Class Notes for Math 1110, Section 5

Timothy Goldberg

Spring 2009

Contents

Remark. These are notes for Section 5 of Math 1110, taught at Cornell University during the spring semester of 2009. This is a first-semester course in single variable calculus. These notes are only meant to *supplement* reading from the textbook, *not replace it*. They do not contain all the material necessary for this class.

1 Day One: January 19

1.1 Introduction

Calculus can be viewed as a collection of concepts and techniques that were invented/discovered to answer the question, "How can we understand the nature of changing quantities?" This differs from other mathematical topics, such as algebra and geometry, which typically study static, unchanging situations.

Math 1110 is a first-semester calculus course. In broad strokes, it covers limits, differentiation, integration, relations among them, and applications of each. I have two objectives for this course. The first is to help you to learn, understand, and apply the basic features of calculus, whose concepts and techniques are the foundation for many other classes in mathematics and other fields. The second objective is more general but equally important. It is to help you understand how mathematics can be used to better grasp, analyze, and solve problems in all sorts of contexts. Mathematics provides a way of thinking about situations and information in a logical and systematic fashion. Helping you to develop this understanding and these skills should be a part of every math course you take.

1.2 Relations between quantities

1.2.1 Equations

The most straightforward example of a relationship between quantities is an equation.

$$A = 2\pi r^2, \qquad x^2 + y^2 = 1, \qquad y^2 = x^3 - x + 1$$

The examples given here each expresses a relationship between only two quantities — between A and r in the first case, and between x and y in the second and third. But certainly there can be arbitrarily many quantities involved in an equation.

The big contribution of Descartes to mathematics was to realize that we can literally picture equations (specifically, their solution sets), by assigning coordinates to points in space. By putting an xy-axis on a plane, we can assign to each point a unique coordinate (x, y). We form the **graph of an equation** involving the variables x and y by coloring in each point in the plane whose coordinates (x, y) satisfy the given equation.

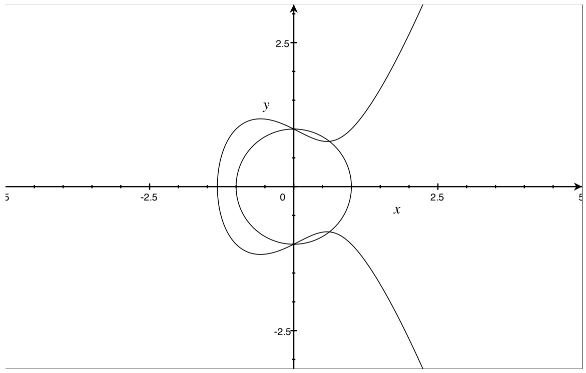

Figure 1: The circle is the graph of the equation $x^2 + y^2 = 1$. The other curve is the graph of the equation $y^2 = x^3 - x + 1$, called an **elliptic curve**.

Notice that it is a minor miracle that the graphs of most equations we encounter form nice curves. If you randomly start coloring in points in the plane, there's no particular reason to think that they would form a curve.

1.2.2 Equations that are functions

The simplest kind of equation is one where one of the variables is alone on one side of the equation.

$$y = x^2 - 1$$

In this equation, the value of y is completely determined by the value of x. We say that "y is a function of x", and we can write $y(x) = x^2 - 1$, or $y = f(x)$ where $f(x) = x^2 - 1$. There are many different notations.

Because each value of x determines a specific value of y, the graph of an equation involving x and y where y is a function of x will have a particularly nice property. Above each value on the x-axis, there can be at most one point on the graph. Otherwise, there would be two values of y that correspond to a single value of x.

Another way to characterize this property is called the **vertical line test**. An equation gives y as a function of x if and only if it is impossible to draw a vertical line in the plane going through more than one point on the graph. For instance, notice that the circle pictured in Figure 1 fails the Vertical Line Test, so it cannot be the graph of a function.

1.3 Functions in general

The concept of a function actually encompasses things much more general than just those defined by equations.

Notation.

A **set** is a collection of objects. An **element** is an object in a set. The notation "$a \in A$" means "a is an element of the set A". The notation "$B \subset A$" means that "the set B is a **subset** of the set A", which simply means that every element of B is also an element of A.

We denote the set of **real numbers** by the symbol $\mathbb{R}$. The set of real numbers includes all integers, fractions of integers, square roots, cube roots, et cetera. It also includes **transcendental numbers**, like π and e. The real numbers exactly consist of all possible decimal numbers, terminating and non-terminating, repeating and non-repeating. The real numbers fill up the entire number line.

Non-examples of real numbers include, but are not limited to, the **imaginary unit** $\mathrm{i} = \sqrt{-1}$, all complex numbers such as $2 + 3\mathrm{i}$, and me, Timothy Goldberg. Also, apples are not real numbers.

Example 1.1.

Consider the open interval $(-1, 3)$, which consists of all real numbers x such that $-1 < x < 3$. Then $0 \in (-1, 3)$, and $(-1, 3) \subset \mathbb{R}$.

Definition 1.2.

A **function** f between two sets A and B is any assignment of an element $f(a) \in B$ to each element $a \in A$. We use the notation "$f\colon A \to B$" to denote such a function. The set A of inputs is called the **domain** of f, and the set B of outputs is called the **codomain** of f.

The set of elements of B which *actually appear* as outputs of f is called the **range**, or **image** of f, denoted by $f(A)$. (This notation is meant to remind you that the range is what you get by dumping the entire domain into the function.) Of course, $f(A) \subset B$.

For later reference, the symbol ":=" means "is defined to be equal to", whereas "=" just means "is equal to".

It may be silly to have separate concepts for the codomain and the range. Why don't we only use the range, the set of things that can actually come out of the function? The reason is that it is sometimes quite difficult to figure out what the range of a given function is. It's usually easier just to say what the output *could be*, without having to peg down exactly what it *is*.

Example 1.3.

1. Most functions we will deal with in this course are **real-valued functions**, which means they are of the form

$$f\colon \mathbb{R} \to \mathbb{R}.$$

   In general, if we are given a function in the form of a formula, like $f(x) = x^2$, then we assume that it's domain is the largest subset of $\mathbb{R}$ for which the formula is defined. For instance, given the formula $g(x) = \sqrt{x-2}$, we assume that the domain of g is the set of $x \in \mathbb{R}$ such that $x - 2 \geq 0$. Hence,

$$g\colon [2, \infty) \to \mathbb{R}.$$

2. Consider the function **mother** defined by

$$\textbf{mother}(x) := \text{the mother of } x.$$

   For example, **mother** (Gwynneth Paltrow) = Blythe Danner. This is a function

$$\textbf{mother}\colon \{\text{people}\} \to \{\text{people}\}.$$

   Of course, we could have set the codomain to be the set {women}. The image of this function is exactly the set of women who have given birth.

3. It's important to realize that even though most function involving numbers that we encounter are given as formulas, there are certainly others for which there isn't a formula. For example, consider the function $\textbf{favorite}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} \textbf{favorite}(x) \quad := \quad & \text{the number of people enrolled in Section 5 of} \\ & \text{Math 1110 this spring whose favorite number is } x. \end{aligned}$$

   Of course, this function will be equal to 0 almost everywhere.

Remark 1.4.

A very important point to keep in mind is that when we write down a formula for a function, like $f(x) = \cos(x^2)$, *the variable we use doesn't matter*. For this reason, it is sometimes called a **dummy variable**. The following formulae all define the same function.

$$\cos(x^2), \quad \cos(z^2), \quad \cos(\theta^2), \quad \cos\left(\triangle^2\right), \quad \cos\left(☺^2\right)$$

1.4 Building new functions from old

Given two real-valued functions $f, g\colon A \to \mathbb{R}$ defined on some set $A \subset \mathbb{R}$, because we can add, subtract, multiply, and divide real numbers, we can construct several new functions.

$$\begin{aligned}
(f+g)(x) &:= f(x) + g(x) \\
(f-g)(x) &:= f(x) - g(x) \\
(f \cdot g)(x) &:= f(x) \cdot g(x) \\
(f/g)(x) &:= \frac{f(x)}{g(x)}
\end{aligned}$$

The domains of $f+g$, $f-g$, and $f \cdot g$ are all A, just like those of the original functions. But since division by zero is undefined, the domain of f/g is the set

$$\{x \in A \text{ such that } g(x) \neq 0\}.$$

This way of constructing new functions should be very familiar to you, so I won't include any example.

A more general way of creating new functions, one which does not depend on the functions being real-valued, is **function composition**. Let A, B, and C be sets, and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Their **composition** is the function $g \circ f\colon A \to C$ defined by

$$(g \circ f)(a) := g\left(f(a)\right)$$

for all $a \in A$. Notice that if you want to form the composition $g \circ f$ of two functions, you have to be sure the output of the **inner function** f is allowed to be an input of the **outer function** g.

Example 1.5.

Let $f(x) = \sin x$ and $g(x) = x^2 + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f\left(x^2+1\right) = \sin\left(x^2+1\right)$$

and

$$(g \circ f)(x) = g(f(x)) = g(\sin x) = (\sin x)^2 + 1 = \sin^2 x + 1.$$

The last way of constructing new functions from old that we will describe is **piecewise-defined functions**. In a piecewise-defined function, the domain is broken up into several pieces, and each piece is assigned a function. Consider the following formula.

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 3x & \text{if } 0 \le x < 2 \\ -2x+10 & \text{if } x \ge 2 \end{cases}$$

This represents a function $f\colon \mathbb{R} \to \mathbb{R}$ whose domain is split up into three pieces:

$$(-\infty, 0), \quad [0,2), \quad \text{and} \quad [2,\infty).$$

When you plug a particular value of x into $f(x)$, the formula you use to evaluate it depends on which piece of the domain contains that value of x. So

$$\begin{aligned} f(-1) &= (-1)^2 = 1, \\ f(1) &= 3(1) = 3, \text{ and} \\ f(3) &= -2(3) + 10 = -6 + 10 = 4. \end{aligned}$$

We will now give two important examples of piecewise-defined functions.

Example 1.6.

1. The **absolute value function**, $|x|$, can be defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}.$$

A little thought, and perhaps working through a couple of examples, will show that this definition agrees with the usual definition of the absolute value of a number x as the distance between x and 0, or as the number x written without its sign.

2. The **signum function**, $\mathbf{sgn}(x)$, can be defined by

$$\mathbf{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

The signum function simply measures the sign of its input, and is undefined at 0. It is not hard to show that

$$\mathbf{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}.$$

2 Day Two: January 21

2.1 Introduction

Today's topic is LIMITS OF FUNCTIONS. To ease into this topic, consider the function $f(x) = \frac{x^2-3x+2}{x-2}$. We can factor the numerator and write

$$f(x) = \frac{x^2-3x+2}{x-2} = \frac{(x-2)(x-1)}{x-2}.$$

It is incredibly tempting to cancel the factor $x-2$ from the numerator and denominator, leaving the function $x-1$. The question is, **what is the difference between the functions $\frac{(x-2)(x-1)}{x-2}$ and $x-1$?**

For any value of x except $x = 2$, these two functions have the same value. For $x = 2$, the first function is undefined but the second simply evaluates to $2 - 1 = 1$. So the only difference between the two functions is their domains. Therefore, the graph of $y = f(x)$ looks exactly like the graph of $y = x - 1$, *except that it is missing the point* $(2, 1)$. See Figure 2.

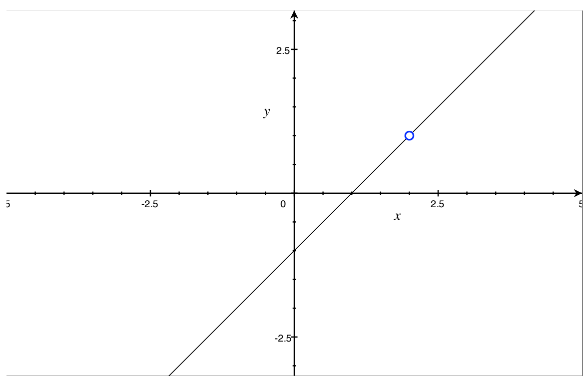

Figure 2: The graph of $f(x) = \frac{(x-2)(x-1)}{x-2}$. It is exactly the line $y = x - 1$ with a hole at the point $(2, 1)$.

Even though the function f is not defined at 2, we can ask about its **behavior** near 2. We can ask about the behavior of the values of $f(x)$ as the number x approaches 2. From our work above, we see that the value of $f(x)$ will approach 1. To record this fact, we use the notation

$$\lim_{x\to 2} f(x) = 1,$$

which is read as "the limit of $f(x)$ as x goes to 2 is 1".

In order to ask about the behavior of a function $f(x)$ as x approaches a particular number $a \in \mathbb{R}$, it is evidently not necessary for f to be defined at a, but it is important that f be defined *for all numbers near a*. Specifically, in order to ask about the limit

$$\lim_{x \to a} f(x),$$

we require that there be an **open interval** $I \subset \mathbb{R}$ such that $a \in \mathbb{R}$, and such that f is defined on the set $I - \{a\}$.

Remark 2.1.

- The notation "$\{a\}$" means the set containing only the element a. The notation "$I - \{a\}$" denotes the set consisting of every element of I *except for a*. Thus,

$$(-1, 2) - \{0\} = \{x \in \mathbb{R} \text{ such that } -1 < x < 2 \text{ and } x \neq 0\}.$$

- By an **open interval**, we mean any finite open interval such as $(-1, 2)$, any half-open interval such as $(-\infty, 1)$ or $(0, \infty)$, or the entire real line $\mathbb{R} = (-\infty, \infty)$.

2.2 Properties of limits

Let $a \in \mathbb{R}$.

1. For any constant $c \in \mathbb{R}$, we have $\lim_{x \to a} c = c$, because the value of a constant function does not depend at all on the input.

2. Because it would be ridiculous to imagine otherwise, we have

$$\lim_{x \to a} x = a.$$

   (Notice that a is a particular number, while x is a dummy variable for both the limit and the function of which we are taking the limit.)

3. Suppose f and g are functions, and that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, for some numbers $L, M \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{x\to a}[f(x)+g(x)] &= L+M, \\
\lim_{x\to a}[f(x)-g(x)] &= L-M, \\
\lim_{x\to a}[f(x)\cdot g(x)] &= L\cdot M, \text{ and} \\
\lim_{x\to a}\frac{f(x)}{g(x)} &= \frac{L}{M} \text{ so long as } M \neq 0.
\end{aligned}$$

So, the limit of a sum is the sum of the limits, the limit of a difference is the difference of the limits, and so forth.

Definition 2.2.

A **polynomial function** (of a single variable) is any function that can be written in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some real numbers $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$. Examples include the functions

$$2, \quad 3x+1, \quad x^3 - \pi x + 2, \quad \text{and} \quad 300x^{335} - 83x^{45} + 2.$$

The class of polynomial functions includes all **linear functions** ($ax+b$) and all **constant functions**.

A **rational function** (of a single variable) is any function that can be written in the form $\frac{f(x)}{g(x)}$ for polynomial functions $f(x)$ and $g(x)$. An example is the function

$$\frac{x-3}{x^2+1}.$$

Setting the denominator $g(x)$ equal to 1, we see that the class of rational functions includes all polynomial functions.

Using the properties of limits we have already stated, we now know how to compute most limits of any rational function. For instance,

$$\begin{aligned}
\lim_{x\to 5}\frac{x-3}{x^2+1} &= \frac{\lim_{x\to 5}(x-3)}{\lim_{x\to 5}(x^2+1)} \\
&= \frac{(\lim_{x\to 5} x) - (\lim_{x\to 5} 3)}{(\lim_{x\to 5} x)\cdot(\lim_{x\to 5} x) + (\lim_{x\to 5} 1)} \\
&= \frac{5-3}{(5)(5)+1} = \frac{2}{26}.
\end{aligned}$$

In the end, all we really did was plug the value $x = 5$ into the function. The only thing that could have gone wrong is if the denominator had ended up being equal to zero. This is exactly how one proves the following theorem.

Theorem 2.3. *Let f be a rational function which is defined at the number $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = f(a).$$

2.3 Algebraic manipulations

What if we are trying to calculate the limit $\lim_{x \to a} f(x)$ of a rational function $f(x)$ that *isn't defined* at $x = a$? Usually, the answer is to try some algebraic manipulation.

Example 2.4.

1. **Question.** Calculate the limit

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}.$$

   Answer. We saw above that $\frac{x^2-3x+2}{x-2} = \frac{(x-2)(x-1)}{x-2}$, and hence $\frac{x^2-3x+2}{x-2} = x - 1$ for all $x \in \mathbb{R} - \{2\}$. Since the limit at 2 only cares what happens *near* 2, and not *at* 2, we know that

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x-1)}{x-2} = \lim_{x \to 2} x - 1 = 2 - 1 = 1.$$

   In general, *if we can perform an algebraic manipulation that only changes our function at a finite number of points, then the limit will remain the same.*

2. **Question.** Let $a \in \mathbb{R}$ be a fixed real number. Calculate the limit

$$\lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}.$$

   Answer. This actually is a rational function, because it can be re-written in the correct form. The trick is to combine the fractions in the numerator by finding a common denominator. We calculate

$$\begin{aligned}
\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a} &= \lim_{x\to a}\frac{\frac{a}{a}\cdot\frac{1}{x}-\frac{x}{x}\cdot\frac{1}{a}}{x-a}\\
&= \lim_{x\to a}\frac{\frac{a}{ax}-\frac{x}{ax}}{x-a}\\
&= \lim_{x\to a}\frac{\frac{a}{ax}-\frac{x}{ax}}{x-a}\\
&= \lim_{x\to a}\frac{\frac{a-x}{ax}}{x-a}\\
&= \lim_{x\to a}\frac{-(x-a)}{(x-a)(ax)}\\
&= \lim_{x\to a}-\frac{1}{ax} = -\frac{1}{a^2}.
\end{aligned}$$

Note that our final answer doesn't make sense if $a = 0$.

3. **Question.** Calculate the limit

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}.$$

Answer. There are certainly several different ways to do this, but the one that comes first to my mind is to multiply the numerator by its **conjugate**. Of course, to balance the fraction we must multiply the denominator by the same thing. We calculate

$$\begin{aligned}
\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16} &= \lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}\\
&= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)}\\
&= \lim_{x\to 4}\frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)}\\
&= \lim_{x\to 4}\frac{1}{(x+4)(\sqrt{x}+2)}\\
&= \frac{1}{(4+4)(\sqrt{4}+2)} = \frac{1}{(8)(4)} = \frac{1}{32}.
\end{aligned}$$

2.4 One-sided limits

Because the real numbers form a line, there are basically two ways to sneak up on a particular real number $a \in \mathbb{R}$: **from above** and **from below**. The behavior of a function as x approaches a may very well depend on *how* x approaches a.

We use the notation "$x \to a^+$" to denote that x is approaching a from above, through numbers larger than a. We use the notation "$x \to a^-$" to denote that x is approaching a from below, through numbers smaller than a. Notice that

$$x \to a^+ \qquad \text{means exactly} \qquad \begin{cases} x \to a \\ \textbf{and} \\ x > a \end{cases}$$

and

$$x \to a^- \qquad \text{means exactly} \qquad \begin{cases} x \to a \\ \textbf{and} \\ x < a \end{cases}.$$

Example 2.5.

1. **Question.** Evaluate the one-sided limits

$$\lim_{x\to 0^+} \mathbf{sgn}(x) \qquad \text{and} \qquad \lim_{x\to 0^-} \mathbf{sgn}(x).$$

Answer. Recall that $\mathbf{sgn}(x) = 1$ if $x > 0$ and $\mathbf{sgn}(x) = -1$ if $x < 0$. For the limit as $x \to 0^+$, we can assume that $x > 0$, so

$$\lim_{x\to 0^+} \mathbf{sgn}(x) = \lim_{x\to 0^+} 1 = 1.$$

Similarly,

$$\lim_{x\to 0^-} \mathbf{sgn}(x) = \lim_{x\to 0^-} -1 = -1.$$

2. **Question.** Examine the behavior of the function $f(x) = \dfrac{|x-2|}{x^2+x-6}$ as $x \to 2$.

Answer. As it is given, it is hard to tell what $f(x)$ looks like. We can simplify it a bit, by using the piecewise definition of the absolute value function. Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Hence

$$\begin{aligned} |x-2| &= \begin{cases} x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases} \\ &= \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{|x-2|}{x^2+x-6} &= \begin{cases} \frac{x-2}{x^2+x-6} & \text{if } x \geq 2 \\ \frac{-(x-2)}{x^2+x-6} & \text{if } x < 2 \end{cases} \\ &= \begin{cases} \frac{x-2}{(x-2)(x+3)} & \text{if } x \geq 2 \\ \frac{-(x-2)}{(x-2)(x+3)} & \text{if } x < 2 \end{cases} \\ &= \begin{cases} \frac{1}{x+3} & \text{if } x \geq 2 \\ -\frac{1}{x+3} & \text{if } x < 2 \end{cases}. \end{aligned}$$

Therefore

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{x+3} = \frac{1}{2+3} = \frac{1}{5}$$

and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} -\frac{1}{x+3} = -\frac{1}{2+3} = -\frac{1}{5}$$

The relationship between one-sided limits and full limits is quite simple. When we write $\lim_{x\to a} f(x) = L$, we mean that the limit of $f(x)$ as x approaches a *in any way whatsoever* is L. Since the only ways that x *can* approach a is from above or from below, we know the following fact.

Fact 2.6.

$$\lim_{x\to a} f(x) = L \qquad \text{exactly if} \qquad \lim_{x\to a^+} f(x) = L \textbf{ AND } \lim_{x\to a^-} f(x) = L.$$

3 Day Three: January 23

3.1 What do limits really mean?

What exactly do we mean when we write

$$\lim_{x \to a} f(x) = L?$$

Intuitively, we mean the following two things.

- As we take numbers x that are increasingly close to a, we obtain numbers $f(x)$ that are increasingly close to L.
- We can make $f(x)$ as close to L as we want, so long as we make x really close to a.

These conditions amount to the following working definition.

> We write "$\lim_{x \to a} f(x) = L$" if the value of $f(x)$ can be made *arbitrarily close* to L by making x *sufficiently close* to a.

We measure the closeness of two numbers by calculating the **distance** between them, and we do this by taking the *absolute value of their difference*. So $f(x)$ and L are close to each other exactly when $|f(x) - L|$ is very small. So we can rewrite our working definition as follows.

> We write "$\lim_{x \to a} f(x) = L$" if the distance $|f(x) - L|$ can be made *arbitrarily small* by making the number $|x - a|$ *sufficiently small.*

The terms "arbitrarily small" and "sufficiently small" are still not very precise. They basically mean, if you tell me how small you want $|f(x) - L|$ to be, I can tell you how small you need to make $|x - a|$ in order to make that happen. There's a succinct way to write this in mathematical language.

Definition 3.1.

We say that **the limit of** $f(x)$ **as** x **approaches** a **is** L, $\lim_{x \to a} f(x)$, if the following statement holds.
For all numbers $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \qquad |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \epsilon.$$

This is the standard mathematical definition of the limit of a function. The number ϵ represents how close you want $f(x)$ to be to L, and the number δ represents how close you need x to be to a to make that happen.

Remark 3.2.

1. That the definition uses the *inequality* $|x - a| < \delta$ instead of an *equality* means that even if x is closer to a than is necessary, the desired result about $f(x)$ and L still holds.

2. That the definition uses the *inequality* $|f(x) - L| < \epsilon$ instead of an *equality* means that even you don't know exactly how close $f(x)$ will be to L, but you know *at least* how close it will be.

3. That there has to be a number δ *for every* ϵ *you could choose* means that no matter how close you want $f(x)$ to be to L, there's a way to make that happen.

4. The inequality $|x - a| < \epsilon$ is equivalent to the double-inequality $-\epsilon < x - a < \epsilon$, which is equivalent to

$$a - \epsilon < x < a + \epsilon.$$

Similarly, the inequality $|f(x) - L| < \epsilon$ can be rewritten as

$$L - \epsilon < f(x) < L + \epsilon.$$

See Figure 3.

You are not required to know this definition for this course. My motivation for including it here and in class is to demonstrate how subtle the idea of a limit , and the remarkable power of mathematical language to capture it precisely and concisely. Even though in most cases the meaning of the limit a function is intuitively clear, in order to prove things about limits and to test our intuition, or just to communicate with people who may not share it, it is absolutely necessary to have a definition that we can really get our hands on. The ability to record and communicate ideas can be just as important as having them in the first place.

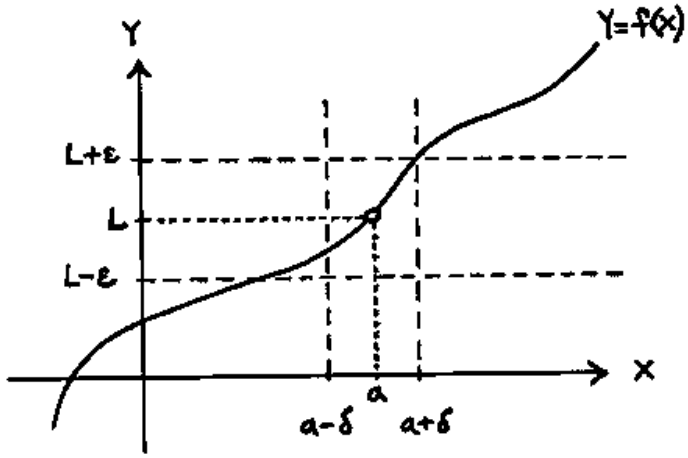

Figure 3: The graph of a function $f(x)$ satisfying the condition: "if $|x-a|<\delta$ then $|f(x)-L|<\epsilon$".

3.2 Limits at infinity

Just as we can ask about the behavior of a function $f(x)$ as x gets *infinitely close* to a number a, we can ask about its behavior as x gets *infinitely large.*

Definition 3.3.

We say **the limit of** $f(x)$ **as** x **approaches infinity is** L, written

$$\lim_{x\to\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Similarly, we say **the limit of** $f(x)$ **as** x **approaches negative infinity is** L, written

$$\lim_{x\to-\infty} f(x) = L,$$

if $f(x)$ can be made arbitrarily close to L by taking x sufficiently large *in the negative direction.*

Example 3.4.

Consider the function $f(x)=\frac{1}{x}$. When you divide 1 by a huge number you get a small result, and the huger the number is the smaller the result. Hence $\lim\limits_{x\to\infty}\dfrac{1}{x}=0$.

The same thing holds if you divide 1 by a hugely negative number, so $\lim_{x\to-\infty} \dfrac{1}{x} = 0$ also. These facts are evident from the graph of $\frac{1}{x}$, shown in Figure 4.

The laws of limits we listed previously hold for limits at infinity as well. Hence, for $\frac{1}{x^2} = \frac{1}{x} \cdot \frac{1}{x}$, for $\frac{1}{x^3} = \frac{1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x}$, and in general for $\frac{1}{x^p}$ for any positive integer p, we have

$$\lim_{x\to\pm\infty} \frac{1}{x^p} = 0.$$

This will prove to be an extremely valuable fact.

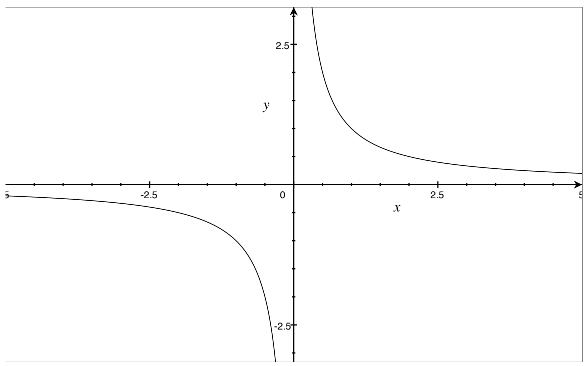

Figure 4: The graph of $\frac{1}{x}$, which demonstrates that $\lim_{x\to\pm\infty} \frac{1}{x} = 0$. If we want to be more precise about *how* the function approaches 0, we could write $\lim_{x\to\infty} \frac{1}{x} = 0^+$ and $\lim_{x\to-\infty} \frac{1}{x} = 0^-$, although usually this isn't necessary.

Definition 3.5. The horizontal line $y = L$ is a **horizontal asymptote** of $f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, or both.

Since there are only two infinite limits to check, a function can have *at most two horizontal asymptotes.* It could instead have no horizontal asymptotes, or it could have only one. If a function has just a single horizontal line as an asymptote, it could be an asymptote in the ∞ direction only, in the $-\infty$ direction only, or in both directions at once. An example of this last situation is the function $\frac{1}{x}$, which has $y = 0$ as a horizontal asymptote *in both directions.*

The graph of a function can cross its horizontal asymptote *any number of times*, including infinitely-many. For example, see Figure 5.

Example 3.6.

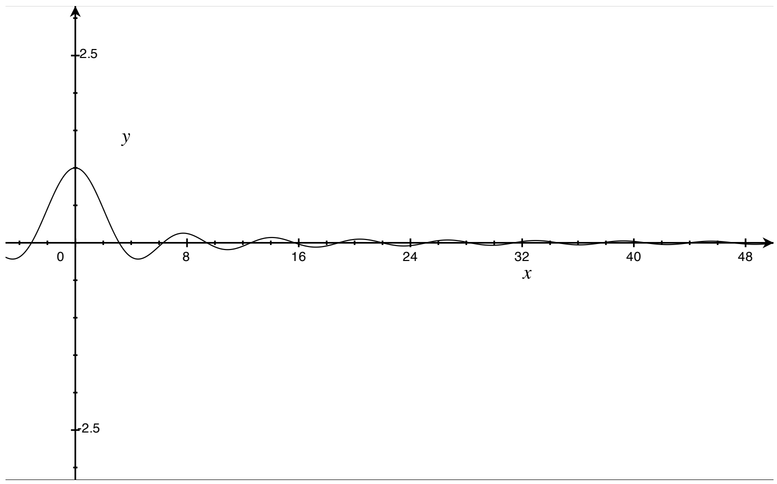

Figure 5: The graph of $\frac{1}{x}\sin x$. This function has the line $y = 0$ as a horizontal asymptote in both directions (which can be proved using the Squeeze Theorem), and its graph crosses this line infinitely-many times (whenever x is an integer multiple of π).

1. **Question.** Find all horizontal asymptotes of the function

$$\frac{x}{\sqrt{x^2+1}}.$$

Answer. We need to find the limits of this function as $x \to \infty$ and $x \to -\infty$. Before doing this, let's rewrite the function as follows:

$$\frac{x}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{x}{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x^2}}} = \frac{x}{|x|\cdot\sqrt{1+\frac{1}{x^2}}} = \frac{\mathbf{sgn}(x)}{\sqrt{1+\frac{1}{x^2}}}.$$

Here we have used the fact that $\sqrt{x^2} = |x|$, and $\mathbf{sgn}(x) = x/|x|$. (We actually cheated a little, since the original function was defined at $x = 0$, whereas the function we obtained is not. But since we are only considering large positive or negative values of x, we don't need to worry about $x = 0$.) In the limit as $x \to \infty$, we can assume $x > 0$, so

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{\mathbf{sgn}(x)}{\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\sqrt{1+0}} = 1.$$

Here we used the fact that $\frac{1}{x^2} \to 0$ as $x \to \infty$, and also the previously-unmentioned fact that we can take limits under square root signs. Similarly,

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{\mathbf{sgn}(x)}{\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to-\infty} -\frac{1}{\sqrt{1+\frac{1}{x^2}}} = -\frac{1}{\sqrt{1+0}} = -1.$$

Therefore, this function has $y = 1$ as a horizontal asymptote in the positive direction, and $y = -1$ as a horizontal asymptote in the negative direction.

2. **Question.** Find all horizontal asymptotes of the function

$$\frac{2x^2-x+3}{3x^2+5}.$$

Answer. The only functions whose behaviors at $\pm\infty$ we know are those of the form $\frac{1}{x^p}$ for some positive integer p. There is a nice trick for transforming any rational function into a combination of terms of this form, and that is to *divide both numerator and denominator by the highest power of x appearing in the denominator*. In the given function, that power term is x^2, and we obtain

$$\frac{2x^2-x+3}{3x^2+5} = \frac{\frac{1}{x^2}\cdot(2x^2-x+3)}{\frac{1}{x^2}\cdot(3x^2+5)} = \frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}}.$$

Therefore

$$\lim_{x\to\pm\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\pm\infty} \frac{2-\frac{1}{x}+\frac{3}{x^2}}{3+\frac{5}{x^2}} = \frac{2-0+0}{3+0} = \frac{2}{3}.$$

Therefore $y = \frac{2}{3}$ is a horizontal asymptote for this function in both directions.

3. **Question.** Find all horizontal asymptotes of the function

$$\frac{5x+2}{2x^3-1}.$$

Answer. In the given function, that power term is x^3, and we obtain

$$\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^3}\cdot(5x+2)}{\frac{1}{x^3}\cdot(2x^3-1)} = \lim_{x\to\pm\infty} \frac{\frac{5}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = \frac{0}{2} = 0.$$

Therefore $y = 0$ is a horizontal asymptote for this function in both directions.

4. **Question.** Calculate the limit

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right).$$

Answer. A little thought reveals that both $\sqrt{x^2+x}$ and x go to ∞ as $x\to\infty$, and so it is incredibly tempting to conclude that this limit equals $\infty-\infty$, which must surely be 0. But this is wrong. **The infinity symbol ∞ represents a behavior, NOT a number.** To calculate this limit, we need to use a little algebra to change the form of the function. We have

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\cdot\left(\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right)\\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}\\
&= \lim_{x\to\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}\\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}\\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x}\\
&= \lim_{x\to\infty}\frac{x}{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x}}+x}\\
&= \lim_{x\to\infty}\frac{\frac{1}{x}\cdot(x)}{\frac{1}{x}\cdot\left(|x|\cdot\sqrt{1+\frac{1}{x}}+x\right)}\\
&= \lim_{x\to\infty}\frac{1}{\mathbf{sgn}(x)\cdot\sqrt{1+\frac{1}{x}}+1}\\
&= \frac{1}{1\cdot\sqrt{1+0}+1}=\frac{1}{2}.
\end{aligned}$$

4 Day Four: January 26

We discussed what it means for a function to be **continuous**, and why continuous functions are so nice. The main definition is that a function $f(x)$ is **continuous** at a point $x = a$ if

$$\lim_{x \to a} = f(a).$$

We noted that almost all the functions we know, except piecewise functions, are continuous.

One of the main reasons continuous functions are nice is that we can calculate their limits as $x \to a$ by simply plugging in $x = a$. Two other nice properties of continuous functions are the **Extreme Value Theorem** (also known as the **Min-Max Theorem**), and the **Intermediate Value Theorem**.

Theorem 4.1 (Extreme Value Theorem)**.** *Let $[a, b]$ be a closed and finite interval, and let f be a continuous function on $[a, b]$. Then there are numbers m and M in the interval $[a, b]$ such that $f(m)$ is a* **minimum** *of f on $[a, b]$ and $f(M)$ is a* **maximum** *of f on $[a, b]$; i.e.*

$$f(m) \leq f(x) \leq f(M)$$

for all x in $[a, b]$.

Theorem 4.1 says not only that a continuous function on a closed and finite interval is **bounded**, but that *it achieves its bounds.*

Theorem 4.2 (Intermediate Value Theorem)**.** *Let $[a, b]$ be a closed and finite interval, and let f be a continuous function on $[a, b]$. If s is any number between $f(a)$ and $f(b)$, then there exists a point c in the interval $[a, b]$ such that*

$$f(c) = s.$$

Theorems 4.1 and 4.2 are together equivalent to the following fact.

Theorem 4.3. *A continuous function maps closed, finite intervals to closed, finite intervals. In other words, if f is continuous on the closed, finite interval $[a, b]$, then there are numbers $c < d$ such that the* **image** *of f is*

$$f\big([a, b]\big) = [c, d].$$

5 Day Five: January 28

The most common way to define a **tangent line** to a graph is as a *limit of secant lines* to the graph. Suppose we want to identify the tangent line to the graph of a function $f(x)$ at the point on the graph with x-coordinate x_0. To define a line, we need a point on the line and the line's slope. The line will contain the point $\big(x_0, f(x_0)\big)$. To determine its slope, consider the secant line of f over the closed interval $[x_0, x]$, which is the line through the points $\big(x_0, f(x_0)\big)$ and $\big(x, f(x)\big)$. As $x \to x_0$, these points will move closer and closer to each other, and the secant line over the interval $[x_0, x]$ will become more and more like the tangent line at x_0. The slope of this secant line is

$$m_{\mathrm{sec},[x_0,x]} := \frac{f(x) - f(x_0)}{x - x_0},$$

which is sometimes called a **difference quotient**. We define the slope of the tangent line at x_0 to be

$$m_{\tan,x_0} := \lim_{x \to x_0} m_{\mathrm{sec},[x_0,x]} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Therefore an equation for this tangent line is

$$y - f(x_0) = m_{\tan,x_0}\big(x - x_0\big).$$

Another common way of writing the definition of the slope of the tangent line is to let $h = x - x_0$. Then $x = x_0 + h$ and saying that $x \to x_0$ is the same as saying that $h \to 0$. Hence the difference quotient may be written as

$$m_{\mathrm{sec},[x_0,x_0+h]} = \frac{f(x_0 + h) - f(x_0)}{h},$$

and the tangent slope may be defined by

$$m_{\tan,x_0} := \lim_{h \to 0} m_{\mathrm{sec},[x_0,x_0+h]} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

To summarize, the tangent line is the limit of secant lines, and the slope of the tangent line is the limit of the slopes of the secant lines.

To figure out what's going on, sometimes we have to consider the two **one-sided limits** of the difference quotient:

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} \qquad \text{and} \qquad \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}.$$

If these are both equal to ∞ or $-\infty$, then the tangent line is a **vertical line**. If these two one-sided limits are *different finite numbers*, then the graph of $f(x)$ has a **corner** at $x = x_0$. If these two one-sided limits are *different infinities*, then the graph of $f(x)$ has a **cusp** at $x = x_0$, which is an extreme version of a corner.

6 Day Six: January 30

The **derivative** of a function f is a new function f' defined by

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = m_{\tan,x}.$$

Of course, this limit may not be defined at all points, so the domain of f' may be a strictly smaller set than the domain of f. The process of calculating f' is called **differentiation**. If $f'(x)$ is defined at $x = x_0$, we say f is **differentiable** at x_0. If $f'(x)$ is not defined at $x = x_0$, we say f is **singular** at x_0.

We noted that the derivative of a constant function is zero, the derivative of a linear function is its slope, and the derivative of $|x|$ is $\mathbf{sgn}(x)$. We also stated the **Power Rule**:

$$\frac{d}{dx}\, x^r = r\, x^{r-1},$$

which holds for any real number r and for any value of x for which the expressions on both sides make sense. We definitely did not prove the Power Rule, although a fairly easy proof in the case that r is a positive integer can be found on page 101 of the textbook.

7 Day Seven: February 2

We covered some basic properties of the derivative, including **Linearity**, the **Product Rule**, and the **Quotient Rule**. These tell us that if f and g are differentiable at x, and if c is any real number, then

$$\begin{aligned}
\frac{d}{dx}\big[c\cdot f(x)\big] &= c\cdot f'(x),\\
&=\\
\frac{d}{dx}\big[f(x)+g(x)\big] &= f'(x)+g'(x),\\
&=\\
\frac{d}{dx}\big[f(x)\cdot g(x)\big] &= f'(x)\cdot g(x)+f(x)\cdot g'(x), \text{ and}\\
&=\\
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{\big[g(x)\big]^2}.
\end{aligned}$$

This last equality only holds, of course, when $g(x)\neq 0$.

We also proved that **differentiability implies continuity**. The other direction of this statement is not true, as evidenced by the function $|x|$, which is continuous at $x=0$ but not differentiable there.

8 Day Eight: February 4

We discussed the Chain Rule,

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x),$$

which holds for any x such that g is differentiable at x and f is differentiable at $g(x)$. The Chain Rule implies that the composition of differentiable functions is differentiable.

Morally speaking, the Chain Rule holds because when we compose two linear functions, we get another linear function whose *slope is the product of the slopes* of the original two lines.

For a function f and a point $x = a$, the **linearization** of f at a is the function whose graph is the tangent line to the graph of f at $x = a$. Since this tangent line has equation

$$y - f(a) = f'(a) \cdot \big(x - a\big),$$

the linearization of f at a is the function

$$L_{f,a}(x) := f'(a) \cdot \big(x - a\big) + f(a).$$

Using the Chain Rule, we can prove the following (possibly) interesting result, which you certainly will not be required to know for this class.

Theorem 8.1. *Suppose the function g is differentiable at $x = a$ and the function f is differentiable at $x = g(a)$. Then*

$$L_{f\circ g,a} = L_{f,g(a)} \circ L_{g,a}.$$

Therefore, the tangent line to $f \circ g$ is the composition of a tangent line to f with a tangent line to g.

Proof. The linearization of g at $x = a$ is

$$L_{g,a}(x) = g'(a) \cdot \big(x - a\big) + g(a),$$

and the linearization of f at $x = g(a)$ is

$$L_{f,g(a)}(x) = f'\big(g(a)\big) \cdot \big(x - g(a)\big) + f\big(g(a)\big).$$

Their composition is

$$\begin{aligned}
L_{f,g(a)}\big(L_{g,a}(x)\big) &= f'\big(g(a)\big)\cdot\big(L_{g,a}(x)-g(a)\big)+f\big(g(a)\big)\\
&= f'\big(g(a)\big)\cdot L_{g,a}(x)-f'\big(g(a)\big)\cdot g(a)+f\big(g(a)\big)\\
&= f'\big(g(a)\big)\cdot\big(g'(a)\cdot\big(x-a\big)+g(a)\big)-f'\big(g(a)\big)\cdot g(a)+f\big(g(a)\big)\\
&= f'\big(g(a)\big)\cdot g'(a)\cdot\big(x-a\big)+f'\big(g(a)\big)\cdot g(a)-f'\big(g(a)\big)\cdot g(a)+f\big(g(a)\big)\\
&= f'\big(g(a)\big)\cdot g'(a)\cdot\big(x-a\big)+f\big(g(a)\big).
\end{aligned}$$

By the Chain Rule, we have $f'\big(g(a)\big)\cdot g'(a)=(f\circ g)'(a)$, and of course $f\big(g(a)\big)=(f\circ g)(a)$. Therefore this last equation above is equal to

$$(f\circ g)'(a)\cdot\big(x-a\big)+(f\circ g)(a),$$

which is exactly $L_{f\circ g,a}$. $\square$

9 Day Nine: February 6

We went through a brief review of trigonometry, and discussed the derivatives of the trigonometric functions **sine**, **cosine**, **tangent**, **secant**, **cosecant**, and **cotangent**. We also discussed the important limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0,$$

which are secretly the derivatives

$$\frac{d}{dx} \sin x \bigg|_{x=0} = \lim_{h\to 0} \frac{\sin h}{h} \qquad \text{and} \qquad \frac{d}{dx} \cos x \bigg|_{x=0} = \lim_{h\to 0} \frac{\cos h - 1}{h}.$$

10 Day Ten: February 9

Rolle's Theorem states that if the graph of a differentiable function over a closed interval comes back to the same y-value at which it started, then it must have had a horizontal tangent line at some point.

Theorem 10.1 (Rolle's Theorem)**.** *Let $[a, b]$ be a closed and finite interval, and let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a number c in the interval (a, b) such that*

$$f'(c) = 0.$$

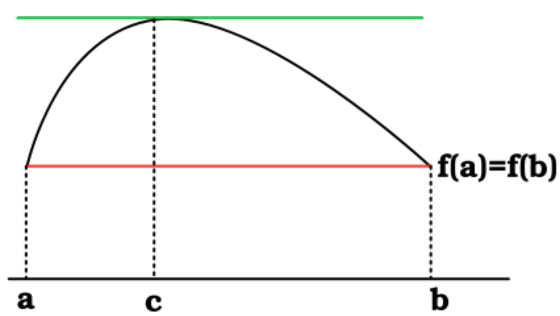

Figure 6: A graphical depiction of Rolle's Theorem.

Example 10.2. Suppose you are tossing a ball up into the air and catch it, and suppose that you toss it and throw it from the same height off the floor. Let $y(t)$ denote the ball's height off the floor t seconds after you throw it, and suppose it spends exactly 3 seconds in the air. Assuming the the height of the ball is a continuous and differentiable function of time, which is probably true as far as we can really measure these things, Rolle's Theorem tells us that there is a time t_0 between 0 and 3 seconds such that $y'(t_0) = 0$.

Of course, this is nothing too exciting. The derivative $y'(t)$ measures the ball's upwards velocity at time t, and we know from experience that when the ball is at its peak height, it's velocity is zero.

Geometrically, Rolle's Theorem is saying that if the secant line of the graph of f over the closed interval $[a, b]$ is horizontal, then the graph has a horizontal tangent line. The **Mean Value Theorem** says that this fact *does not depend* on the secant line being horizontal.

Theorem 10.3 (Mean Value Theorem)**.** *Let $[a, b]$ be a closed and finite interval, and let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is some point c in the interval (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words, there is a tangent line to the graph of f that is **parallel** *to the secant line of the graph over the interval $[a, b]$.*

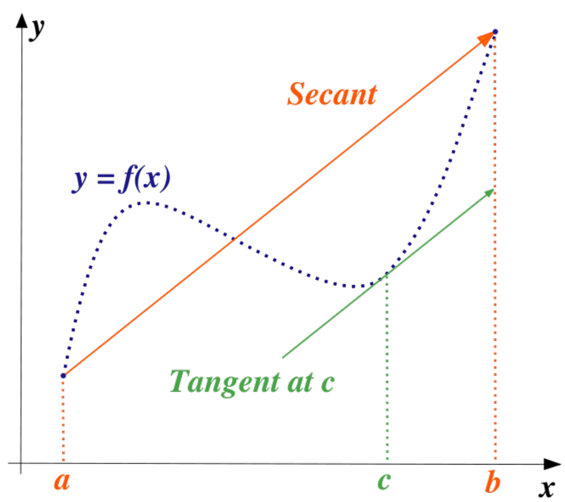

Figure 7: A graphical depiction of the Mean Value Theorem.

Remark 10.4.

1. Notice that Rolle's Theorem is *exactly* the Mean Value Theorem in the case that the secant slope is zero, which happens precisely when $f(b) - f(a) = 0$, i.e. when $f(a) = f(b)$.

2. Like the Extreme Value Theorem and the Intermediate Value Theorem, Rolle's Theorem and the Mean Value Theorem are both examples of **existence theorems**. They assure us that a point with a certain property *exists*, but they give us no clue as to how to find it.

3. Both Rolle's Theorem and the Mean Value Theorem tell us that there is *at least one* tangent line to the function's graph that is parallel to the secant line over the whole interval, but *there could be any number* of such tangent lines. For instance, in Figure 7 there are two points on the graph of $f(x)$ whose tangent lines are parallel to the red secant line.

Example 10.5.

Suppose Tim wants to drive from his apartment in Ithaca to the Carousel Center Mall in Syracuse. Let $x(t)$ be the distance in miles Tim has traveled from his apartment t hours after he started on his adventure. According to Google maps, the trip is about 55.6 miles and should take about 1.25 hours, and let us assume that it is and that it does. Then

$$x(0) = 0 \qquad \text{and} \qquad x(1.25) = 55.6.$$

Recall that Tim's **average speed** on this trip is the quotient of the total distance and the total time taken, which is $55.6/1.25 = 44.48$ miles per hour. Of course, this is also the slope of the secant line of the graph of $x(t)$ over the t-interval $[0, 1.25]$:

$$\frac{x(1.25) - x(0)}{1.25 - 0} = 44.48.$$

Assuming Tim's distance traveled is a continuous and differentiable function of time, the Mean Value Theorem tells us that there is some time $t = t_0$ such that $x'(t_0) = 44.48$. Just as the slope of the secant line over a time interval represents the **average speed** over that period of time, the slope of the tangent line at a particular time represents the **instantaneous speed** at that time. We conclude that there must have been a time during Tim's trip when he was driving *exactly* 44.48 miles per hour.

This demonstrates a general principal, that most of us probably all know without being aware of it. *During any trip, there is always a point when the speed at which we are driving is exactly the same as our average speed over the entire trip.* Naturally, sometimes we may drive faster than the average, and sometimes slower, but at some point we have to hit it right on the nose.

Example 10.6.

Question. Prove that if $x > 0$ then $x > \sin x$.

Answer. A priori, there is no reason to think that the Mean Value Theorem will help us with this. The very clever trick is to realize that if $x > 0$, then saying that $\sin x < x$ is exactly the same as saying that $\frac{\sin x}{x} < 1$, and that

$$\frac{\sin x}{x} = \frac{\sin x - 0}{x - 0} = \frac{\sin x - \sin 0}{x - 0}$$

is the slope of the secant line for the sine graph over the closed interval $[0, x]$.

Let's do the easy case first. If $x > 1$, then since $\sin x \leq 1$ we know that $x > \sin x$.

Now suppose that $x \leq 1$. Since the sine function is differentiable everywhere, we can apply the Mean Value Theorem to it over the closed interval $[0, x]$. We conclude that there is some number c in the open interval $(0, x)$ such that the tangent slope to the sine curve at c is equal to the secant slope over the interval $[0, x]$, which we saw above is $\frac{\sin x}{x}$. The tangent slope in question is

$$\frac{d}{dx} \sin x \bigg|_{x=c} = \cos x|_{x=c} = \cos c,$$

so $\frac{\sin x}{x} = \cos c$. Now, recall that $\cos\theta$ is *strictly less than* 1 if $0 < \theta < 2\pi$. Since $0 < cx \leq 1 < \pi$, we know $\cos c < 1$. Therefore

$$\frac{\sin x}{x} < 1,$$

so

$$\sin x < x.$$

This completes the solution.

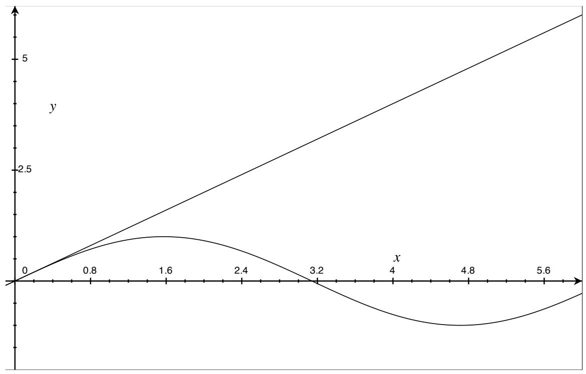

Figure 8: If $x > 0$, then $x > \sin x$.

11 Day Eleven: February 10

11.1 Increasing and decreasing

Definition 11.1. Let f be a function defined on an interval I.

- f is **increasing** on I if for all $x, y \in I$,

$$x < y \implies f(x) < f(y).$$

- f is **decreasing** on I if for all $x, y \in I$,

$$x < y \implies f(x) > f(y).$$

- f is **nondecreasing** on I if for all $x, y \in I$,

$$x < y \implies f(x) \leq f(y).$$

- f is **nonincreasing** on I if for all $x, y \in I$,

$$x < y \implies f(x) \geq f(y).$$

- f is **stationary** on I if for all $x, y \in I$ we have $f(x) = f(y)$.

(The symbol $\implies$ means "implies that". For example, "$x < y \implies f(x) < f(y)$" means "if $x < y$ then $f(x) < f(y)$".)

Remark 11.2.

1. If $y = f(x)$, then to say that f is **increasing** means that if we make x bigger, then y gets bigger. The meaning of **decreasing** can be explained similarly.

2. The difference between **increasing** and **nondecreasing** is the following. Suppose $y = f(x)$ and we make x bigger. If f is increasing, then y will definitely get bigger, but if f is only nondecreasing, then y will either get bigger *or stay the same*. The difference between **decreasing** and **nonincreasing** can be explained similarly.

3. Saying that f is **stationary** is the same as saying that f is constant.

Theorem 11.3. *Let f be a function defined on an interval I. Let J be the set of* **interior points** *of I, (i.e. J consists of all points of I that aren't endpoints). Suppose f is continuous on I and differentiable on J.*

- *If $f' > 0$ on J then f is* **increasing** *on I.*
- *If $f' < 0$ on J then f is* **decreasing** *on I.*
- *If $f' \geq 0$ on J then f is* **nondecreasing** *on I.*
- *If $f' \leq 0$ on J then f is* **nonincreasing** *on I.*
- *If $f' = 0$ on J then f is* **stationary** *on I.*

Remark 11.4.

1. It is fair to say that this theorem seems obvious, because we know that positive derivative means that tangent lines have positive slope, which means that the graph of the function is going up as x increases, and similarly (but opposite) for negative derivative. However, just because this theorem seems obvious for *every function and curve we can imagine* does not mean it is true for every function and curve that there is. Some of the most amazing topics in mathematics arose from people trying to prove something "obvious" that turned out not to be true. An example of this is the development of **non-Euclidean geometries**, which can be pretty awesome.

2. The proof of this theorem relies on the Mean Value Theorem, and it is suggested by Question 1 of Quiz 2. Basically, we are trying to determine whether $f(x) < f(y)$ or $f(x) > f(y)$ when $x < y$, which is the same as determining whether $f(y) - f(x) > 0$ or $f(y) - f(x) < 0$ when $x < y$. Since we know that $y - x > 0$, if we can figure out the sign of the fraction

$$\frac{f(y) - f(x)}{y - x}$$

then we know the sign of $f(y) - f(x)$. The Mean Value Theorem, applied to the closed interval $[x, y]$, tells us that this fraction is equal to $f'(c)$ for some c in the interval (x, y). Therefore, if we know the sign of the derivative, then we can figure out whether $f(x) < f(y)$ or vice versa.

3. We already knew that the derivative of a constant function is zero. This theorem tells us the converse, that if a function has zero derivative then it is constant.

Example 11.5.

Question. For what values of x is $f(x) = x^3 - 12x + 1$ increasing? For what values is it decreasing?

Answer. We compute

$$\begin{aligned} f'(x) &= 3x^2 - 12 \\ &= 3(x^2 - 4) \\ &= 3(x-2)(x+2). \end{aligned}$$

We need to find where f' is positive and where its negative. Since $f'(x) = 0$ only when $x = \pm 2$, and a continuous function on an interval can only change sign when it passes through zero, we know f' will have constant sign on $(-\infty, -2)$, and constant sign on $(-2, 2)$, and constant sign on $(2, \infty)$. There are several ways to determine what these signs are, including plugging in points, constructing number lines for the factors of $f'(x)$ and multiplying their signs together, and simply inspecting the graph of $f'(x)$ (see Figure 9). By whichever method, we learn that $f'(x) > 0$ when $x < -2$ or $x > 2$, and $f'(x) < 0$ when $-2 < x < 2$. Therefore f is increasing on the x-intervals $(-\infty, -2)$ and $(2, \infty)$, and decreasing on the x-interval $(-2, 2)$.

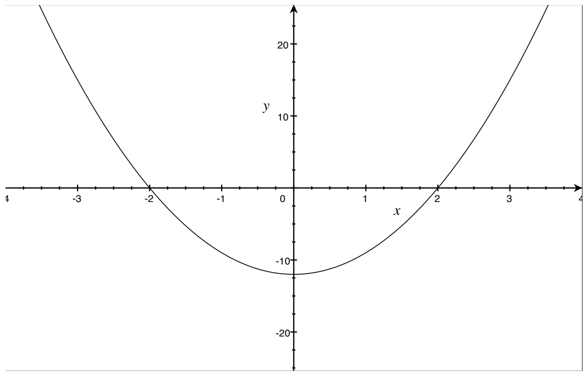

Figure 9: The graph of $f'(x) = 3x^2 - 12$, from which one can deduce that $f'(x) > 0$ when $x < -2$ or $x > 2$ and $f'(x) < 0$ when $-2 < x < 2$.

11.2 Approximating small changes

Suppose $y = f(x)$, and we know the value $y_0 = f(x_0)$ of f at x_0. If we add Δx to x_0, how much does the value of the function f change from y_0? In other words, what is

$$\Delta y := f(x_0 + \Delta x) - f(x_0)?$$

In this section, we will learn how to estimate the function value $f(x_0 + \Delta x)$ by estimating Δy.

Suppose the graph of f is a line of slope m. For a line, we know that the ratio $\frac{\Delta y}{\Delta x}$ of any change in y corresponding to a specified change in x is constant, and equals the slope. Therefore, if f represents a line of slope m, then $\frac{\Delta y}{\Delta x} = m$, so $\Delta y = m \cdot \Delta x$, so

$$f(x_0 + \Delta x) = f(x_0) + m \cdot \Delta x.$$

Now let us go back to f being an arbitrary function. Note that

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} \\ &= \text{the slope of the secant line over the interval } [x_0, x_0 + \Delta x]. \end{aligned}$$

Now, if Δx is small, then this **secant slope** is a good approximation of the **tangent slope** at x_0. Therefore $\frac{\Delta y}{\Delta x} \approx f'(x_0)$, so

$$\Delta y \approx f'(x_0) \cdot \Delta x$$

and

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x.$$

Notice that whereas we usually use the secant slope to approximate the tangent slope, in this case we are doing things the other way around.

To summarize, suppose we know the value $f(x_0)$ of a function f at a point x_0, and we want to know the value of f at the point $x_0 + \Delta x$. If Δx is small and $f'(x_0)$ exists, then

$$\Delta y \approx f'(x_0) \cdot \Delta x,$$

where Δy is the difference between $f(x_0 + \Delta x)$ and $f(x_0)$. Therefore

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x.$$

Example 11.6.

Question. Suppose you draw a circle of radius r_0 cm. By approximately how many cm^2 will the area of this circle increase if you increase the radius by 2 cm?

Answer. The radius r of a circle and its area A are related by the equation $A = \pi r^2$. The relationship between ΔA and Δr is

$$\Delta A \approx \frac{dA}{dr} \cdot \Delta r = 2\pi r\, \Delta r.$$

In this example, we are starting with $r = r_0$ and $\Delta r = 2$, so

$$\Delta A \approx 2\pi r\, \Delta r = 2\pi r_0 \cdot 2 = 4\pi r_0.$$

Therefore the area will increase by approximately $4\pi r_0\,\text{cm}^2$.

The importance of a change Δx depends on the context, i.e. on how big x is. If you are measuring the length of a room, a mistake of 10 ft. is pretty important, but this same mistake is far less important if you are measuring the length of a highway. Therefore we make the following definition.

Definition 11.7. Let x be a quantity, and let Δx be a specific change in that quantity. The **relative change** in x is $\frac{\Delta x}{x}$, and the **percentage change** in x is $100 \cdot \frac{\Delta x}{x}$.

In relative change, we are taking our starting value of x as *one unit*, and the relative change represents *the number of these units* by which the quantity is changing. Therefore, a relative change of 1 means that our quantity has doubled.

The percentage change is simply the relative change represented as a percentage. Therefore, a percentage change of 100% means that our quantity has doubled.

Example 11.8.

Question. Suppose you increase the radius of a circle by 3%. By what percentage have you increased the area of the circle?

Answer. As in the previous example, we have $A = \pi r^2$ and $\Delta A \approx \frac{dA}{dr} \cdot \Delta r = 2\pi r\, \Delta r$. The relative change in A is

$$\frac{\Delta A}{A} = \frac{2\pi r\, \Delta r}{\pi r^2} = 2 \cdot \frac{\Delta r}{r}.$$

Since the relative change in r is $\frac{\Delta r}{r} = 3\% = \frac{3}{100}$, we conclude that the relative change in A is

$$\frac{\Delta A}{A} = 2 \cdot \frac{\Delta r}{r} = 2 \cdot \frac{3}{100} = \frac{6}{100},$$

so the area of the circle increased by 6%.

12 Day Twelve: February 13

12.1 Rates of change

By now, you may already be familiar with the following definitions.

Definition 12.1. Let $f(x)$ be a function.

- The **average rate of change** of the function f with respect to x over the x-interval $[a, a+h]$ is the slope of the secant line over this interval:

$$\frac{f(a+h)-f(a)}{h}.$$

- The **instantaneous rate of change** of the function f with respect to x at the x-value a is the slope of the tangent line at this point (if it exists):

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Example 12.2.

Question. How fast is the area of a circle increasing with respect to its radius when the radius is $5\,\mathrm{m}$?

Answer. The rate of change of the area with respect to the radius is

$$\frac{dA}{dr} = 2\pi r.$$

When $r = 5\,\mathrm{m}$, this rate of change is $2\pi \cdot 5 = 10\pi\,\mathrm{m}^2/\mathrm{m}$. (Note that the units we use for this rate of change are **units of area per units of radius**.

Related to the interpretation of derivatives as measuring instantaneous rates of change, we have the following terminology. If $f'(x_0) = 0$, we say that the function f is **stationary** at x_0, and we call the point x_0 a **critical point** of the function f. Sometimes the corresponding point $\big(x_0, f(x_0)\big)$ on the graph of f is also called a critical point.

Graphically, a critical point of f represents a location where the graph of f has a horizontal tangent line.

12.2 Sensitivity to change

Verbatim from the textbook, "when a small change in x produces a large change in the value of a function $f(x)$, we say that the function is very **sensitive** to changes in x". Recall the approximation formula

$$\Delta y \approx f'(x_0) \cdot \Delta x.$$

Here Δy is the change in function value from x_0 to $x_0 + \Delta x$, i.e. $\Delta y = f(x_0 + \Delta x) - f(x_0)$. From this approximation, we see that generally speaking, *the larger the derivative $f'(x_0)$, the more sensitive the function $f(x)$ to changes in x near $x = x_0$*. Therefore, the derivative $f'(x)$ measures the **sensitivity to change** of $f(x)$ with respect to x.

Example 12.3 (Example 4, page 134 from the textbook)**.** Suppose a pharmacologist studying a drug that has been developed to lower blood pressure determines experimentally that the average reduction R in blood pressure from a daily dosage of x mg of the drug is

$$R = 24.2\left(1 + \frac{x-13}{\sqrt{x^2 - 26x + 529}}\right) \text{ mmHg}.$$

(The units used here for measuring blood pressure are millimeters of mercury.) The daily dose of many medications is 5 mg, 15 mg, or 35 mg. A natural question to ask is this. *For patients on which of these daily doses of the medication will a small increase in dosage have the greatest effect?*

What we are really asking about is the *sensitivity in R to changes in x*, which is measured by the derivative $\frac{dR}{dx}$. Through arduous computation (as shown in the textbook), one can compute that

$$\left.\frac{dR}{dx}\right|_{x=5} \approx 0.998, \quad \left.\frac{dR}{dx}\right|_{x=15} \approx 1.254, \quad \text{and} \quad \left.\frac{dR}{dx}\right|_{x=35} \approx 0.355.$$

The greatest sensitivity of R to change in x occurs when $x = 15$ mg, at which point an increase in dosage by just 1 mg will yield an average reduction in blood pressure of approximately

$$\big(1\,\text{mg}\big)\big(1.254\,\text{mmHg/mg}\big) = 1.254\,\text{mmHg}.$$

There are many applications of derivatives to economics. Some of these are described in the textbook on pages 134–135. Generally speaking, this material is not terribly important in this course, so I will not spend any time on it here. **But**, your homework includes a couple of questions about this material, so you should probably give it a read. If you will be required to know it for any exam in this course, I will be sure to let you know.

12.3 Higher order derivatives

The derivative of a function f is a new function, which we denote by f'. What we have done once, we can do again, and define f'' to be the derivative of the f'. We call f'' the **second derivative** of f. If $y = f(x)$, there are many different ways to denote the second derivative, including

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\, f(x) = D_x^2\, f(x).$$

The notation $\frac{d^2y}{dx^2}$ comes from

$$y'' := \left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right) y = \left(\frac{d}{dx}\right)^2 y = \frac{d^2y}{dx^2},$$

and the fact that *we treat dx as a single symbol*, so that its square is dx^2.

Of course, we can go further, and consider the third derivative of f, denoted f''', the fourth derivative of f, denoted f'''', and so forth. In general, for any positive integer n we can consider the n**th derivative** of f. If $y = f(x)$, the nth derivative is denoted

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}\, f(x) = D_x^n\, f(x).$$

For convenience, we define the 0**th derivative** of f just to be itself, $f^{(0)} := f$. This is what you get when you differentiate f zero times!

Remark 12.4.

1. Just as the derivative of f may not be defined at every point in the domain of f, the second derivative f'' may not be defined at every point in the domain of f'. Thus

   $$\cdots \mathrm{Dom}(f''') \subset \mathrm{Dom}(f'') \subset \mathrm{Dom}(f') \subset \mathrm{Dom}(f).$$

   If $f^{(n)}(x_0)$ exists, we say that f in n **times differentiable** at x_0.

2. Recall that if f is differentiable at x_0, then f must be continuous at x_0. For the same reason, if f is *twice* differentiable at x_0, then we know that f' is continuous at x_0, and in particular that $f'(x_0)$ exists.

Example 12.5.

1. Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases},$$

whose graph is given in Figure 10. It is not too hard to show that $f'(x) = |x|$, and hence $f''(x) = \mathbf{sgn}(x)$. Therefore $f(x)$ is differentiable at $x = 0$, *but not twice differentiable* there.

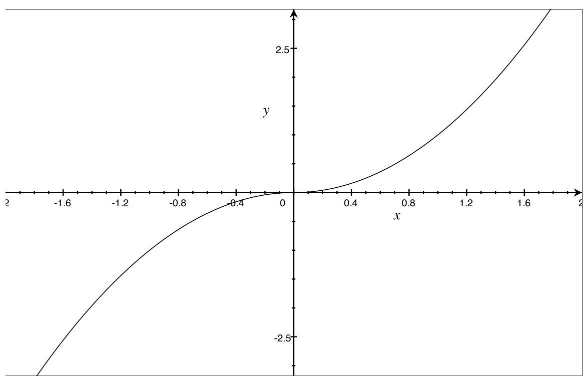

Figure 10: The graph of $f(x)$ from Example 12.5 (a). This function is differentiable at $x = 0$, but not twice differentiable there.

2. Suppose $f(x) = x^3 + 3x^2 + 5$. Then

$$\begin{aligned} f'(x) &= 3x^2 + 6x, \\ f''(x) &= 6x + 6, \\ f'''(x) &= 6, \\ f^{(4)}(x) &= 0, \text{ and} \\ f^{(5)}(x) &= 0. \end{aligned}$$

Since the derivative of 0 is 0, we see that $f^{(n)}(x) = 0$ for all integers $n > 3$.

3. Suppose $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \cos x, \\ f''(x) &= -\sin x, \\ f'''(x) &= -\cos x, \\ f^{(4)}(x) &= \sin x, \\ f^{(5)}(x) &= \cos x, \end{aligned}$$

and the pattern continues. Therefore we can calculate that

$$\frac{d^{1001}}{dx^{1001}} \sin x = \frac{d}{dx} \sin x = \cos x,$$

because $1001 = 4 \times 250 + 1$ and every four derivative of the sine function come back to the sine function.

Besides being mathematically interesting, higher order derivatives are *indescribably useful* in applications. Perhaps the most common instance of this is in physics, which is the subject that calculus was essentially invented to study in the first place.

Suppose an object moves in a straight line. For convenience, let's slap an x-axis onto this line of motion, and at time t denote the **position** of the object on this axis by $x(t)$. The **velocity** of the object at time t is

$$v(t) := x'(t) = \frac{d}{dt} x(t),$$

and the **acceleration** of the object at time t is

$$a(t) := v'(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} x(t).$$

Thus, the velocity is the rate of change of position with respect to time, and the acceleration is the rate of change of velocity with respect to time, which is the *rate of change of the rate of change* of position with respect to time. If this x-axis is in its standard position, with large positive values to the right and large negative values to the left, then the signs of $x(t)$, $v(t)$, and $a(t)$ can be interpreted as shown in the following table.

$x(t) > 0$	object is to the right of the origin
$x(t) < 0$	object is to the left of the origin
$x(t) = 0$	object is at the origin (at this instant)
$v(t) > 0$	object is moving to the right
$v(t) < 0$	object is moving to the left
$v(t) = 0$	object is not moving (at this instant)
$a(t) > 0$	object's motion to the right is speeding up object's motion to the left is slowing down
$a(t) < 0$	object's motion to the left is speeding up object's motion to the right is slowing down
$a(t) = 0$	object is moving at constant speed (at this instant)

At the beginning of this course, we spoke of how relationships between quantities can sometimes be expressed as equations. We can sometimes express relationships between quantities *and their rates of change* as **differential equations**. A **differential equation** is an equation involving *an unknown function and its derivatives*, such as

$$y' = y.$$

(In fact, this differential equation is extremely important, and we will discuss it at length in the near future.) Again, notice that whereas for usual equations we are trying to solve for *an unknown number*, in differential equations we are trying to solve for *an unknown function.*

Solving differential equations can be *incredibly* difficult — indeed, a significant portion of the most interesting topics in mathematics have their roots in the study of differential equations. In this course, with very few exceptions you will not be required to solve any differential equations. What you *will* be expected to be able to do is verify that a given function is a solution to a particular differential equation.

Example 12.6.

Question. Verify that for any constants A, B, and k, the function $y = A\cos(kt) + B\sin(kt)$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} + k^2y = 0.$$

Answer. Using the Chain Rule, we compute

$$\frac{dy}{dx} = -kA\sin(kt) + kB\cos(kt)$$

and

$$\frac{d^2y}{dx^2} = -k^2A\cos(kt) - k^2B\sin(kt).$$

Therefore

$$\begin{aligned}
\frac{d^2y}{dx^2} + k^2y &= -k^2A\cos(kt) - k^2B\sin(kt) + k^2\left(A\cos(kt) + B\sin(kt)\right) \\
&= -k^2A\cos(kt) + k^2A\cos(kt) - k^2B\sin(kt) + k^2B\sin(kt) \\
&= 0
\end{aligned}$$

as desired.

13 Day Thirteen: February 16

We reviewed for Prelim One.

14 Day Fourteen: February 18

14.1 Some announcements

Today: office hours 3:30 PM – 4:30 PM

Tomorrow: office hours 11:30 AM – 12:30 PM

Sometime soon: Answers to Prelim One will be posted on Moodle.

Monday, February 23: Graded Prelims will be returned to you.

Monday, March 2: Prelim corrections are due.

Sometime after March 3: Full solutions to Prelim One will be posted on Moodle.

14.2 Some warm-up problems

Example 14.1.

Question. Suppose that a square is changing size. At the instant that its area is $100\,\mathrm{m}^2$, ...

1. ... what is the rate of change of the square's area with respect to the length of its sides?
2. ... what is the rate of change of the square's area with respect to the length of its diagonals?

Answer. Let A be the area of the square, let x be the length of the square's sides, and let d be the length of the square's diagonals. We know that $A = x^2$, so $\frac{dA}{dx} = 2x$. When $100 = A = x^2$ we have $x = \sqrt{100} = 10$ (where we use the positive square root since x is measuring the length of something), so

$$\left.\frac{dA}{dx}\right|_{x=10} = 2x|_{x=10} = 2 \cdot 10 = 20\,\mathrm{m}^2/\mathrm{m}.$$

Notice that if we cut the square diagonally, we obtain an isosceles right triangle with hypotenuse length d and leg length x. By the Pythagorean Theorem we have $d^2 = x^2 + x^2 = 2x^2$. Therefore $A = x^2 = \frac{1}{2}d^2$. Therefore $\frac{dA}{dd} = \frac{1}{2} \cdot 2d = d$. When $100 = A = \frac{1}{2}d^2$ we have $d = \sqrt{200} = 10\sqrt{2}$, so

$$\left.\frac{dA}{dd}\right|_{d=10\sqrt{2}} = 10\sqrt{2}\,\mathrm{m}^2/\mathrm{m}.$$

Example 14.2.

Question. Prove that for any real number k, $y = \tan kx$ is a solution to the differential equation

$$\frac{d^2y}{dx^2} = 2k^2y(1+y^2).$$

(HINT: Remember that $\sec^2\theta = 1 + \tan^2\theta$.)

Answer. We have $\frac{dy}{dx} = k\sec^2 kx$, so

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\,k\sec^2 kx \\ &= 2k\sec kx \cdot \left(\frac{d}{dx}\,\sec kx\right) \\ &= 2k\sec kx \cdot \sec kx \cdot \tan kx \cdot \left(\frac{d}{dx}\,kx\right) \\ &= 2k^2\sec^2 kx\,\tan kx.\end{aligned}$$

Using the trigonometric identity mentioned above, note that this is the same as

$$\begin{aligned}2k^2y(1+y^2) &= 2k^2\tan kx\,\left(1+\tan^2 kx\right) \\ &= 2k^2\tan kx\,\sec^2 kx,\end{aligned}$$

so the differential equation is satisfied.

14.3 More higher order derivatives

Sometimes we can figure out a general formula, sometimes called a **closed formula**, for the nth derivative of a particular function. This can be very useful when we start talking about using Taylor polynomials to approximate functions.

Many of these formulas make use of the **factorial function**, which for any positive integer n is denoted $n!$ and defined by

$$n! := n(n-1)\cdot(n-2)\cdot\cdots\cdot 3\cdot 2\cdot 1.$$

Despite the comedic value of doing so, the symbol "$n!$" is not read by shouting "n" loudly.

By definition, we put $0! := 1$. This may seem strange, except that $n!$ is the number of ways of ordering n objects, and there is really only one way to order *no* objects. There, we just did it. And oops, we did it again!

Example 14.3.

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$, and in general

$$\begin{aligned} f^{(k)}(x) &= n(n-1)(n-2)\cdots\big((n-(k-1)\big)x^{n-k} \\ &= \begin{cases} \frac{n!}{(n-k)!}x^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}. \end{aligned}$$

2. Suppose $y = \frac{1}{1+x} = (1+x)^{-1}$. Then

$$\begin{aligned} y' &= -(1+x)^{-2} \\ y'' &= -(-2)(1+x)^{-3} = 2(1+x)^{-3} \\ y''' &= -3(2)(1+x)^{-4} = -3!(1+x)^{-4} \text{ and} \\ y^{(4)} &= -4(-3!)(1+x)^{-5} = 4!(1+x)^{-5}. \end{aligned}$$

It seems pretty clear that

$$y^{(n)} = (-1)^n\, n!(1+x)^{-(n+1)} = \frac{(-1)^n\, n!}{(1+x)^{n+1}},$$

although it takes a little work to prove mathematically that this really is the right formula.

Note our use of $(-1)^n$ to give us 1 if n is even and -1 if n is odd. This is a handy little trick.

3. Suppose $f(x) = \sin(ax + b)$, where a and b are some constant real numbers. Then

$$\begin{aligned} f'(x) &= a\cos(ax+b), \\ f''(x) &= -a^2\sin(ax+b) = -a^2 f(x), \\ f'''(x) &= -a^3\cos(ax+b) = -a^3 f'(x), \\ f^{(4)}(x) &= a^4\sin(ax+b) = a^4 f(x), \text{ and} \\ f^{(4)}(x) &= a^5\cos(ax+b) = a^5 f'(x). \end{aligned}$$

The derivatives alternate between $f(x) = \sin(ax + b)$ and $f'(x) = \cos(ax + b)$ at each step, change sign every other step, and at the nth step have a factor of a^n out front. Therefore, it is safe to assume that

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax+b) & \text{if } n = 2k \text{ for some } k = 0, 1, 2, \ldots \\ (-1)^k a^n \cos(ax+b) & \text{if } n = 2k+1 \text{ for some } k = 0, 1, 2, \ldots \end{cases}.$$

Saying that $n = 2k$ for some $k = 0, 1, 2, \ldots$ is another way of saying that n is a nonnegative *even* integer, since every even integer can be written this way. Similarly, $n = 2k + 1$ means that n is a nonnegative *odd* integer. Notice that the use of $(-1)^k$ assures us that the sign will only change at*every other* step, whereas using $(-1)^n$ would make the sign change at *every* step.

14.4 Implicit functions

If a curve in the plane is actually the graph of a function, we know how to compute the slopes of its tangent lines, which in turn can give us lots of useful information about the curve. The goal of **implicit differentiation** is to be able to calculate the slopes of tangent lines to curves that aren't necessarily the graphs of functions, but are given by some equation in x and y.

A simple example of such a thing is the equation

$$y^2 = x,$$

whose corresponding curve in the xy-plane is a parabola opening to the right. We know that this curve is not the graph of any function $y = f(x)$, because it fails the **Vertical Line Test** (infinitely-many times, in fact). But, one thing we can do is

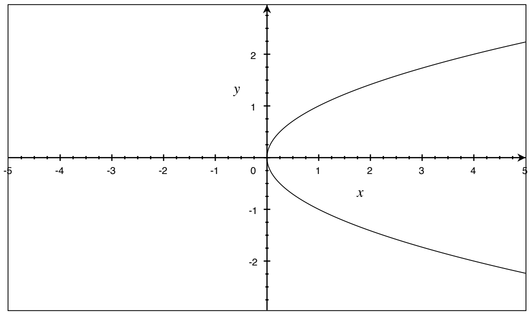

Figure 11: The curve corresponding to the equation $y^2 = x$. Notice that it is not the graph of any function $y = f(x)$.

solve for y, obtaining the two equations

$$y_1 = \sqrt{x} \qquad \text{and} \qquad y_2 = -\sqrt{x}.$$

These are functions, whose derivatives we can readily calculate:

$$y_1' = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad y_2' = -\frac{1}{2\sqrt{x}}.$$

Notice what is happening here.

Let $\mathcal{C}$ be the curve given by the equation $y^2 = x$.

- The part of $\mathcal{C}$ with $y \geq 0$ is the graph of the function $y = \sqrt{x}$.
- The part of $\mathcal{C}$ with $y \leq 0$ is the graph of the function $y = -\sqrt{x}$.

Definition 14.4. Given an equation in x and y, is there a way to restrict the values of x and y so that the equation defines y as a function of x? In other words, by making restrictions on the values of x and y, can we solve the equation for y?

If the answer is "yes", then we say that the equation **defines y implicitly as a function of x**.

This is a more general situation then where an equation in x and y is *already solved* for y, in which case we say that the equation **defines y explicitly as a function of x**. An example is the equation

$$y = x^2 + \sin x,$$

which clearly represents a function of x.

15 Day Fifteen: February 20

Remark 15.1. The examples in this day's notes are actually a bit different from those we did in class. In particular, there are a few more examples here than I managed to present live in class.

15.1 Introduction

Today we will expand our differentiation techniques to enable us to determine the slopes of tangent lines to a curve corresponding to an arbitrary equation involving x and y. Often, a piece of such a curve is actually the graph of a function, in which case this function is called an **implicit function** defined by the equation. If we can figure out what these implicit functions are, then we can simply differentiate them to obtain the tangent slopes. Unfortunately, this is often essentially impossible. Using **implicit differentiation**, we will be able to find a formula for the slope $\frac{dy}{dx}$ of the tangent line without having to find any implicit functions. The price we pay is that this formula almost always involves both x and y.

15.2 What is an implicit function?

Almost any equation in x and y defines a curve in the xy-plane, consisting of the points (x, y) that satisfy the given equation. For example, the unit circle centered at the origin consists of all points (x, y) such that $x^2 + y^2 = 1$. Of course, the graph

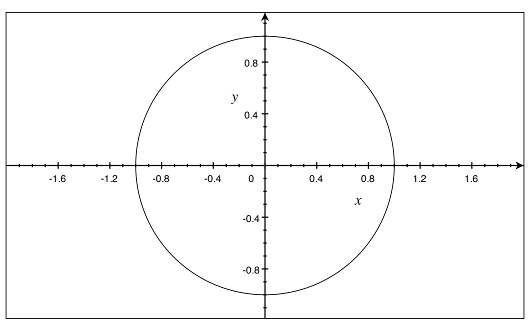

Figure 12: The curve formed by points (x, y) satisfying the equation $x^2 + y^2 = 1$.

of any function f defines an equation, and this equation determines a curve. This curve is exactly the graph of the function f, by definition. For instance, the function $f(x) = \sin x$ determines the equation $y = \sin x$, and this equation determines the graph of f in the xy-plane. Most equations, such as $x^2 + y^2 = 1$, are not of this form. However, many times *a piece* of the curve corresponding to the equation might be the graph of a function. We can visually recognize this occurrence by checking whether *just that piece* of the curve passes the Vertical Line Test.

For the example of the unit circle, clearly the top semicircle is the graph of some function, as is the bottom semicircle. We know from experience that these pieces of the curve are given as the graphs of the functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$, respectively. If we are given a formula for a function f, how can we check whether its graph fits in the curve corresponding to a particular equation *without having to draw any pictures*?

For convenience, notice that any equation in x and y can be rewritten in the form $F(x, y) = 0$ for some function $F(x, y)$ of two variables. For example, the equation $y^2 = x$ can be written as

$$F(x, y) = 0, \qquad \text{where } F(x, y) = y^2 - x,$$

and the equation $x^2 + y^2 = 1$ can be written as

$$G(x, y) = 0, \qquad \text{where } G(x, y) = x^2 + y^2 - 1.$$

Definition 15.2. A function $y = f(x)$ is **implicitly defined** by the equation $F(x, y) = 0$ if

$$F\big(a, f(a)\big) = 0$$

for all a in the domain of f. This means exactly that every point $(x, y) = \big(x, f(x)\big)$ on the graph of $y = f(x)$ is a solution of the equation $F(x, y) = 0$. We sometimes say that f is an **implicit function** defined by the equation $F(x, y) = 0$.

Example 15.3.

1. Consider the equation $y^2 = x$ from last class. We can write this as $F(x, y) = 0$, where $F(x, y) = y^2 - x$. We noted previously that the graphs of the functions $f_1(x) = \sqrt{x}$ and $f_2(x) = -\sqrt{x}$ are all contained in the curve corresponding to $y^2 = x$. Observe that f_1 and f_2 are indeed implicitly defined by this equation, because

$$F\big(x, f_1(x)\big) = F\big(x, \sqrt{x}\big) = (\sqrt{x})^2 - x = x - x = 0$$

and

$$F\big(x, f_2(x)\big) = F\big(x, -\sqrt{x}\big) = (-\sqrt{x})^2 - x = x - x = 0.$$

2. Let $G(x, y) = x^2 - y^2 - 1$, so that the equation $G(x, y) = 0$ describes the unit circle centered at the origin. Let $g_1(x) = \sqrt{1 - x^2}$ and $g_2(x) = -\sqrt{1 - x^2}$. Then

$$G\big(x, g_1(x)\big) = G\big(x, \sqrt{1 - x^2}\big) = x^2 + (\sqrt{1 - x^2})^2 - 1 = x^2 + 1 - x^2 - 1 = 0$$

and

$$G\big(x, g_2(x)\big) = G\big(x, -\sqrt{1 - x^2}\big) = x^2 + (-\sqrt{1 - x^2})^2 - 1 = x^2 + 1 - x^2 - 1 = 0.$$

Clearly, the easiest way to get an implicit function out of an equation is to solve that equation for y. This is exactly how one obtains f_1 and f_2 from $F(x, y) = 0$ and g_1 and g_2 from $G(x, y) = 0$ in the previous example. Unfortunately, it is not hard at all to come up with an equation which cannot easily be solved for y, but which nonetheless determines a curve that obviously has pieces that are the graphs of some functions. See Figure 13, for instance.

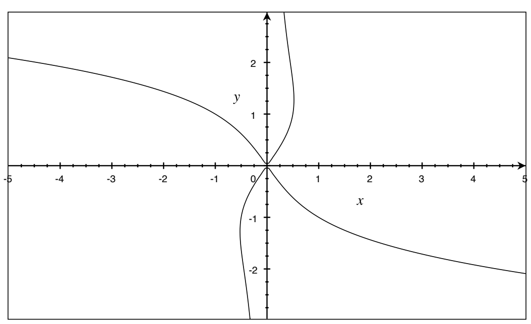

Figure 13: The curve corresponding to the equation $xy^3 - y^2 + 2x^2 = 0$. Certain pieces of this curve must be the graphs of some functions, but it is unclear what these functions are.

In this class, we have learned how to use information about the slopes of the tangent lines to curves to study those curves. So far, we only really know how to determine the slope of a tangent line to the graph of a function. When a piece of a curve can be represented as the graph of a function, we can determine the slope of tangent lines to that piece by differentiating the function. But as we saw, it is not always easy, or indeed *even possible*, to figure out what the implicit function is! But by a minor miracle, *we can essentially calculate the derivative of the implicit function without having to know what it is!* This process is called **implicit differentiation**.

15.3 Implicit differentiation

Beginning with an equation, such as $y^2 = x$, the process of finding a formula for the slopes of tangent lines to the curve represented by this equation, which we call **implicit differentiation**, proceeds in two relatively easy steps.

Step 1. *Differentiate both sides of the equation with respect to x (i.e. take $\frac{d}{dx}$ of each side and set the results equal to each other), treating y as a function of x.*

For example, if y is a function of x, then $\frac{d}{dx}\,y = \frac{dy}{dx}$, so by the Chain Rule we have

$$\frac{d}{dx}\,y^2 = 2y\left(\frac{d}{dx}\,y\right) = 2y\,\frac{dy}{dx}.$$

(Sometimes we write y' for $\frac{dy}{dx}$, to save labour.) So the equation $y^2 = x$ leads to the following equations.

$$\begin{aligned} y^2 &= x \\ \frac{d}{dx}\,y^2 &= \frac{d}{dx}\,x \\ 2y\,\frac{dy}{dx} &= 1 \end{aligned}$$

Step 2. *Solve for $\frac{dy}{dx} = y'$.*

In our example, we have $2y\,\frac{dy}{dx} = 1$, so

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Remark 15.4. You probably noticed that the result of implicit differentiation is a formula for $\frac{dy}{dx}$ *in terms of both x and y*. In theory, if y really is a function of x, then this formula for $\frac{dy}{dx}$ really only depends on x, but that relies on our being able to figure out a formula for the implicit function y as a function of x. To avoid doing this, we pay the price of having to a formula involving both x and y, but it is a small price to pay in most situations.

Example 15.5.

Above, we found that for the curve represented by the equation $y^2 = x$, we have $\frac{dy}{dx} = \frac{1}{2y}$. Let's check this formula.

Suppose we want the slope of the tangent line to $y^2 = x$ at the point $(4, 2)$, noting that this point *actually is* on this curve. This point lies on the portion of the curve that is given as the graph of the function $f(x) = \sqrt{x}$, so the slope of the tangent when $x = 4$ is

$$f'(4) = f'(x)\big|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Using our result from implicit differentiation, we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=(4,2)} = \left.\frac{1}{2y}\right|_{(x,y)=(4,2)} = \frac{1}{2\cdot 2} = \frac{1}{4}.$$

Our answers match!

It is not too hard to verify that this will happen for any x in the domain of $f(x) = \sqrt{x}$. Note that

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(x,f(x)\right)} = \left.\frac{1}{2y}\right|_{(x,y)=(x,\sqrt{x})} = \frac{1}{2\sqrt{x}} = f'(x).$$

Example 15.6.

1. **Question:** Find the slope of the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

   Answer: First note that this point *actually is* on the circle, because $(-3)^2 + 4^2 = 9 + 16 = 25$. We could do this problem by using the implicit function $y = \sqrt{1 - x^2}$, but that would involve differentiating a square root and a pesky Chain Rule. Implicit differentiation is easier.

$$\begin{aligned}
x^2 + y^2 &= 25 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(25) \\
2x + 2y\,\frac{dy}{dx} &= 0 \\
2y\,\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{2x}{2y} = -\frac{x}{y}
\end{aligned}$$

Therefore

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-3,4)} = -\left.\frac{x}{y}\right|_{(x,y)=(-3,4)} = -\frac{-3}{4} = \frac{3}{4}.$$

2. **Question:** Find $\frac{dy}{dx}$ if $y\,\sin x = x^3 + \cos y$.

Answer: Here we do not even have the option of solving for y to find an implicit function. We have to use implicit differentiation.

$$\begin{aligned}
y\,\sin x &= x^3 + \cos y \\
\frac{d}{dx}\left(y\,\sin x\right) &= \frac{d}{dx}\left(x^3 + \cos y\right) \\
\frac{dy}{dx}\cdot \sin x + y\,\cos x &= 3x^2 - \sin y\,\frac{dy}{dx} \\
\sin x\,\frac{dy}{dx} + \cos y\,\frac{dy}{dx} &= 3x^2 - y\,\cos x \\
(\sin x + \cos y)\,\frac{dy}{dx} &= 3x^2 - y\,\cos x \\
\frac{dy}{dx} &= \frac{3x^2 - y\,\cos x}{\sin x + \cos y}
\end{aligned}$$

Notice how we used the Product Rule to compute $\frac{d}{dx}\left(y\,\sin x\right)$.

3. **Question:** Find an equation of the line tangent to the curve $x^2 + xy + 2y^3 = 4$ at the point $(-2, 1)$.

Answer: Note that this point is indeed on the curve, since $(-2)^2 + (-2)(1) + 2(1)^3 = 4 - 2 + 2 = 4$. Taking $\frac{d}{dx}$ of each side of the equation, we have the following.

$$\begin{aligned} 2x + y + x\,\frac{dy}{dx} + 6y^2\,\frac{dy}{dx} &= 0 \\ (x + 6y^2)\,\frac{dy}{dx} &= -(2x + y) \\ \frac{dy}{dx} &= -\frac{2x + y}{x + 6y^2} \end{aligned}$$

Therefore
$$\left.\frac{dy}{dx}\right|_{(x,y)=(-2,1)} = -\frac{2(-2) + 1}{-2 + 6(1)^2} = -\frac{-3}{4} = \frac{3}{4}.$$

Therefore this tangent line has slope $\frac{3}{4}$ and contains the point $(-2, 1)$, so it is given by the equation

$$y - 1 = \frac{3}{4}\,(x + 2) \qquad \text{or} \qquad y = \frac{3}{4}\,x + \frac{5}{2}.$$

4. **Question:** For any constants a and b, the equations $x^2 - y^2 = a$ and $xy = b$ represent **hyperbolas**. Show that they intersect at right angles, i.e. that at any point where the curves intersect, their tangent lines are perpendicular.

   Answer: The tangent slope to the curve $x^2 - y^2 = a$ is given by the equation $2x - 2y\,\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = \frac{x}{y}$. The tangent slope to $xy = b$ is given by $y + x\,\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y}{x}$.
   Suppose the two curves intersect at the point (x_0, y_0). The tangent slopes at this point are $\frac{x_0}{y_0}$ and $-\frac{y_0}{x_0}$. Since

   $$\frac{x_0}{y_0} \cdot -\frac{y_0}{x_0} = -1,$$

   the tangent lines at this point are perpendicular to each other.

We can also use implicit differentiation to calculate higher derivatives of y with respect to x, although this can get messy very quickly.

Example 15.7.

1. **Question:** If $xy + y^2 = 2x$, what is y''?

Answer: First we calculate y'.

$$\begin{aligned} y + xy' + 2yy' &= 2 \\ y'(x+2y) &= 2 - y \\ y' &= \frac{2-y}{x+2y} \end{aligned}$$

Just as $y' = \frac{d}{dx}\, y$, note that $y'' = \frac{d}{dx}\, y'$. Therefore

$$\begin{aligned} y'' &= \frac{d}{dx}\, y' \\ &= \frac{d}{dx}\, \frac{2-y}{x+2y} \\ &= \frac{(x+2y) \cdot \frac{d}{dx}\,(2-y) - (2-y) \cdot \frac{d}{dx}\,(x+2y)}{(x+2y)^2} \\ &= \frac{(x+2y)(2-y') - (2-y)(1+2y')}{(x+2y)^2}. \end{aligned}$$

This might seem to be the best we can do, but notice that this is a formula for y'' involving *three quantities*: x, y, **and** y'. But we already have a formula for y' in terms of x and y, so let's use it! Our final formula for y'' is

$$\begin{aligned} y'' &= \frac{(x+2y)(2-y') - (2-y)(1+2y')}{(x+2y)^2} \\ &= \frac{(x+2y)\left(2 - \frac{2-y}{x+2y}\right) - (2-y)\left(1 + 2 \cdot \frac{2-y}{x+2y}\right)}{(x+2y)^2}. \end{aligned}$$

This formula is a little messy, but unless there is any particular reason to simplify it, we can just leave it as it is.

2. **Question:** Find $\frac{d^2y}{dx^2}$ if $x^2 + xy + 2y^3 = 4$.

Answer: We calculated in part 3. of Example 15.5 that

$$\frac{dy}{dx} = -\frac{2x+y}{x+6y^2}.$$

Therefore

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{2x+y}{x+6y^2}\right) \\
&= -\frac{(x+6y^2)\left(2+\frac{dy}{dx}\right)-(2x+y)\left(1+12y\,\frac{dy}{dx}\right)}{(x+6y^2)^2} \\
&= -\frac{(x+6y^2)\left(2-\frac{2x+y}{x+6y^2}\right)-(2x+y)\left(1-12y\cdot\frac{2x+y}{x+6y^2}\right)}{(x+6y^2)^2}.
\end{aligned}$$