All Sierpińskian integers are represented by pairs of integers, a representation somewhat similar to complex numbers, but with different rules of addition and multiplication, as illustrated by

$$3' \oplus 4' = (2,0)_+ \oplus (1,2)_+ = 7' = (3,0)_+$$

(numbers represented in decimal form).

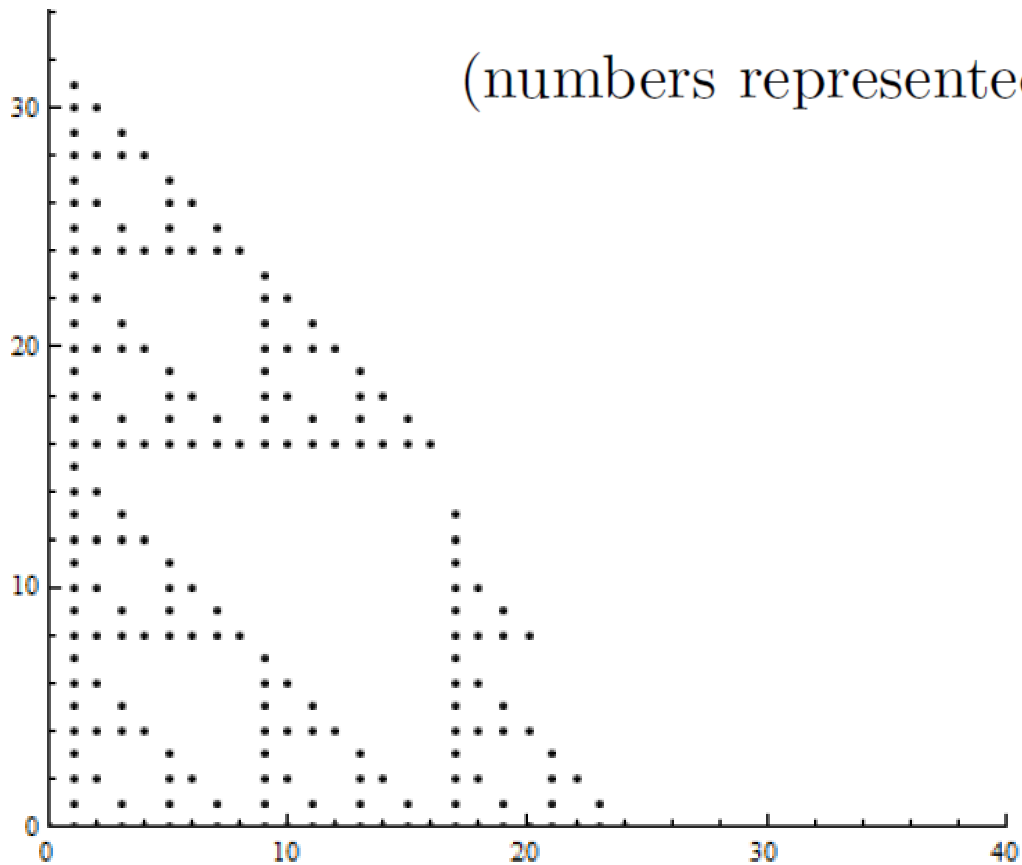
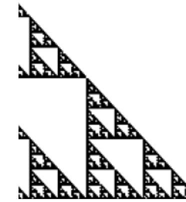
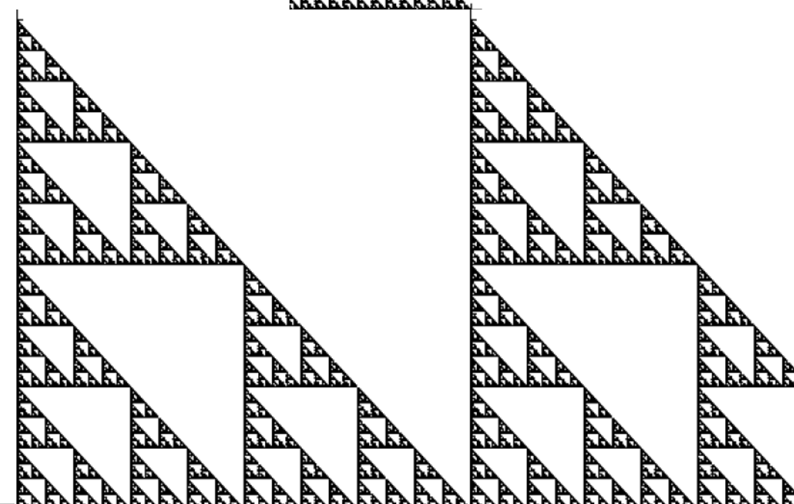
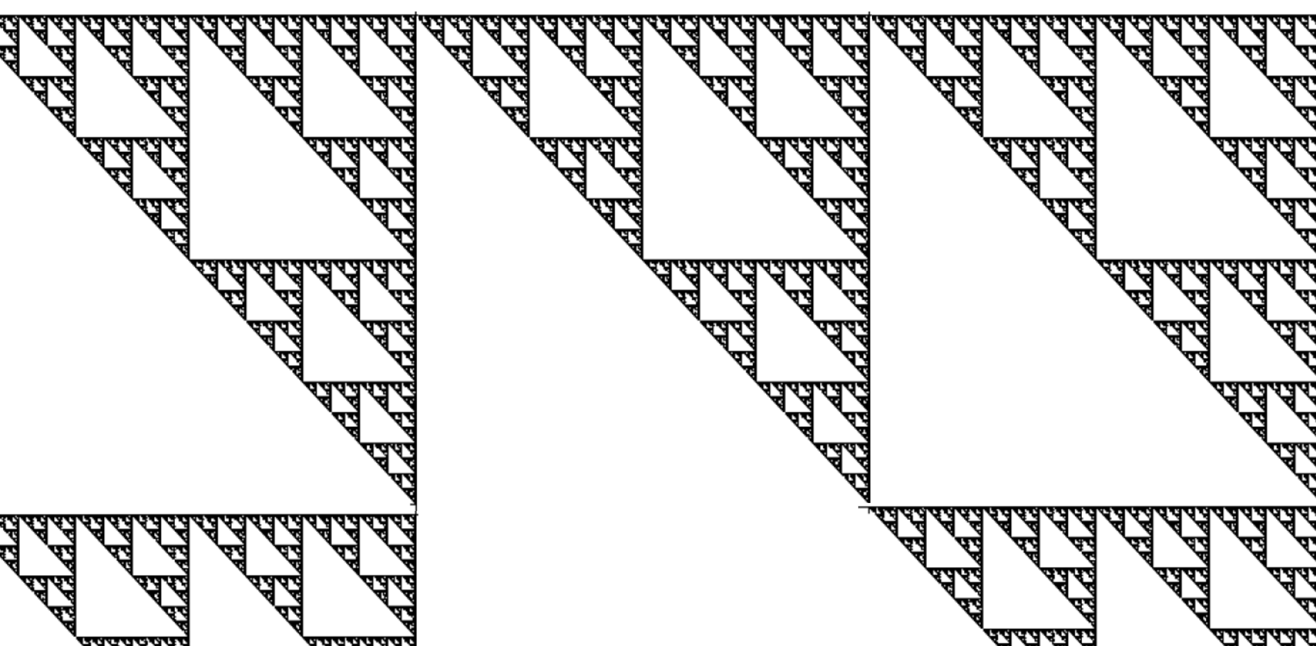


FIG. 2: The image of the first 200 natural numbers, $f^{-1}(\{1,\ldots,200\})$. All natural numbers are mapped into the positive side of the oriented plane.

Our algorithm defines an injective map g_+ of $\mathbb{R}_+$ into a two-sided plane, with the above mentioned identifications. Let us extend g_+ to g by $g(|x|) = g_+(|x|)$, $g(-|x|) = -g_+(|x|)$. The image $S = g(\mathbb{R})$ is our definition of the Sierpiński set. Denoting $f = g^{-1}$, $f : S \to \mathbb{R}$ we obtain

$$
\begin{aligned}
x \oplus y &= f^{-1}\big(f(x) + f(y)\big), \\
x \ominus y &= f^{-1}\big(f(x) - f(y)\big), \\
x \odot y &= f^{-1}\big(f(x) f(y)\big), \\
x \oslash y &= f^{-1}\big(f(x) / f(y)\big).
\end{aligned}
$$

$f^{-1}(0) = (0, 0)$

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