# Fractal zeta functions and geometric combinatorial problems for self-similar subsets of $\mathbb{Z}^n$.

Ben Lichtin

joint work with Driss Essouabri (Saint-Etienne)

# 2 examples:

**Pascal's triangle mod $p$**

$$Pas(p) := \left\{ (m_1, m_2) \in \mathbb{N}_0^2 : \binom{m_1}{m_2} \not\equiv 0 \pmod{p} \right\}.$$

---

**Pascal's pyramid (mod $p$).**

- For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, first set

$$Multnom_n(\alpha) := \binom{\alpha_1 + \cdots + \alpha_n}{\alpha_1, \ldots, \alpha_n} = \frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1! \ldots \alpha_n!}.$$

Define:

$$\mathcal{M}_n(p) := \{\alpha \in \mathbb{N}_0^n : Multnom_n(\alpha) \not\equiv 0 \pmod{p}\}.$$

# 2 examples:

**Pascal's triangle mod $p$**

$$Pas(p) := \left\{ (m_1, m_2) \in \mathbb{N}_0^2 : \binom{m_1}{m_2} \not\equiv 0 \pmod{p} \right\}.$$

**Pascal's pyramid (mod $p$).**

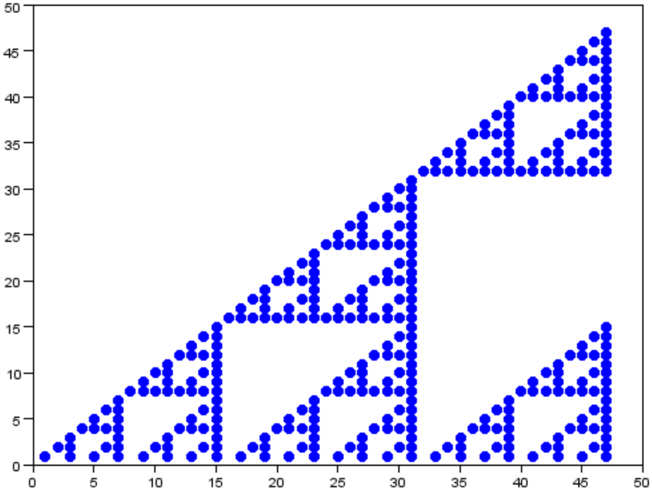
- For any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, first set

$$Multnom_n(\boldsymbol{\alpha}) := \binom{\alpha_1 + \cdots + \alpha_n}{\alpha_1, \ldots, \alpha_n} = \frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1! \ldots \alpha_n!}.$$

Define:

$$\mathcal{M}_n(p) := \{\boldsymbol{\alpha} \in \mathbb{N}_0^n : Multnom_n(\boldsymbol{\alpha}) \not\equiv 0 \pmod{p}\}.$$

$Pas(2)$ :



# Upper Minkowski dimension of discrete sets.

**Definition:**

$q =$ positive definite quadratic form on $\mathbb{R}^n$

$\|\mathbf{x}\| = q^{1/2}(\mathbf{x})$.

$\mathcal{F} =$ discrete subset of $\mathbb{R}^n$.

$\mathcal{F}$ has finite "upper Minkowski dimension" if :

$$e(\mathcal{F}) := \overline{\lim_{R\to\infty}} \frac{\log\left(\#\mathcal{F}\cap B(R)\right)}{\log R} \in [0,\infty).$$

where:

$$B(R) := \{\mathbf{m} \in \mathbb{R}^n;\ \|\mathbf{m}\| \leq R\}.$$

Define the zeta function of $\mathcal{F}$

$$\zeta(\mathcal{F}; s) := \sum_{\mathbf{m}\in\mathcal{F}'} \|\mathbf{m}\|^{-s} \qquad (\mathcal{F}' = \mathcal{F} \setminus \{\mathbf{0}\}).$$

Properties:

- converges absolutely in $\{\Re(s) > e(\mathcal{F})\}$.
- $e(\mathcal{F})$ = abscissa of convergence .
- Always assume $e(\mathcal{F}) \in (0, \infty)$.

$$\{\|\mathbf{m}\| : \mathbf{m} \in \mathcal{F}'\} = \{\lambda_1 < \lambda_2 < \cdots\}; \qquad r_k = \#\{\mathbf{m} : \|\mathbf{m}\| = \lambda_k\}$$

$$\sigma = Re\, s > e(\mathcal{F}) \implies \zeta(\mathcal{F}, s) = \sum_k \frac{r_k}{\lambda_k^s}.$$

Define the zeta function of $\mathcal{F}$

$$\zeta(\mathcal{F}; s) := \sum_{\mathbf{m} \in \mathcal{F}'} \|\mathbf{m}\|^{-s} \qquad (\mathcal{F}' = \mathcal{F} \setminus \{\mathbf{0}\}).$$

Properties:

- converges absolutely in $\{\Re(s) > e(\mathcal{F})\}$.
- $e(\mathcal{F}) =$ abscissa of convergence .
- Always assume $e(\mathcal{F}) \in (0, \infty)$.

$$\{\|\mathbf{m}\| : \mathbf{m} \in \mathcal{F}'\} = \{\lambda_1 < \lambda_2 < \cdots\}; \qquad r_k = \#\{\mathbf{m} : \|\mathbf{m}\| = \lambda_k\}$$

$$\sigma = Re\, s > e(\mathcal{F}) \implies \zeta(\mathcal{F}, s) = \sum_k \frac{r_k}{\lambda_k^s}.$$

Define the zeta function of $\mathcal{F}$

$$\zeta(\mathcal{F}; s) := \sum_{\mathbf{m} \in \mathcal{F}'} \|\mathbf{m}\|^{-s} \qquad (\mathcal{F}' = \mathcal{F} \setminus \{\mathbf{0}\}).$$

Properties:

- converges absolutely in $\{\Re(s) > e(\mathcal{F})\}$.
- $e(\mathcal{F})$ = abscissa of convergence .
- Always assume $e(\mathcal{F}) \in (0, \infty)$.

$$\{\|\mathbf{m}\| : \mathbf{m} \in \mathcal{F}'\} = \{\lambda_1 < \lambda_2 < \cdots\}; \qquad r_k = \#\{\mathbf{m} : \|\mathbf{m}\| = \lambda_k\}$$

$$\sigma = Re\, s > e(\mathcal{F}) \implies \zeta(\mathcal{F}, s) = \sum_k \frac{r_k}{\lambda_k^s}.$$

# Tauberian Applications

Define

$\mathcal{F}_x := \mathcal{F}' \cap \{\|\mathbf{m}\| < x\} =$ *"x−ball" of $\mathcal{F}$.*

$\#\mathcal{F}_x = \sum_{\lambda_k < x} r_k =$ average of coefficients of $\zeta(\mathcal{F}, s)$.

---

**Problem:** Describe $\#\mathcal{F}_x$ as $x \to \infty$.

Tauberian theory is useful if information outside halfplane of absolute convergence is available.

---

**Key Point:** Self-similarity of $\mathcal{F}$ helps provide such information.

# Tauberian Applications

Define

$\mathcal{F}_x := \mathcal{F}' \cap \{\|\mathbf{m}\| < x\} =$ *"x−ball" of* $\mathcal{F}$.

$\#\mathcal{F}_x = \sum_{\lambda_k < x} r_k =$ average of coefficients of $\zeta(\mathcal{F}, s)$.

---

**Problem:** Describe $\#\mathcal{F}_x$ as $x \to \infty$.

Tauberian theory is useful if information outside halfplane of absolute convergence is available.

---

**Key Point:** Self-similarity of $\mathcal{F}$ helps provide such information.

# Tauberian Applications

Define

$\mathcal{F}_x := \mathcal{F}' \cap \{\|\mathbf{m}\| < x\} =$ *"x−ball" of* $\mathcal{F}$.

$\#\mathcal{F}_x = \sum_{\lambda_k < x} r_k =$ average of coefficients of $\zeta(\mathcal{F}, s)$.

---

**Problem:** Describe $\#\mathcal{F}_x$ as $x \to \infty$.

Tauberian theory is useful if information outside halfplane of absolute convergence is available.

---

**Key Point:** Self-similarity of $\mathcal{F}$ helps provide such information.

# Compatible self-similar discrete sets.

- "Similarity" $=$ affine map $\mathbf{x} \to c\,T\mathbf{x} + \mathbf{b}$
- $c > 1$ and $T$ is invariant by $O(q)$
- Finite set $\mathbf{f} = (f_i)_1^r$ $(f_i = c_i T_i + \mathbf{b}_i)$ of similarities is **"compatible"** if

$$T_i \circ T_j = T_j \circ T_i \quad \forall i, j$$

---

**Definition.** *$\mathcal{F}$ is compatible self-similar*

- $\exists \{T_i\}_1^r$ compatible s.t.

defining $f_i = c_i T_i + \mathbf{b}_i$:

- $f_i(\mathcal{F}) \cap f_{i'}(\mathcal{F})$ finite if $i \neq i'$
- $\mathcal{F} \equiv \cup_{i=1}^r f_i(\mathcal{F})$ is finite ( symmetric difference).

# Compatible self-similar discrete sets.

- "Similarity" $=$ affine map $\mathbf{x} \to c\,T\mathbf{x} + \mathbf{b}$
- $c > 1$ and $T$ is invariant by $O(q)$
- Finite set $\mathbf{f} = (f_i)_1^r$ $(f_i = c_i T_i + \mathbf{b}_i)$ of similarities is **"compatible"** if

$$T_i \circ T_j = T_j \circ T_i \quad \forall i, j$$

---

**Definition.** *$\mathcal{F}$ is compatible self-similar*

- $\exists \{T_i\}_1^r$ compatible s.t.

defining $f_i = c_i T_i + \mathbf{b}_i$:

- $f_i(\mathcal{F}) \cap f_{i'}(\mathcal{F})$ finite if $i \neq i'$
- $\mathcal{F} \equiv \cup_{i=1}^r f_i(\mathcal{F})$ is finite ( symmetric difference).

Compatibility implies:

- $\exists\ \mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n) =$ orthonormal basis of $\mathbb{C}^n (= \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n)$ s.t. each $T_j$ is diagonal.

- $\forall i\ \exists \boldsymbol{\lambda}_i = (\lambda_{i,1}, \dots, \lambda_{i,n}) \in (S^1)^n$ s.t.

$$T_i^*(\mathbf{e}_j) = \lambda_{i,j} \mathbf{e}_j \quad \forall j \quad (T_i^* = \text{adjoint}).$$

- $Scal(\mathbf{f}) := \{c_i\}_1^r$.

---

**Theorem 1 [E-L, Adv. Math. 2013].** *Compatible self-similar $\mathcal{F} \subset \mathbb{R}^n$ satisfies:*

$$\zeta(\mathcal{F}; s) = \sum_{\mathbf{m} \in \mathcal{F}'} \frac{1}{\|\mathbf{m}\|^s}$$

*has a meromorphic continuation to $\mathbb{C}$ with moderate growth and poles in:*

$$\mathcal{P}(\mathcal{F}) = \bigcup_{\beta \in \mathbb{N}_0^n} \bigcup_{k \in \mathbb{N}_0} \left\{ s - k : \sum_{i=1}^r \boldsymbol{\lambda}_i^{\beta} c_i^{-s} = 1 \right\} \quad (\boldsymbol{\lambda}^{\beta} := \prod_k \lambda_k^{\beta_k}).$$

Compatibility implies:

- $\exists\ \mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_n) =$ orthonormal basis of $\mathbb{C}^n (= \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n)$ s.t. each $T_j$ is diagonal.

- $\forall i\ \exists \boldsymbol{\lambda}_i = (\lambda_{i,1}, \dots, \lambda_{i,n}) \in (S^1)^n$ s.t.

$$T_i^*(\mathbf{e}_j) = \lambda_{i,j}\mathbf{e}_j \quad \forall j \quad (T_i^* = \text{adjoint}).$$

- $Scal(\mathbf{f}) := \{c_i\}_1^r$.

---

**Theorem 1 [E-L, Adv. Math. 2013].** *Compatible self-similar $\mathcal{F} \subset \mathbb{R}^n$ satisfies:*

$$\zeta(\mathcal{F}; s) = \sum_{\mathbf{m} \in \mathcal{F}'} \frac{1}{\|\mathbf{m}\|^s}$$

*has a meromorphic continuation to $\mathbb{C}$ with moderate growth and poles in:*

$$\mathcal{P}(\mathcal{F}) = \bigcup_{\boldsymbol{\beta} \in \mathbb{N}_0^n} \bigcup_{k \in \mathbb{N}_0} \left\{ s - k : \sum_{i=1}^r \boldsymbol{\lambda}_i^{\boldsymbol{\beta}} c_i^{-s} = 1 \right\} \quad (\boldsymbol{\lambda}^{\boldsymbol{\beta}} := \prod_k \lambda_k^{\beta_k}).$$

**Publicity for forthcoming article.**

- **Compatibility hypothesis <u>not needed</u> for our results.**
- Arguments are fairly different from compatibility case.

# Fractal dimension.

**Theorem 2. [E-L, 2013]**
$\mathcal{F} \subset \mathbb{R}^n$ = *compatible self-similar set. Then*

$e(\mathcal{F})$ = **fractal dimension of** $\mathcal{F}$ = **the unique real solution of**

$$\sum_{i=1}^{r} c_i^{-s} = 1.$$

---

• 2 types of self similar sets:

**Latticelike:** $\mathbb{Z}\{\log c_i\}_i$ = discrete subgroup of $\mathbb{R}$;

**E.G.** $Pas(p), \mathcal{M}_n(p)$ are latticelike.

**Non latticelike:** $\mathbb{Z}\{\log c_i\}_i$ is dense in $\mathbb{R}$.

# Fractal dimension.

**Theorem 2. [E-L, 2013]**
$\mathcal{F} \subset \mathbb{R}^n$ = *compatible self-similar set. Then*

$e(\mathcal{F})$ = **fractal dimension of** $\mathcal{F}$ = **the unique real solution of**

$$\sum_{i=1}^{r} c_i^{-s} = 1.$$

---

- 2 types of self similar sets:

**Latticelike:** $\mathbb{Z}\{\log c_i\}_i$ = discrete subgroup of $\mathbb{R}$;

**E.G.** $Pas(p), \mathcal{M}_n(p)$ are latticelike.

**Non latticelike:** $\mathbb{Z}\{\log c_i\}_i$ is dense in $\mathbb{R}$.

## Properties of $Pas(p)$:

- Similarities:

$$f_i = p \cdot Id_2 + \mathbf{b}_i \qquad \mathbf{b}_i \in \left\{(u,v) \in \{0,1,\ldots,p-1\}^2\right\}.$$

Set

$$\theta_p = \frac{\log(\frac{p(p+1)}{2})}{\log p}.$$

---

**Theorem 3 [E.- Jap. J. of Math. (2005)].**

- $\zeta\,(Pas(p); s)$ *converges in* $\{\sigma > \theta_p\}$.
- $e(Pas(p)) = \theta_p$ .

- $\zeta\,(Pas(p); s)$ *has a meromorphic continuation to* $\mathbb{C}$ *with moderate growth and simple poles in:*

$$\left\{\theta_p - k + \frac{2\pi i h}{\log p} : (k,h) \in \mathbb{N}_0 \times \mathbb{Z}_0\right\}.$$

**Properties of $Pas(p)$:**

- Similarities:

$$f_i = p \cdot Id_2 + \mathbf{b}_i \qquad \mathbf{b}_i \in \left\{(u,v) \in \{0,1,\dots,p-1\}^2\right\}.$$

Set

$$\theta_p = \frac{\log(\frac{p(p+1)}{2})}{\log p}.$$

---

**Theorem 3 [E.- Jap. J. of Math. (2005)].**

- $\zeta\,(Pas(p);s)$ *converges in* $\{\sigma > \theta_p\}$.
- $e(Pas(p)) = \theta_p$ .

- $\zeta\,(Pas(p);s)$ *has a meromorphic continuation to* $\mathbb{C}$ *with moderate growth and simple poles in:*

$$\left\{\theta_p - k + \frac{2\pi i h}{\log p} : (k,h) \in \mathbb{N}_0 \times \mathbb{Z}_0\right\}.$$

# Pascal's pyramid (mod $p$).

Define

$$\theta_{n,p} = \frac{\log\left(\binom{p+n-1}{p-1}\right)}{\log p}\,.$$

**Theorem 4 [E-L, 2013]**

$$\zeta(\mathcal{M}_n(p), s) = \sum_{\boldsymbol{\alpha}\in\mathcal{M}_n(p)} \|\boldsymbol{\alpha}\|^{-s}$$

*has a meromorphic continuation to* $\mathbb{C}$ *with moderate growth and simple poles in:*

$$\left\{\theta_{n,p} - k + \frac{2\pi i h}{\log p} : (k,h) \in \mathbb{N}_0 \times \mathbb{Z}_0\right\},$$

*and*

$$e(\mathcal{M}_n(p)) = \theta_{n,p}\,.$$

(I) $\sigma > \theta_p = \frac{\log(\frac{p(p+1)}{2})}{\log p} \implies \zeta(Pas(p), s) = \sum_k \frac{r_k(p)}{\lambda_k^s}$ where:

$$\{\lambda_k\}_k := \{\|(m_1, m_2)\| : (m_1, m_2) \in Pas(p)\}$$
$$r_k(p) := \#\{(m_1, m_2) \in Pas(p) : \|(m_1, m_2)\| = \lambda_k\}$$

Average of coefficients: $\#(Pas(p)_x) := \sum_{\lambda_k < x} r_k(p)$

---

**Theorem 5.[E-L, 2013].**
$\exists\ \ \eta > 0$ *and a convergent Fourier series* $f_p(u) \neq 0$ *s.t.*

$$\#(Pas(p)_x) = x^{\theta_p} \cdot f_p\left(\frac{logx}{logp}\right) + O(x^{\theta_p - \eta}) \quad x \to \infty.$$

(I) $\sigma > \theta_p = \frac{\log(\frac{p(p+1)}{2})}{\log p} \implies \zeta(Pas(p), s) = \sum_k \frac{r_k(p)}{\lambda_k^s}$ where:

$$\{\lambda_k\}_k := \{\|(m_1, m_2)\| : (m_1, m_2) \in Pas(p)\}$$
$$r_k(p) := \#\{(m_1, m_2) \in Pas(p) : \|(m_1, m_2)\| = \lambda_k\}$$

Average of coefficients: $\#(Pas(p)_x) := \sum_{\lambda_k < x} r_k(p)$

---

**Theorem 5.[E-L, 2013].**
$\exists \;\; \eta > 0$ *and a convergent Fourier series* $f_p(u) \neq 0$ *s.t.*

$$\#(Pas(p)_x) = x^{\theta_p} \cdot f_p\left(\frac{logx}{logp}\right) + O(x^{\theta_p - \eta}) \quad x \to \infty.$$

(II) $\sigma > \theta_{n,p} = \frac{\log\left(\binom{p+n-1}{p-1}\right)}{\log p} \implies \zeta(\mathcal{M}_n(p), s) = \sum_k \frac{R_k(p)}{\Lambda_k^s}$ where:

$$\{\Lambda_k\}_k := \{\|\boldsymbol{\alpha}\| : \boldsymbol{\alpha} \in \mathcal{M}_n(p)\}$$
$$R_k(p) := \#\{\boldsymbol{\alpha} \in \mathcal{M}_n(p) : \|\boldsymbol{\alpha}\| = \Lambda_k\}$$

Average of coefficients: $\#(\mathcal{M}_n(p)_x) := \sum_{\Lambda_k < x} R_k(p)$

---

A new proof of result of Barbolosi-Grabner (1996).

**Theorem 6.[E-L, 2013].**
*$\exists\ \eta > 0$ and convergent Fourier series $F_{n,p}(u) \neq 0$ s.t.*

$$\#(\mathcal{M}_n(p)_x) = x^{\theta_{n,p}} F_{n,p}\left(\frac{\log x}{\log p}\right) + O(x^{\theta_{n,p}-\eta}).$$

(II) $\quad \sigma > \theta_{n,p} = \frac{\log\left(\binom{p+n-1}{p-1}\right)}{\log p} \implies \zeta(\mathcal{M}_n(p), s) = \sum_k \frac{R_k(p)}{\Lambda_k^s}$ where:

$$\{\Lambda_k\}_k := \{\|\boldsymbol{\alpha}\| : \boldsymbol{\alpha} \in \mathcal{M}_n(p)\}$$
$$R_k(p) := \#\{\boldsymbol{\alpha} \in \mathcal{M}_n(p) : \|\boldsymbol{\alpha}\| = \Lambda_k\}$$

Average of coefficients: $\#(\mathcal{M}_n(p)_x) := \sum_{\Lambda_k < x} R_k(p)$

---

A new proof of result of Barbolosi-Grabner (1996).

**Theorem 6.[E-L, 2013].**
$\exists\ \eta > 0$ *and convergent Fourier series* $F_{n,p}(u) \neq 0$ *s.t.*

$$\#(\mathcal{M}_n(p)_x) = x^{\theta_{n,p}} F_{n,p}\left(\frac{\log x}{\log p}\right) + O(x^{\theta_{n,p}-\eta}).$$

**Examples of metric invariants (w.r.t. $O(q)$):**

$(k = 2)$ : Distinct distances, angles

---

$(k \geq 3)$ : $k-$point configurations, $k-$chains,
volumes of $n-$simplices $(k = n + 1)$.

"$k-$Configuration"
$= \{(\|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_1 - \mathbf{x}_3\|, \ldots, \|(\mathbf{x}_{k-1} - \mathbf{x}_k\|) : \mathbf{x}_i \in \mathcal{F}\ \forall i\}$.

"$k-$Chain"
$= \{(\|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_2 - \mathbf{x}_3\|, \ldots, \|\mathbf{x}_{k-1} - \mathbf{x}_k\|) : \mathbf{x}_i \in \mathcal{F}\ \forall i\}$.

---

**"Geometric combinatorial" problems:**

For metric invariant on $(\mathbb{R}^n)^k$ find a *lower bound* for :

*# distinct values of invariant determined by elements of* $\mathcal{F}_x^k$ $(x \gg 1)$.

# Averages for other weights — Metric Invariants

**Examples of metric invariants (w.r.t. $O(q)$):**

$(k=2)$ : Distinct distances, angles

---

$(k \geq 3)$ : $k-$point configurations, $k-$chains,
volumes of $n-$simplices $(k = n+1)$.

"$k-$Configuration"
$= \left\{ \left( \|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_1 - \mathbf{x}_3\|, \ldots, \|(\mathbf{x}_{k-1} - \mathbf{x}_k\| \right) : \mathbf{x}_i \in \mathcal{F} \ \forall i \right\}.$

"$k-$Chain"
$= \left\{ \left( \|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_2 - \mathbf{x}_3\|, \ldots, \|\mathbf{x}_{k-1} - \mathbf{x}_k\| \right) : \mathbf{x}_i \in \mathcal{F} \ \forall i \right\}.$

---

**"Geometric combinatorial" problems:**

For metric invariant on $(\mathbb{R}^n)^k$ find a *lower bound* for :

$\#$ *distinct values of invariant determined by elements of* $\mathcal{F}_x^k$ $(x \gg 1)$.

**Examples of metric invariants (w.r.t. $O(q)$):**

$(k=2)$ : Distinct distances, angles

$(k \geq 3)$ : $k-$point configurations, $k-$chains,
volumes of $n-$simplices $(k = n+1)$.

"$k-$Configuration"
$= \left\{ \left( \|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_1 - \mathbf{x}_3\|, \dots, \|(\mathbf{x}_{k-1} - \mathbf{x}_k\| \right) : \mathbf{x}_i \in \mathcal{F}\ \forall i \right\}$.

"$k-$Chain"
$= \left\{ \left( \|\mathbf{x}_1 - \mathbf{x}_2\|, \|\mathbf{x}_2 - \mathbf{x}_3\|, \dots, \|\mathbf{x}_{k-1} - \mathbf{x}_k\| \right) : \mathbf{x}_i \in \mathcal{F}\ \forall i \right\}$.

**"Geometric combinatorial" problems:**

For metric invariant on $(\mathbb{R}^n)^k$ find a *lower bound* for :

*$\#$ distinct values of invariant determined by elements of $\mathcal{F}_x^k$* $(x \gg 1)$.

**Erdös distinct distance problem (1946)**:

$q(\mathbf{x}) := \sum_i x_i^2 \qquad \|\mathbf{x}\| := (\sum_i x_i^2)^{1/2}$.

• If $X$ is a finite subset of $\mathbb{R}^n$, then the set of its distinct distances $\Delta(X) := \{\|x - y\|;\ x, y \in X\}$ should satisfy:

$$\#\Delta(X) \gg_\varepsilon \#X^{\frac{2}{n}-\varepsilon} \quad \text{as } \#X \to \infty.$$

• $n = 2$: Guth-Katz (2014) (with slightly better exponent of Erdòs).

**Erdös distinct distance problem (1946)**:

$q(\mathbf{x}) := \sum_i x_i^2 \qquad \|\mathbf{x}\| := (\sum_i x_i^2)^{1/2}$.

• If $X$ is a finite subset of $\mathbb{R}^n$, then the set of its distinct distances $\Delta(X) := \{\|x-y\|;\ x, y \in X\}$ should satisfy:

$$\#\Delta(X) \gg_\varepsilon \#X^{\frac{2}{n}-\varepsilon} \quad \text{as } \#X \to \infty.$$

---

• $n = 2$: Guth-Katz (2014) (with slightly better exponent of Erdòs).

Distance set in any $x-$ball of $\mathcal{F}$ :

$$\Delta(\mathcal{F}_x) := \{\|\mathbf{m}_1 - \mathbf{m}_2\| : \mathbf{m}_i \in \mathcal{F}_x \ \ \forall i = 1, 2\}.$$

**Theorem 7 [E-L, PLMS 2015].**
*$\mathcal{F}$ = compatible self-similar subset of $\mathbb{Z}^n$ $(n \geq 2)$;*
*$e(\mathcal{F}) > n - 2$.*

$$\Longrightarrow \quad \forall \varepsilon > 0, \ \ \#\Delta(\mathcal{F}_x) \gg_\varepsilon \#(\mathcal{F}_x)^{1-\frac{n-2}{e(\mathcal{F})}-\varepsilon} \quad (x \to \infty).$$

**Corollaries.**

For (compatible) self-similar $\mathcal{F} \subset \mathbb{Z}^n$:

(1) $n = 2$ and $e(\mathcal{F}) > 0$ implies Erdös distance conjecture
*but only for all sufficiently large $x-$ balls*:

$$\forall \varepsilon > 0, \quad \#\Delta(\mathcal{F}_x) \gg_\varepsilon \#(\mathcal{F}_x)^{1-\varepsilon} \qquad (x \to \infty).$$

(2) $\theta_p > 0 \implies \#\Delta\left(Pas(p)_x\right) \gg_\varepsilon \left[\#Pas(p)_x\right]^{1-\varepsilon}$.

(3) $\exists a_n$ s.t. $p > a_n \implies \theta_{n,p} > n-2$.

$$\therefore p > a_n \implies \#\Delta\left(\mathcal{M}_n(p)_x\right) \gg_\varepsilon \left[\#\mathcal{M}_n(p)_x\right]^{1-\frac{n-2}{\theta_{n,p}}-\varepsilon}.$$

**Corollaries.**

For (compatible) self-similar $\mathcal{F} \subset \mathbb{Z}^n$:

(1) $n = 2$ and $e(\mathcal{F}) > 0$ implies Erdös distance conjecture
*but only for all sufficiently large $x-$ balls*:

$$\forall \varepsilon > 0, \quad \#\Delta(\mathcal{F}_x) \gg_\varepsilon \#(\mathcal{F}_x)^{1-\varepsilon} \qquad (x \to \infty).$$

(2) $\theta_p > 0 \implies \#\Delta\left(Pas(p)_x\right) \gg_\varepsilon \left[\#Pas(p)_x\right]^{1-\varepsilon}$.

(3) $\exists a_n$ s.t. $p > a_n \implies \theta_{n,p} > n-2$.

$$\therefore p > a_n \implies \#\Delta\left(\mathcal{M}_n(p)_x\right) \gg_\varepsilon \left[\#\mathcal{M}_n(p)_x\right]^{1-\frac{n-2}{\theta_{n,p}}-\varepsilon}.$$

**Corollaries.**

For (compatible) self-similar $\mathcal{F} \subset \mathbb{Z}^n$:

(1) $n = 2$ and $e(\mathcal{F}) > 0$ implies Erdös distance conjecture
*but only for all sufficiently large $x-$ balls*:

$$\forall \varepsilon > 0, \quad \#\Delta(\mathcal{F}_x) \gg_\varepsilon \#(\mathcal{F}_x)^{1-\varepsilon} \qquad (x \to \infty).$$

(2) $\theta_p > 0 \implies \#\Delta\left(Pas(p)_x\right) \gg_\varepsilon \left[\#Pas(p)_x\right]^{1-\varepsilon}$.

(3) $\exists a_n$ s.t. $p > a_n \implies \theta_{n,p} > n - 2.$

$$\therefore p > a_n \implies \#\Delta\left(\mathcal{M}_n(p)_x\right) \gg_\varepsilon \left[\#\mathcal{M}_n(p)_x\right]^{1-\frac{n-2}{\theta_{n,p}}-\varepsilon}.$$

# Sketch of proof of Theorem 7.

**Step 1**:
*Introduce* $2-$*variable zeta-function:*

$$\zeta_{dis}(\mathcal{F},\mathbf{s}) = \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in(\mathcal{F}')^2} \frac{\|\mathbf{m}_1-\mathbf{m}_2\|^2}{\|\mathbf{m}_1\|^{s_1}\|\mathbf{m}_2\|^{s_2}}.$$

- Converges absolutely in $\{\mathbf{s}\in\mathbb{C}^2;\ \sigma_i > D\ \forall i\}$ $\quad (D := e(\mathcal{F})+2)$.

---

Set:

$$\begin{aligned}
\Delta(\mathcal{F}_x) &:= \{\|\mathbf{m}_1-\mathbf{m}_2\|;\ (\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{F}_x^2\}\\
&:= \{\delta_1,\dots,\delta_{N_x}\} \quad (N_x = \#\Delta(\mathcal{F}_x))\\
\{\|\mathbf{m}\| : \mathbf{m}\in\mathcal{F}_x\} &:= \{\lambda_1,\dots,\lambda_{M_x}\}\\
\nu_{i,k_1,k_2} &:= \#\{(\mathbf{m}_1,\mathbf{m}_2) : \|\mathbf{m}_j\| = \lambda_{k_j}\ \forall j,\\
&\qquad \|\mathbf{m}_1-\mathbf{m}_2\| = \delta_i\}.
\end{aligned}$$

---

$\therefore \sigma_1,\sigma_2 > D \implies \zeta_{dis}(\mathcal{F},\mathbf{s}) = \sum_{(k_1,k_2)} \frac{\sum_i \delta_i^2\cdot\nu_{i,k_1,k_2}}{\lambda_{k_1}^{s_1}\lambda_{k_2}^{s_2}}.$

# Sketch of proof of Theorem 7.

**Step 1**:
*Introduce 2−variable zeta-function:*

$$\zeta_{dis}(\mathcal{F}, \mathbf{s}) = \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in(\mathcal{F}')^2} \frac{\|\mathbf{m}_1 - \mathbf{m}_2\|^2}{\|\mathbf{m}_1\|^{s_1}\|\mathbf{m}_2\|^{s_2}}.$$

- Converges absolutely in $\{\mathbf{s} \in \mathbb{C}^2;\ \sigma_i > D\ \forall i\} \quad (D := e(\mathcal{F}) + 2)$.

---

Set:

$$\begin{aligned}
\Delta(\mathcal{F}_x) &:= \{\|\mathbf{m}_1 - \mathbf{m}_2\|;\ (\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{F}_x^2\} \\
&:= \{\delta_1, \ldots, \delta_{N_x}\} \quad (N_x = \#\Delta(\mathcal{F}_x)) \\
\{\|\mathbf{m}\| : \mathbf{m} \in \mathcal{F}_x\} &:= \{\lambda_1, \ldots, \lambda_{M_x}\} \\
\nu_{i,k_1,k_2} &:= \#\{(\mathbf{m}_1, \mathbf{m}_2) : \|\mathbf{m}_j\| = \lambda_{k_j}\ \forall j, \\
&\qquad \|\mathbf{m}_1 - \mathbf{m}_2\| = \delta_i\}.
\end{aligned}$$

---

$\therefore \sigma_1, \sigma_2 > D \implies \zeta_{dis}(\mathcal{F}, \mathbf{s}) = \sum_{(k_1,k_2)} \frac{\sum_i \delta_i^2 \cdot \nu_{i,k_1,k_2}}{\lambda_{k_1}^{s_1}\lambda_{k_2}^{s_2}}.$

# Sketch of proof of Theorem 7.

**Step 1**:

*Introduce 2−variable zeta-function:*

$$\zeta_{dis}(\mathcal{F}, \mathbf{s}) = \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in(\mathcal{F}')^2} \frac{\|\mathbf{m}_1 - \mathbf{m}_2\|^2}{\|\mathbf{m}_1\|^{s_1}\|\mathbf{m}_2\|^{s_2}}.$$

- Converges absolutely in $\{\mathbf{s} \in \mathbb{C}^2;\ \sigma_i > D\ \forall i\}$ $\quad (D := e(\mathcal{F}) + 2)$.

---

Set:

$$\begin{aligned}
\Delta(\mathcal{F}_x) &:= \{\|\mathbf{m}_1 - \mathbf{m}_2\|;\ (\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{F}_x^2\} \\
&:= \{\delta_1, \ldots, \delta_{N_x}\} \qquad (N_x = \#\Delta(\mathcal{F}_x)) \\
\{\|\mathbf{m}\| : \mathbf{m} \in \mathcal{F}_x\} &:= \{\lambda_1, \ldots, \lambda_{M_x}\} \\
\nu_{i,k_1,k_2} &:= \#\{(\mathbf{m}_1, \mathbf{m}_2) : \|\mathbf{m}_j\| = \lambda_{k_j}\ \forall j, \\
&\qquad\qquad \|\mathbf{m}_1 - \mathbf{m}_2\| = \delta_i\}.
\end{aligned}$$

---

$$\therefore \sigma_1, \sigma_2 > D \implies \zeta_{dis}(\mathcal{F}, \mathbf{s}) = \sum_{(k_1,k_2)} \frac{\sum_i \delta_i^2 \cdot \nu_{i,k_1,k_2}}{\lambda_{k_1}^{s_1}\lambda_{k_2}^{s_2}}.$$

**Step 2:**

*Upper bound for coefficient averages* $\mathcal{A}(x) := \sum_{\forall j\, \lambda_{k_j} < x} \sum_i \delta_i^2 \cdot \nu_{i,k_1,k_2}$.

**Claim 1:** $\boxed{\mathcal{A}(x) \ll_\varepsilon N_x \cdot x^{e(\mathcal{F})+n+\epsilon}}$.

- $\mu_i(x) := \#\{(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{F}_x^2 : \|\mathbf{m}_1 - \mathbf{m}_2\| = \delta_i\}$.
- $\mathcal{A}(x) = \sum_{i=1}^{N_x} \delta_i^2 \mu_i(x)$.
- *Uniform bound for $\mu_i(x)$: (via integral points on spheres).*

$\forall \mathbf{m}_2 \in \mathcal{F}_x$:

$$\#\{\mathbf{m}_1 \in \mathbb{Z}^n;\ \|\mathbf{m}_1 - \mathbf{m}_2\|^2 = t^2,\ \|\mathbf{m}_1\| \le x\} \ll_\varepsilon t^{n-2+\varepsilon}$$

where implied constant is *uniform* in $m_2$.

$$\text{Setting } t = \delta_i \implies \mu_i(x) \ll_\varepsilon \sum_{\mathbf{m}_2 \in \mathcal{F}_x} t^{n-2+\varepsilon} \ll_\varepsilon x^{e(\mathcal{F})+n-2+\varepsilon}.$$

$$\therefore\ \mathcal{A}(x) \ll_\varepsilon N_x \cdot x^{e(\mathcal{F})+n+\varepsilon}.$$

**Step 2:**
*Upper bound for coefficient averages* $\mathcal{A}(x) := \sum_{\forall j\, \lambda_{k_j} < x} \sum_i \delta_i^2 \cdot \nu_{i,k_1,k_2}$.

**Claim 1:** $\boxed{\mathcal{A}(x) \ll_\varepsilon N_x \cdot x^{e(\mathcal{F})+n+\epsilon}}$.

---

- $\mu_i(x) := \#\{(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{F}_x^2 : \|\mathbf{m}_1 - \mathbf{m}_2\| = \delta_i\}$.
- $\mathcal{A}(x) = \sum_{i=1}^{N_x} \delta_i^2 \mu_i(x)$.
- *Uniform bound for $\mu_i(x)$: (via integral points on spheres).*

$\forall \mathbf{m}_2 \in \mathcal{F}_x$:

$$\#\{\mathbf{m}_1 \in \mathbb{Z}^n;\ \|\mathbf{m}_1 - \mathbf{m}_2\|^2 = t^2,\ \|\mathbf{m}_1\| \leq x\} \ll_\varepsilon t^{n-2+\varepsilon}$$

where implied constant is *uniform* in $m_2$.

$$\text{Setting } t = \delta_i \implies \mu_i(x) \ll_\varepsilon \sum_{\mathbf{m}_2 \in \mathcal{F}_x} t^{n-2+\varepsilon} \ll_\varepsilon x^{e(\mathcal{F})+n-2+\varepsilon}.$$

$$\therefore\ \mathcal{A}(x) \ll_\varepsilon N_x \cdot x^{e(\mathcal{F})+n+\varepsilon}.$$

# Step 3:

*Lower bound* for $\mathcal{A}(x)$ - *Harder part*

**Claim 2:** $\boxed{\mathcal{A}(x) \gg_{\varepsilon} x^{2e(\mathcal{F})+2-\varepsilon}}$.

---

*So:*

$$\text{Claims } 1 + 2 \implies x^{2e(\mathcal{F})+2-\varepsilon} \ll_{\varepsilon} \mathcal{A}(x) \ll_{\varepsilon} x^{e(\mathcal{F})+n+\varepsilon} \cdot N_x$$

$$\therefore \quad N_x := \#\Delta(\mathcal{F}_x) \gg_{\varepsilon} x^{e(\mathcal{F})-n+2-\varepsilon} \gg_{\varepsilon} (\#\mathcal{F}_x)^{1-\frac{n-2}{e(\mathcal{F})}-\varepsilon}.$$

# Step 3:

*Lower bound for $\mathcal{A}(x)$ - Harder part*

**Claim 2:** $\boxed{\mathcal{A}(x) \gg_\varepsilon x^{2e(\mathcal{F})+2-\varepsilon}}$.

---

*So:*

$$\text{Claims } 1+2 \implies x^{2e(\mathcal{F})+2-\varepsilon} \ll_\varepsilon \mathcal{A}(x) \ll_\varepsilon x^{e(\mathcal{F})+n+\varepsilon} \cdot N_x$$

$$\therefore \quad N_x := \#\Delta(\mathcal{F}_x) \gg_\varepsilon x^{e(\mathcal{F})-n+2-\varepsilon} \gg_\varepsilon (\#\mathcal{F}_x)^{1-\frac{n-2}{e(\mathcal{F})}-\varepsilon}.$$

## Idea behind Proof of Claim 2

- Write

$$\begin{aligned}\zeta_{dis}(\mathcal{F},\mathbf{s}) &= \zeta(\mathcal{F},s_1)\zeta(\mathcal{F},s_2-2)+\zeta(\mathcal{F},s_1-2)\zeta(\mathcal{F},s_2)-2\zeta_3(\mathbf{s})\\ &= \zeta_1+\zeta_2-2\zeta_3\end{aligned}$$

where

$$\zeta_3(\mathbf{s}) := \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{F}'\times\mathcal{F}'} \frac{\langle \mathbf{m}_1,\mathbf{m}_2\rangle}{\|\mathbf{m}_1\|^{s_1}\|\mathbf{m}_2\|^{s_2}} = \sum_{i=1}^{n}\zeta_{3,i}(s_1)\zeta_{3,i}(s_2).$$

- $\zeta_1$ is analytic in $\{\sigma_1 > D-2, \sigma_2 > D\}$
- $\zeta_2$ is analytic in $\{\sigma_1 > D, \sigma_2 > D-2\}$
- $\zeta_3$ is analytic in $\{\sigma_i > D-1\ \forall i\}$.
- Each $\zeta_i$ is meromorphic on $\mathbb{C}^2$ with moderate growth.

## Idea behind Proof of Claim 2

- Write

$$\begin{aligned}\zeta_{dis}(\mathcal{F},\mathbf{s}) &= \zeta(\mathcal{F},s_1)\zeta(\mathcal{F},s_2-2)+\zeta(\mathcal{F},s_1-2)\zeta(\mathcal{F},s_2)-2\zeta_3(\mathbf{s})\\ &= \zeta_1+\zeta_2-2\zeta_3\end{aligned}$$

where

$$\zeta_3(\mathbf{s}) := \sum_{(\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{F}'\times\mathcal{F}'} \frac{\langle \mathbf{m}_1,\mathbf{m}_2\rangle}{\|\mathbf{m}_1\|^{s_1}\|\mathbf{m}_2\|^{s_2}} = \sum_{i=1}^{n} \zeta_{3,i}(s_1)\zeta_{3,i}(s_2).$$

---

- $\zeta_1$ is analytic in $\{\sigma_1 > D-2, \sigma_2 > D\}$
- $\zeta_2$ is analytic in $\{\sigma_1 > D, \sigma_2 > D-2\}$
- $\zeta_3$ is analytic in $\{\sigma_i > D-1\ \forall i\}$.
- Each $\zeta_i$ is meromorphic on $\mathbb{C}^2$ with moderate growth.

- **("Initial") Poles of $\zeta_1, \zeta_2, \zeta_3$:**
- $\zeta_1$ has "first" pole (real part) at $Q_1 = (D-2, D)$;
- $\zeta_2$ has "first" pole (real part) at $Q_2 = (D, D-2)$;
- *Real part of "first"* pole $(u_i, v_i)$ of $\zeta_{3,i}(s_1)\zeta_{3,i}(s_2)$ *lies in or below* *Int* $[Q_1, Q_2]$.

# Step 4:

*Perron formula - if Latticelike*

$\boldsymbol{\xi} = (\xi, \xi) \in \mathbb{R}^2$ set $(\boldsymbol{\xi}) := (\xi) \times (\xi)$. Then $\xi > D(= e(\mathcal{F}) + 2) \implies$

$$\mathcal{H}(x_1, x_2) := \sum_{\{\mathbf{m}_i \in \mathcal{F}' : \|\mathbf{m}_i\| < x_i \ \forall i\}} \|\mathbf{m}_1 - \mathbf{m}_2\|^2 \cdot \prod_i \left(1 - \frac{\|\mathbf{m}_i\|}{x_i}\right)$$

$$= \int_{(\boldsymbol{\xi})} (\zeta_1 + \zeta_2 - 2\zeta_3) \cdot \prod_i \frac{x_i^{s_i}}{s_i(s_i + 1)} ds_1 ds_2$$

---

Next: Iterate Cauchy residue method.
For $\zeta_1$ resp. $\zeta_2$ move to $(Q_1 - \eta)$ resp. $(Q_2 - \eta)$ $(\eta \ll 1)$ etc.

- $\implies \exists A, B$ s.t. $A + B < 2D - 2$ and

$$\mathcal{H}(x_1, x_2) = x_1^{D-2} x_2^{D} \phi_1(\log x_1, \log x_2) + x_1^{D} x_2^{D-2} \phi_2(\log x_1, \log x_2)$$

$$+ \sum_{i=1}^{t} \pm x_1^{u_i} x_2^{v_i} \psi_i(\log x_1, \log x_2) + O(x_1^{A-\varepsilon} \cdot x_2^{B-\varepsilon})$$

where $\phi_1$, $\phi_2, \psi_i$ are *non zero* almost periodic functions.

# Step 4:

*Perron formula - if Latticelike*

$\boldsymbol{\xi} = (\xi, \xi) \in \mathbb{R}^2$ set $(\boldsymbol{\xi}) := (\xi) \times (\xi)$. Then $\xi > D(= e(\mathcal{F}) + 2) \implies$

$$\mathcal{H}(x_1, x_2) := \sum_{\{\mathbf{m}_i \in \mathcal{F}' : \|\mathbf{m}_i\| < x_i \ \forall i\}} \|\mathbf{m}_1 - \mathbf{m}_2\|^2 \cdot \prod_i \left(1 - \frac{\|\mathbf{m}_i\|}{x_i}\right)$$

$$= \int_{(\boldsymbol{\xi})} (\zeta_1 + \zeta_2 - 2\zeta_3) \cdot \prod_i \frac{x_i^{s_i}}{s_i(s_i + 1)} ds_1 ds_2$$

---

Next: Iterate Cauchy residue method.

For $\zeta_1$ resp. $\zeta_2$ move to $(Q_1 - \boldsymbol{\eta})$ resp. $(Q_2 - \boldsymbol{\eta})$ $(\eta \ll 1)$ etc.

- $\implies \exists A, B$ s.t. $A + B < 2D - 2$ and

$$\mathcal{H}(x_1, x_2) = x_1^{D-2} x_2^D \phi_1(\log x_1, \log x_2) + x_1^D x_2^{D-2} \phi_2(\log x_1, \log x_2)$$
$$+ \sum_{i=1}^{t} \pm x_1^{u_i} x_2^{v_i} \psi_i(\log x_1, \log x_2) + O(x_1^{A-\varepsilon} \cdot x_2^{B-\varepsilon})$$

where $\phi_1$, $\phi_2, \psi_i$ are *non zero* almost periodic functions.

# Step 5:

*Monomialisation-if Latticelike*

- Polygon from $\mathcal{H}$ monomials: vertices at $Q_1, Q_2$.

---

- Choose, say, $Q_1 = (D-2, D)$. $\forall \kappa' > \kappa > 1$ show:

$\exists\ \mathbf{x}_k = (x_{k,1}, x_{k,2})$ s.t. $x_{k,1}^{\kappa} < x_{k,2} < x_{k,1}^{\kappa'}$,

$$\text{and} \begin{cases} \phi(\log x_{k,1}, \log x_{k,2})) \gg 1 \\ \\ x_{k,1}^{D} x_{k,2}^{D-2} + \sum_i x_{k,1}^{u_i} x_{k,2}^{v_i} = o(x_{k,1}^{D-2} x_{k,2}^{D}) = o(\mathbf{x}_k^{Q_1}) \end{cases}$$

---

- $\therefore\ \mathcal{H}(x_{k,1}, x_{k,2}) = \mathbf{x}_k^{Q_1} \phi(\log x_{k,1}, \log x_{k,2}) + o(\mathbf{x}_k^{Q_1}) \gg \mathbf{x}_k^{Q_1}$.

$$\therefore \boxed{\mathcal{A}(x) \gg_{\varepsilon} x^{2e(\mathcal{F})+2-\varepsilon}} \tag{1}$$

---

- **Non latticelike case:** *Variant of monomialisation proves (1) but uses "screens" - needed due to possible pole clustering on* $\sigma = D, D-2$.

# Step 5:

*Monomialisation-if Latticelike*

- Polygon from $\mathcal{H}$ monomials: vertices at $Q_1, Q_2$.

- Choose, say, $Q_1 = (D-2, D)$. $\quad \forall \kappa' > \kappa > 1 \quad$ <u>show</u>:

$\exists\ \mathbf{x}_k = (x_{k,1}, x_{k,2})$ s.t. $x_{k,1}^{\kappa} < x_{k,2} < x_{k,1}^{\kappa'}$,

$$\textit{and} \begin{cases} \phi\big(\log x_{k,1}, \log x_{k,2}\big)\big) \gg 1 \\ \\ x_{k,1}^{D} x_{k,2}^{D-2} + \sum_i x_{k,1}^{u_i} x_{k,2}^{v_i} = o(x_{k,1}^{D-2} x_{k,2}^{D}) = o(\mathbf{x}_k^{Q_1}) \end{cases}$$

- $\therefore\ \mathcal{H}(x_{k,1}, x_{k,2}) = \mathbf{x}_k^{Q_1}\phi(\log x_{k,1}, \log x_{k,2}) + o(\mathbf{x}_k^{Q_1}) \gg \mathbf{x}_k^{Q_1}$.

$$\therefore \boxed{\mathcal{A}(x) \gg_{\varepsilon} x^{2e(\mathcal{F})+2-\varepsilon}} \qquad (1)$$

- **Non latticelike case:** *Variant of monomialisation proves (1) but uses "screens" - needed due to possible pole clustering on $\sigma = D, D-2$.*

# Step 5:

*Monomialisation-if Latticelike*

- Polygon from $\mathcal{H}$ monomials: vertices at $Q_1, Q_2$.

- Choose, say, $Q_1 = (D-2, D)$. $\forall \kappa' > \kappa > 1$ <u>show</u>:

$\exists\ \mathbf{x}_k = (x_{k,1}, x_{k,2})$ s.t. $x_{k,1}^{\kappa} < x_{k,2} < x_{k,1}^{\kappa'}$,

$$\textit{and} \begin{cases} \phi\big(\log x_{k,1}, \log x_{k,2})\big) \gg 1 \\ \\ x_{k,1}^{D} x_{k,2}^{D-2} + \sum_i x_{k,1}^{u_i} x_{k,2}^{v_i} = o(x_{k,1}^{D-2} x_{k,2}^{D}) = o(\mathbf{x}_k^{Q_1}) \end{cases}$$

- $\therefore\ \mathcal{H}(x_{k,1}, x_{k,2}) = \mathbf{x}_k^{Q_1} \phi(\log x_{k,1}, \log x_{k,2}) + o(\mathbf{x}_k^{Q_1}) \gg \mathbf{x}_k^{Q_1}$.

$$\therefore \boxed{\mathcal{A}(x) \gg_{\varepsilon} x^{2e(\mathcal{F})+2-\varepsilon}} \tag{1}$$

- **Non latticelike case:** *Variant of monomialisation proves (1) but uses "screens" - needed due to possible pole clustering on $\sigma = D, D-2$.*

# Step 5:

*Monomialisation-if Latticelike*

- Polygon from $\mathcal{H}$ monomials: vertices at $Q_1, Q_2$.

---

- Choose, say, $Q_1 = (D-2, D)$. $\forall \kappa' > \kappa > 1$ <u>show</u>:

$\exists\, \mathbf{x}_k = (x_{k,1}, x_{k,2})$ s.t. $x_{k,1}^{\kappa} < x_{k,2} < x_{k,1}^{\kappa'}$,

$$\textit{and} \begin{cases} \phi\big(\log x_{k,1}, \log x_{k,2})\big) \gg 1 \\ \\ x_{k,1}^{D} x_{k,2}^{D-2} + \sum_i x_{k,1}^{u_i} x_{k,2}^{v_i} = o(x_{k,1}^{D-2} x_{k,2}^{D}) = o(\mathbf{x}_k^{Q_1}) \end{cases}$$

---

- $\therefore\ \mathcal{H}(x_{k,1}, x_{k,2}) = \mathbf{x}_k^{Q_1} \phi(\log x_{k,1}, \log x_{k,2}) + o(\mathbf{x}_k^{Q_1}) \gg \mathbf{x}_k^{Q_1}$.

$$\therefore \boxed{\mathcal{A}(x) \gg_{\varepsilon} x^{2e(\mathcal{F})+2-\varepsilon}} \tag{1}$$

---

- **Non latticelike case:** *Variant of monomialisation proves (1) but uses "screens" - needed due to possible pole clustering on* $\sigma = D, D-2$.

• **Forthcoming article proves Erdös distinct distance problem for $x-$balls** *of any self-similar* **subset of $\mathbb{Z}^2$.**

• **Need use Ingham's Tauberian theorem (1965) instead of Cauchy Residues and "screens" if non latticelike.**

# Angles

For any $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'$ set:

$$\theta(\mathbf{m}_1, \mathbf{m}_2) := \text{ angle formed between } \mathbf{m}_1, \mathbf{m}_2.$$

$$Ang(x) := \#\{cos\big(\theta(\mathbf{m}_1, \mathbf{m}_2)\big);\ \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}_x\}.$$

*Find bound on $e(\mathcal{F})$ s.t. $Ang(x) \nearrow +\infty$.*

**Theorem 8 [PLMS 2015].**
*$e(\mathcal{F}) > n-2$ and $n \geq 4 \implies$*

$$Ang(x) \gg_\varepsilon [\#\mathcal{F}(x)]^{1-\frac{n-2}{e(\mathcal{F})}-\varepsilon} \quad (x \gg 1).$$

- Uses Angle Zeta function:

$$\zeta_{ang}(\mathcal{F}; s_1, s_2) := \sum_{\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'} \frac{\left\langle \frac{\mathbf{m}_1}{\|\mathbf{m}_1\|}, \frac{\mathbf{m}_2}{\|\mathbf{m}_2\|} \right\rangle^2}{\|\mathbf{m}_1\|^{s_1} \|\mathbf{m}_2\|^{s_2}}.$$

# Angles

For any $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'$ set:

$$\theta(\mathbf{m}_1, \mathbf{m}_2) := \text{ angle formed between } \mathbf{m}_1, \mathbf{m}_2.$$

$$Ang(x) := \#\{cos\big(\theta(\mathbf{m}_1, \mathbf{m}_2)\big);\ \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}_x\}.$$

*Find bound on $e(\mathcal{F})$ s.t. $Ang(x) \nearrow +\infty$.*

**Theorem 8 [PLMS 2015].**
*$e(\mathcal{F}) > n - 2$ and $n \geq 4$* $\implies$

$$Ang(x) \gg_\varepsilon [\#\mathcal{F}(x)]^{1-\frac{n-2}{e(\mathcal{F})}-\varepsilon} \quad (x \gg 1)\,.$$

- Uses Angle Zeta function:

$$\zeta_{ang}(\mathcal{F}; s_1, s_2) := \sum_{\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{F}'} \frac{\left\langle \frac{\mathbf{m}_1}{\|\mathbf{m}_1\|}, \frac{\mathbf{m}_2}{\|\mathbf{m}_2\|} \right\rangle^2}{\|\mathbf{m}_1\|^{s_1} \|\mathbf{m}_2\|^{s_2}}.$$