

## A FUNKY SINE CURVE SANDWICH

### 1 The original funky sine curve

In class we encountered the *funky sine curve*,  $y = \sin(1/x)$ , which oscillates madly between  $-1$  and  $1$  as  $x$  approaches  $0$ . This happens because when we take any tiny open  $x$ -interval around  $0$ , like  $(-\epsilon, \epsilon)$  for some tiny tiny  $\epsilon > 0$ , and put it into the function  $1/x$ , (excluding  $0$ , of course), the output is the union of two infinite rays,

$$\begin{aligned} x &\mapsto 1/x, \\ (-\epsilon, \epsilon) - \{0\} &\mapsto (-\infty, -1/\epsilon) \cup (1/\epsilon, \infty). \end{aligned}$$

When we put these rays into the sine function, we get an infinite number of oscillations without end. But the function  $\sin(1/x)$  has to do all of these oscillations in the tiny interval  $(-\epsilon, \epsilon)$ ! The result can be seen in Figure 1.

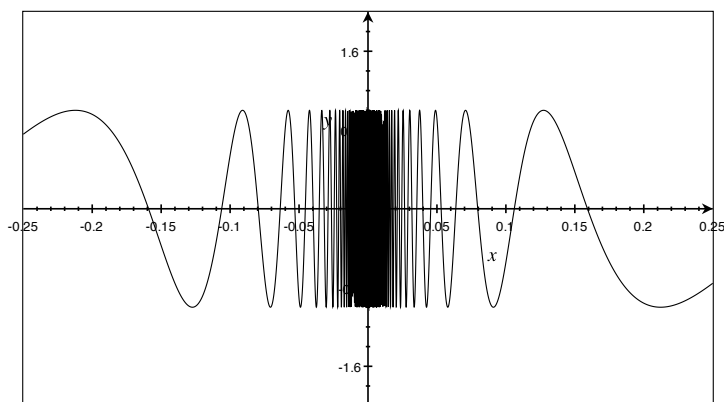


Figure 1: The funky sine curve,  $y = \sin(1/x)$ .

Recall that  $\lim_{x \rightarrow 0} \sin(1/x)$  does not exist, because the function's values *do not approach a single number* as  $x$  approaches  $0$ .

## 2 A dampened funky sine curve

Since the curve  $y = \sin(1/x)$  bounces back and forth between  $-1$  and  $1$ , it is easy to see that  $y = 2\sin(1/x)$  will bounce between  $-2$  and  $2$ , that  $y = 1000\sin(1/x)$  will bounce between  $-1000$  and  $1000$ , and so on. These numbers,  $1$ ,  $2$ , and  $1000$ , are called the **amplitudes** of the sinusoidal (i.e. sine-like) curves.

Consider now the curve  $y = x \sin(1/x)$ . Instead of bouncing back and forth between two *horizontal lines*, this will bounce between the lines  $y = x$  and  $y = -x$ . It has a non-constant amplitude! It still oscillates madly, but its oscillations are **dampened** by the function  $x$ , as seen in Figure 2.

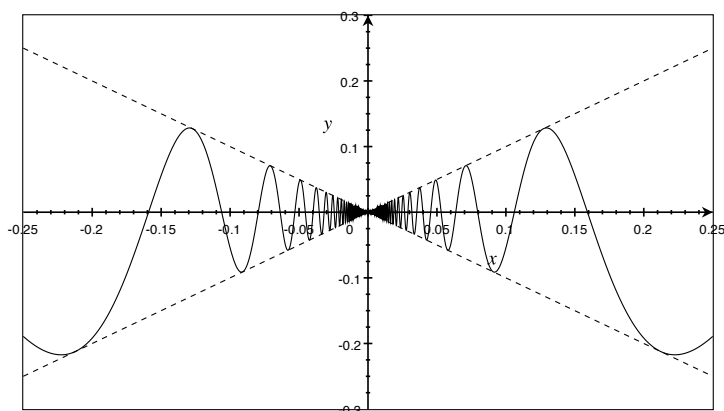


Figure 2: The funky sine curve dampened by the function  $x$ ,  $y = x \sin(1/x)$ .

This dampening factor has the following important effect.

**Claim.**  $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ .

*Proof #1.* From its graph, we know that

$$-|x| \leq x \sin(1/x) \leq |x|$$

for all  $x \neq 0$ . Note also that  $\lim_{x \rightarrow \infty} -|x| = \lim_{x \rightarrow \infty} |x| = 0$ . By the Sandwich Theorem (a.k.a. the Squeeze Theorem), this implies that

$$\lim_{x \rightarrow 0} x \sin(1/x) = 0.$$

QED

### 3 What if we don't know what the graph looks like?

Suppose now that we had no idea what the graph of the dampened funky sine curve  $y = x \sin(1/x)$  looked like. We can still figure out what's happening at 0.

**Claim.**  $\lim_{x \rightarrow 0} x \sin(1/x) = 0$ .

*Proof #2.* First recall that

$$-1 \leq \sin(1/x) \leq 1 \quad \text{for all } x \neq 0. \quad (1)$$

There are two cases.

**Case 1.** Suppose  $x > 0$ . Since multiplication by a positive number *preserves* inequalities, we can multiply each piece of Inequality (1) by  $x$  and obtain

$$-x \leq x \sin(1/x) \leq x \quad \text{for all } x > 0. \quad (2)$$

**Case 2.** Suppose  $x < 0$ . Since multiplication by a negative number *reverses* inequalities, we can multiply each piece of Inequality (1) by  $x$  and obtain

$$-x \geq x \sin(1/x) \geq x,$$

i.e.

$$x \leq x \sin(1/x) \leq -x \quad \text{for all } x < 0. \quad (3)$$

Now, recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{and} \quad -|x| = \begin{cases} -x & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}.$$

Putting together Inequalities (2) and (3), we conclude that

$$-|x| \leq x \sin(1/x) \leq |x| \quad \text{for all } x \neq 0.$$

We can now use the Sandwich Theorem, exactly as in Proof #1.

QED