Tangent lines to a parametrized curve

Question. Consider the curve $\mathcal{C}$ parametrized by

$$x = t^3 - 3t, \quad y = t^2, \quad -2 \leq t \leq 2.$$

(a) What values of t correspond to the point $(0, 3)$ on the curve $\mathcal{C}$? What does it mean if more than one value of t gives you the same point?

(b) Find equations for all tangent lines to $\mathcal{C}$ at the point $(0, 3)$.

(c) Sketch the curve $\mathcal{C}$ along with the tangent lines you determined in part (b). Use arrows to indicate the direction in which this parametrization traces out the curve.

(d) Compute the second derivative $\frac{d^2y}{dx^2}$ of the curve $\mathcal{C}$ at the values of t you found in part (a) above.

Solution.

(a) We set $t^3 - 3t = 0$ and $t^2 = 3$. Factoring the first equation, we obtain

$$0 = t^3 - 3t = t(t^2 - 3) = t(t - \sqrt{3})(t + \sqrt{3}),$$

so $t = 0, \sqrt{3}, -\sqrt{3}$. Only the second two values are also solutions to $t^2 = 3$, so the values of t which yield the point $(0, 3)$ are $\boxed{t = \sqrt{3}, -\sqrt{3}}$. This means that as we trace out the curve $\mathcal{C}$, we go through the point $(0, 3)$ twice, so the curve crosses itself.

(b) Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 3},$$

we have

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{2\sqrt{3}}{3(\sqrt{3})^2 - 3} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

and

$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{-2\sqrt{3}}{3(-\sqrt{3})^2 - 3} = \frac{-2\sqrt{3}}{6} = -\frac{\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$$

Since both tangent lines contain the point $(0, 3)$, they have equations $\boxed{y - 3 = \frac{1}{\sqrt{3}}\,x}$ and $\boxed{y - 3 = -\frac{1}{\sqrt{3}}\,x}$.

(c) To sketch the curve $\mathcal{C}$, we will evaluate the parametrization at some different values of t.

t	$x = t^3 - 3t$	$y = t^2$	(x, y)
-2	-2	4	$(-2, 4)$
$-\sqrt{3}$	0	3	$(0, 3)$
-1	2	1	$(2, 1)$
0	0	0	$(0, 0)$
1	-2	1	$(-2, 1)$
$\sqrt{3}$	0	3	$(0, 3)$
2	2	4	$(2, 4)$

If we plot these points (labeled with the corresponding values of t), connect the dots, and draw arrows to indicate the direction of increasing values of t, we obtain the picture shown in Figure 1.

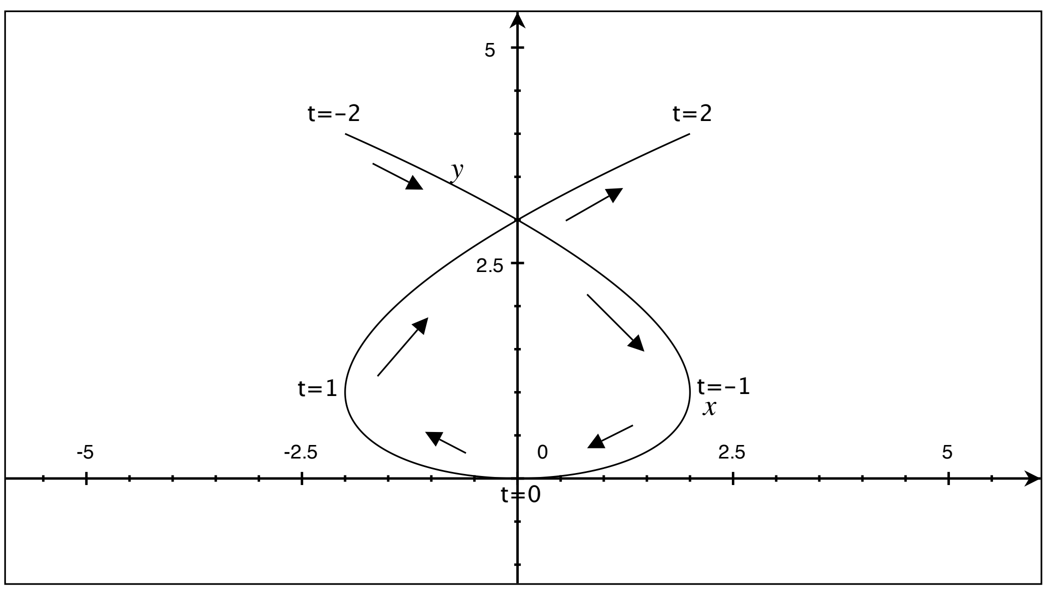

Figure 1: The curve $\mathcal{C}$, with information about how it was parametrized.

Oops! I forgot to put in the tangent lines. Sorry! I think you can see where they should go.

(d) First we compute

$$\frac{d}{dt}\left(\frac{2t}{3t^2-3}\right) = \frac{(2)(3t^2-3)-(2t)(6t)}{(3t^2-3)^2} = \frac{6t^2-6-12t^2}{(3t^2-3)^2} = \frac{-6t^2-6}{(3t^2-3)^2}.$$

Then since

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}(dy/dx)}{dx/dt} \\ &= \frac{(-6t^2-6)/(3t^2-3)^2}{3t^2-3} \\ &= -\frac{6t^2+6}{(3t^2-3)^3},\end{aligned}$$

we have

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pm\sqrt{3}} = -\frac{6(\pm\sqrt{3})^2+6}{(3(\pm\sqrt{3})^2-3)^3} = -\frac{6(3)+6}{(3(3)-3)^3} = -\frac{24}{(6)^3} = \boxed{-\frac{1}{9}}.$$