Triangulations with valence bounds

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This poster explains some interconnections of geometry, topology, and combinatorics of manifold triangulations. Simple combinatorial proofs of the following facts are sketched:

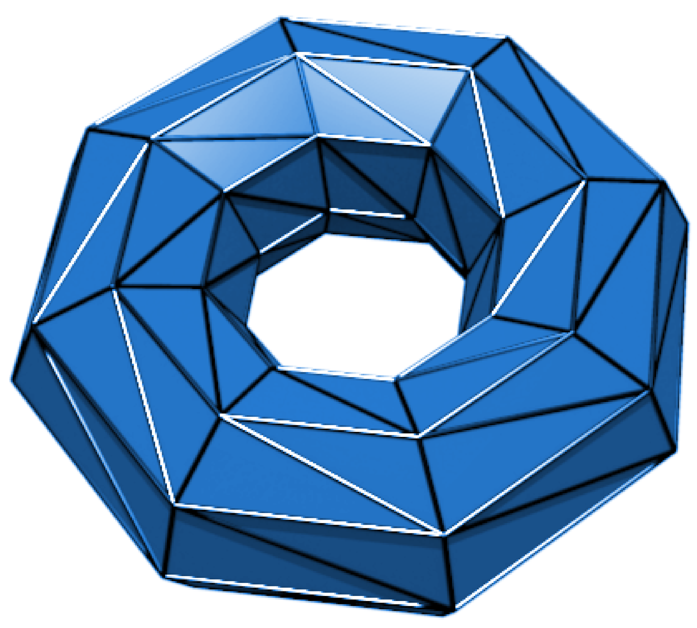
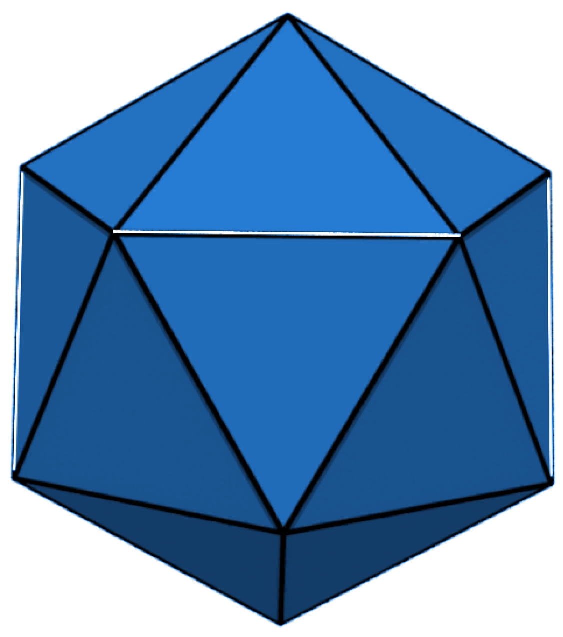
1. every orientable surface admits an equivelar triangulation,
2. every closed 3-manifold admits a triangulation with edge valences ≤ 6,
3. there are three combinatorial types of vertex links that suffice to triangulate any closed 3-manifold,
4. any PL-manifold can be triangulated with valences of codimension-two faces bounded by nine.

Here a triangulation will always be a simplicial complex.

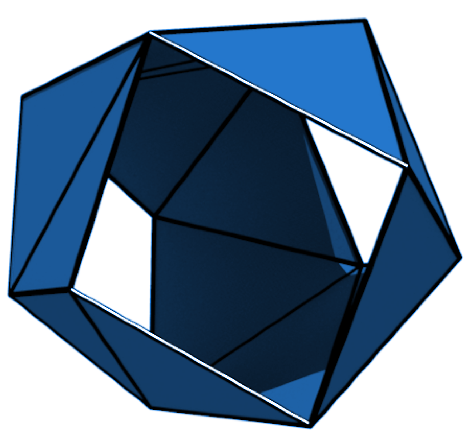
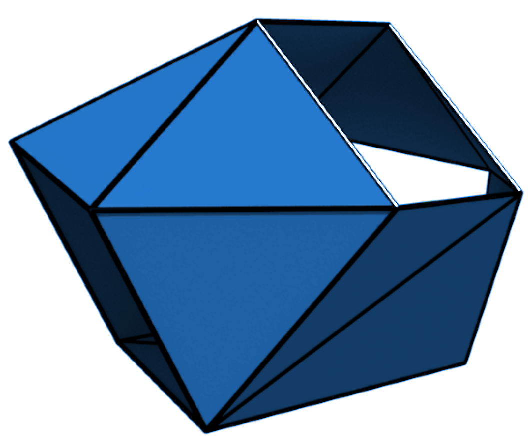
Geometrization in dimension 2

A triangulation of a surface is called *equivelar* if every vertex has the same degree.

Theorem. Every closed orientable surface has an equivelar triangulation.

The torus can be triangulated with constant degree 6, the sphere with constant degree 3, 4, or 5. Orientable hyperbolic surfaces can be triangulated with constant degree 7. Just glue copies of the triangulation of a pair of pants (below) together in an appropriate way.

Gluing two of these pairs of pants together yields an equivelar triangulation of the surface of genus 2. Every other orientable hyperbolic surface is a finite cover of this surface, so the triangulation lifts to equivelar triangulations of all surfaces with genus $g \geq 2$.

The non-orientable surface with Euler characteristic -1 does not admit an equivelar triangulation.

By inducing the correct metric on each triangle and lifting non-orientable surfaces to their orientable double covers we obtain:

Corollary (Geometrization)**.** Every surface admits a constant curvature metric.

Valence bounds in dimension 3

The *valence* of an edge in a 3-dimensional triangulation is the number of facets it is contained in. This is a discrete analogue of curvature.

Theorem (Brady, McCammond, Meier, 2004 [1])**.** Every closed orientable 3-manifold admits a triangulation with valences $4, 5$ and 6.

Earlier results concerning valence bounds were obtained by Cooper & Thurston.

Theorem (Cooper & Thurston, 1988 [2])**.** Every closed orientable 3-manifold can be tiled face-to-face by cubes with 3, 4, or 5 cubes around any edge. Edges contained in 3, respectively 5, cubes form embedded circles that go straight through vertices.

By considering the barycentric subdivision of this tiling, Cooper & Thurston noted the following corollary.

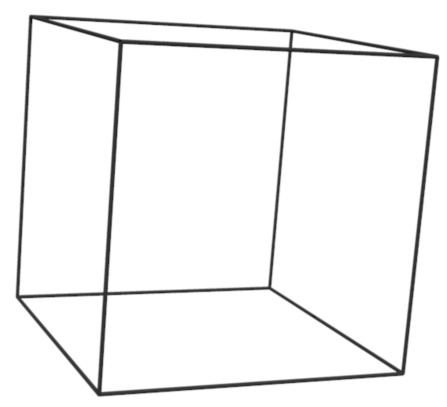
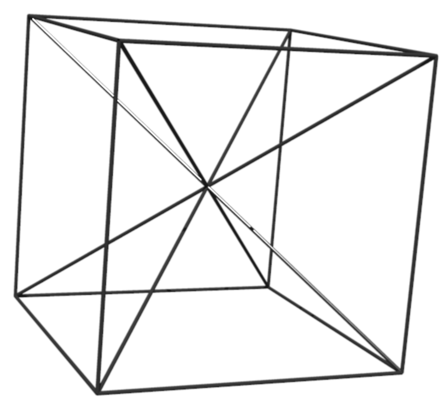
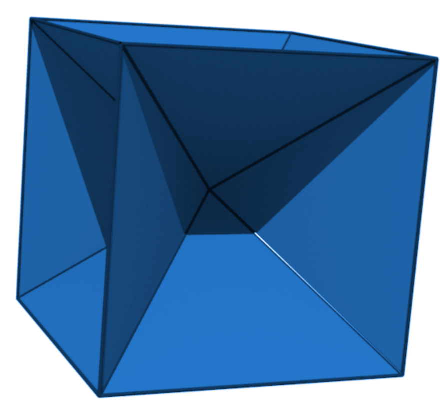
Corollary (Cooper & Thurston, 1988 [2])**.** Every closed orientable 3-manifold can be triangulated with valence bounded by 10. This triangulation uses only five fixed combinatorially different kinds of vertex links.

Here we modify the tiling in a different way to obtain:

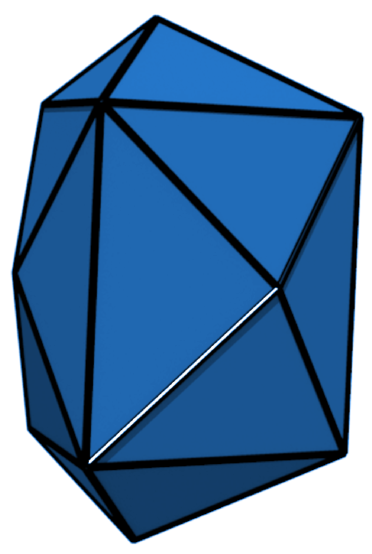
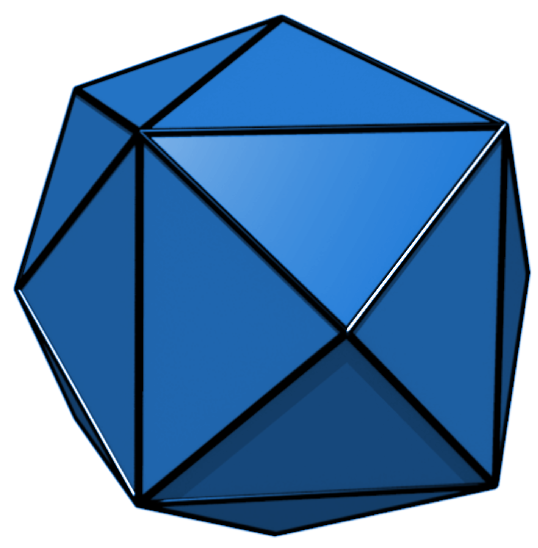
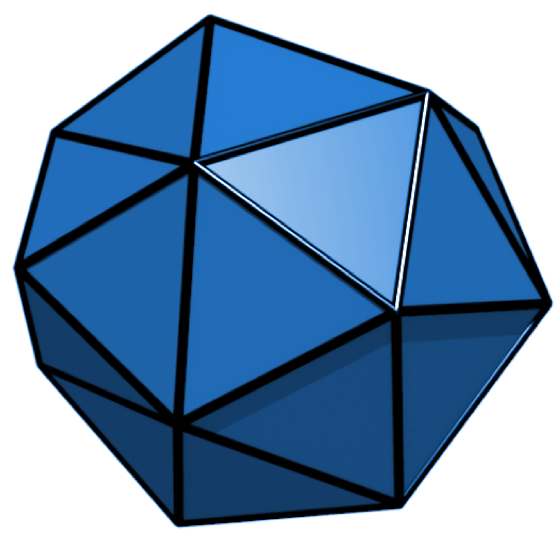
Corollary. Every closed orientable 3-manifold can be triangulated such that each triangle has two edges of valence 6 and one edge of valence 3, 4, or 5. Moreover this triangulation uses only three fixed combinatorially different kinds of vertex links.

The second part of this corollary was noticed already by Kevin Walker (unpublished, personal communication). Triangulations where each triangle has edges of valence 6, 6, and k are spherical for $k = 3$, Euclidean for $k = 4$, and hyperbolic for $k = 5$. Mixing these allows us to triangulate any closed orientable 3-manifold. By flipping edges one can also obtain the result of Brady, McCammond & Meier.

Sketch of proof. Tile the manifold by cubes as in the theorem above. Put a vertex at the barycenter of each cube and add a triangle for each edge of the cube with apex the barycenter. Delete all quadrilaterals. This gives a tiling by octahedra. Triangulate each octahedron by introducing an edge orthogonal to the deleted quadrilateral. □

The three vertex links that suffice to triangulate any closed orientable 3-manifold are obtained from stacking each face in a triangular prism, a cube, and a pentagonal prism.

Valence bounds in higher dimensions

A face of dimension $d-2$ in a d-dimensional triangulation is called *subridge*. The *valence* of a subridge is the number of facets it is contained in.

Theorem. Every PL-manifold admits a triangulation with valences of subridges bounded by 9.

Sketch of proof. Start with any PL-triangulations. Then the combinatorial dual is a strongly regular and simple CW-complex with simple facets. Triangulate the 2-skeleton such that vertices have degree at most nine. Triangulate the 3-skeleton by stacking every 3-dimensional face. Continue inductively for each skeleton. □

References

[1] Noel Brady, Jon McCammond, and John Meier, *Bounding edge degrees in triangulated 3-manifolds*, Proceedings of the AMS **132** (2004), no. 1, 291–298.

[2] Daryl Cooper and William P. Thurston, *Triangulating 3-manifolds using five vertex link types*, Topology **27** (1988), no. 1, 23–25.

[3] Murray Elder, Jon McCammond, and John Meier, *Combinatorial conditions that imply wordhyperbolicity for 3-manifolds*, Topology **42** (2003), no. 6, 1241–1259.

[4] Florian Frick, Frank H. Lutz, and John M. Sullivan, *Simplicial manifolds with small valence*, In preparation.

[5] Kevin Walker, personal communication, 2013-02-12.

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