Games and invariants

Instructor notes for leading a module in Math explorers' club

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Goals

(a) Understanding what an invariant is, through playing games.

(b) Thinking about winning strategies in games through invariants.

Pedagogy

(a) Work out the problems (labelled as questions in this document) on the board, and encourage participation from students.

(b) Set up activities/games on different tables in the room or on chalk boards (as required) and have different groups of students take part in them.

(c) As the students gain comfort with the activities, make minor changes to some parameters of the game to see if they can spot the concept behind the activity.

Day 1

Question 1

(a) From the list of numbers $\{1, 2, 3, 4\}$, erase any two numbers and replace it by the sum of the two numbers. What number do we end up with if we keep repeating this process?

(b) Did the order of erasing the numbers matter in the previous question? What if we do the same, starting with the list of numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

(c) What number do we end up with if we start with $\{1, 2, ..., 100\}$?

Comment: The sum of all the numbers is an invariant.

Question 2

(a) We start with the list of numbers $\{1, 2, 3, 4, 5\}$ and this time we are allowed to erase any two numbers a and b and replace it by $a + b - 1$. What number do we end up with if we keep repeating this process?

(b) If we start with the list $\{1, 2, ..., 100\}$, what is the new answer?

Comment: (Sum of numbers - number of numbers) is an invariant.

Question 3

We start with $\{1, 2, ..., 100\}$ and keep removing two numbers at a time as before. At each step what we do differently is that we replace the removed numbers by their difference. What can we say about the number that we end up with?

Comment: We end up with an even number.

Question 4

A dragon has 100 heads. A knight can cut off $15, 17, 20$, or 5 heads with one blow of the sword. In each of these cases, $24, 2, 14$, or 17 new heads grow on its shoulder. If all the heads are cut off, the dragon dies. Can the dragon ever die?

Comment: The number of heads always leaves a remainder 1 when divided by 3.

Activity 1: Flipping cups

Setup: Place 15 cups upside down on the table. The objective is to get all the cups standing right side up. There is a restriction that exactly 4 cups should be flipped at a time.

Idea: Parity of number of upside down cups is an invariant.

Activity 2: Pieces in a bag

Setup: This is a game between two players. Place 7 black pieces and 8 white pieces in a bag on a table. Players alternate removing two pieces at a time. If they are both white, return one of the pieces to the bag. If they are of two different colors, the black piece is returned to the bag. If the bag is left with only one black piece, the first player wins. If the bag is left with only one white piece, the second player wins.

Idea: The number of black pieces in the bag is always odd. The first player always wins.

Activity 3: Coins and clips on a chess board

Setup: This is a game between two players. The first player can place two coins on any two squares of the chess board. The second player has to place 31 clips on the chess board such that all the square of the chess board are covered. Each clip covers exactly two adjacent squares. If the second player can't cover the chess board then the first player wins.

Idea: The first player wins if the coins are placed on squares of the same color (as each clip will cover a black and a white square). Otherwise the second player wins (Gomory's theorem).

Activity 4: Chess pieces, coins and clips

Setup: There are 13 chess pieces, 15 coins and 17 clips. Any two different pieces can be removed and replaced with a piece of the third kind. The objective is to get all pieces to be the same.

Idea: The remainder when (number of chess pieces - number of coins) is divided by 3 is a constant. The objective can never be attained.

Activity 5: Sectors of a circle - on the board

Setup: A circle is drawn on the board and divided into six sectors. The numbers $1, 0, 1, 0, 0, 0$ are written on them in this order. The objective is to get all the numbers to be the same. The only allowed operation is to add one to any two adjacent sectors.

Idea: Alternating sum of alternate sectors (i.e. $1 - 0 + 1 - 0 + 0 = 2$) is a constant.

Activity 6: Euler characteristic for planar graphs - on the board

Setup: Draw some planar graphs and ask the students to calculate $v - e + f$ (i.e. number of vertices - number of edges + number of faces). They win a prize if they can draw a connected planar graph whose $v - e + f$ is bigger than 1.

Idea: This Euler characteristic is an invariant. Nobody wins a prize.

Activity 7: Signs - on the board

Setup: This is a game between two players. Player 1 writes atleast ten signs on the board each of which is either $+$ or $-$. Player 2 erases any two signs and replaces them with a $+$ sign if they were equal and a $-$ sign if they were unequal. Player 2 repeatedly does this until there is only one sign on the board. Player 1 wins if it is a $+$ sign and player 2 wins if it is a $-$ sign.

Idea: Thinking of the signs as $+1$ and -1, the product of the numbers is an invariant.

Day 2

Question 5

We have a 8×8 table with one of the boxes colored black and the rest colored white. If we can flip all the colors of boxes of any row or column, can we ever get all the boxes to be white?

Comment: The number of black boxes is always odd.

Question 6

We have a 3×3 table with the top left box colored black and the rest colored white. If we can flip all the colors of boxes of any row or column, can we ever get all the boxes to be white?

Comment: The number of black boxes among the highlighted ones is always odd.

$*$	$*$	$*$
$*$	$*$	$*$
$*$	$*$	$*$

Question 7

Starting with the list of numbers $\{1, 2, 3, 4, 5, 6, 7\}$, we are allowed to swap any two adjacent entries. Can we get back the original list after 5 swaps?

Comment: After each swap, the number of inversions (i.e. the pairs of numbers where the bigger number comes before the smaller number) changes by 1.

Activity 8: Pieces on sectors

Setup: A circular sheet is divided into six sectors and placed on a table. A chess piece is placed on each sector. The objective is to get all the pieces on the same sector. The only allowed operation is to take any two pieces and move each of them to any of their respective adjacent sectors.

Idea: The objective cannot be attained. The alternating sum of number of pieces on alternate sectors always leaves a reminder 0 when divided by 4.

Activity 9: Pennies on a circle.

Setup: This is a game between two players. In the first version of the game, 12 pennies are placed in a circle on a table. Players alternate in removing one penny or two adjacent pennies. Whoever removes the last penny wins the game. In the second version of the game, there are 11 pennies instead of 12.

Idea: Player two can always win the game. In the first version of the game, player 2 can symmetrically mimic player 1. In the second version of the game, player 2 will remove one penny in the first move if player 1 had removed two pennies and vice versa. The first move for player 2 is at the diametrically opposite side of the circle as player 1's first move. For the rest of the game, player two can symmetrically mimic player 1.

Activity 10: Nim

Setup: This is a game between two players. There are three piles with $1, 2$ and 2 coins respectively. Players take turns removing any number of coins from any pile. The player who removes the last coin loses.

Idea: Player 1 can always win the game.

Activity 11: Nim (alternate version) - on the board

Setup: This is a game between two players. Player 1 writes 1 on the board. Players take turns in adding 1, 2 or 3 without crossing 21. The player to write 21 loses the game.

Idea: Player 2 can always win the game by ensuring the number is always a multiple of 4.

Activity 12: Coloring nodes on simple graph - on the board.

Setup: Draw a connected graph with 8 nodes and 7 edges. This is a game between two players. Players take turns coloring uncolored nodes. Player 1 colors red and player 2 colors blue. Player 1 wins if there are odd number of red-blue edges. Player 2 wins if there are even number of red-blue edges.

Idea: Player 1 can always force a win. The color of the two end nodes determine the parity of the number of red-blue edges.

Activity 13: Matrix with ± 1 - on the board.

Setup: Draw the following table on the board:

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

The only allowed operation is to flip the sign of all entries in any row, column or entries parallel to one of the diagonals. The objective is to get all the entries of the table to be 1 by repeatedly performing the allowed operation.

Idea: Product of the highlighted entries is an invariant:

1	$\color{red}{1}$	$\color{red}{1}$	1
$\color{red}{1}$	1	1	$\color{red}{1}$
$\color{red}{1}$	1	1	$\color{red}{1}$
1	$\color{red}{-1}$	$\color{red}{1}$	1

Activity 14: Changing entries in a table - on the board.

Setup: Draw the following tables on the board:

1	2	3
4	5	6
7	8	9

7	8	9
6	2	4
3	5	1

The only allowed operation is to add (or subtract) an integer from any two adjacent entries in the first table. The objective is to go from the first table to the second by repeatedly performing the allowed operation.

Idea: (Sum of the red squares - sum of the other squares) is a constant.

$*$	$*$	$*$
$*$	$*$	$*$
$*$	$*$	$*$

References

[1] Engel, A; *Problem Solving strategies, Springer, 1999*

[2] Fomin, D; Genkin, S; Itenberg I; *Mathematical Circles (Russian Experience), Mathematical World, Vol. 7*